Complexity Analysis for Termination by a Well-Quasi-Order

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Outline

- Vector Addition Systems
- Reachability
- Termination
- Complexity

well-quasi-orders (wqo) as a means to prove termination complexity

- combinatorial analysis of controlled bad sequences

- application reachability in vector addition systems (VAS)
Outline

**Length Function Theorem (Figueira, Figueira, S., and Schnoebelen, '11)**

Bad sequences over $\mathbb{N}^d$ controlled by primitive recursive functions are of at most Ackermannian length.

**Upper Bound Theorem (Leroux and S., '15)**

Reachability in vector addition systems is in cubic Ackermann.
VECTOR ADDITION SYSTEMS
Vector Addition Systems

Springfield Power Plant

Can we produce unbounded electricity with no left-over uranium waste?

produce electricity

recycle uranium

(1,1)

(-1,-2)

(q,0,1)

(q,ω,0) is reachable
Can we produce unbounded electricity with no leftover uranium waste? Yes, \((q, \omega, 0)\) is reachable.
**IMPORTANCE OF THE PROBLEM**

**Discrete Resources**

- *modelling*: items, money, energy, molecules, …
- *distributed computing*: active threads in thread pool
- *data*: isomorphism types in data logics and data-centric systems

**Central Decision Problem**

Large number of problems interreducible with VAS reachability
 IMPORTANCE OF THE PROBLEM

**Discrete Resources**

- **modelling**: items, money, energy, molecules, ...
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**Central Decision Problem**

Large number of problems interreducible with VAS reachability
**Importance of the Problem**

- 1962: C. A. Petri: Petri nets
- 1976: R. J. Lipton: EXPSpace lower bound
- 1981: E. W. Mayr: decidability by decomposition
- 1982: J.-L. Lambert: decidability by decomposition
- 1992: S. R. Kosaraju: decidability by decomposition
- 2011: J. Leroux: decidability by Presburger inductive invariants
- 2015: this talk: cubic Ackermann upper bound
Perfect Marked Witness Graphs

$q, 0, 1 \rightarrow (q, \omega, \omega) \rightarrow q, \omega, 0

(1, 1)

(-1, -2)
**Perfect Marked Witness Graphs**

Characteristic System

\[0 + 1 \cdot a - 1 \cdot b = c\]

\[1 + 1 \cdot a - 2 \cdot b = 0\]

Homogeneous System

\[1 \cdot a - 1 \cdot b = c\]

\[1 \cdot a - 2 \cdot b = 0\]

\[a, b, c > 0\]
Perfect Marked Witness Graphs

Characteristic System

\[
0 + 1 \cdot a - 1 \cdot b = c \\
1 + 1 \cdot a - 2 \cdot b = 0
\]

Homogeneous System

\[
1 \cdot a - 1 \cdot b = c \\
1 \cdot a - 2 \cdot b = 0 \\
a, b, c > 0
\]

Solutions over \( \mathbb{N} \)

\((1,1,0)\) 
\((4,2,2)\)
**PERFECT MARKED WITNESS GRAPHS**

**Characteristic System**

\[
\begin{align*}
0 + 1 \cdot a - 1 \cdot b &= c \\
1 + 1 \cdot a - 2 \cdot b &= 0
\end{align*}
\]

**Homogeneous System**

\[
\begin{align*}
1 \cdot a - 1 \cdot b &= c \\
1 \cdot a - 2 \cdot b &= 0
\end{align*}
\]

\[a, b, c > 0\]

**Solutions over \(\mathbb{N}\)**

\[(1,1,0) \quad (4,2,2)\]

**Euler’s Lemma** on associated system of equations

- Perform once

  \[(0,-1)\]

- Repeat

  \[(2,0)\]
Perfect Marked Witness Graphs
Perfect Marked Witness Graphs

Pumpable Paths: up to and down from $(\omega, \omega)$

$pump\ up + pump\ down + remainder = solution\ of\ homogeneous\ system$

(we picked a large enough solution)
**Perfect Marked Witness Graphs**
**Perfect Marked Witness Graphs**

**Perfectness (aka Theta Condition)**

0. solution of characteristic system:

1. solution of homogeneous system:

2. up/down pumpable paths: and

⇒ implies existence of a run
Perfect Marked Witness Graphs

Checking Perfectness

0. solution of characteristic system
   ▶ in \( \text{NPTime} \)

1. solution of homogeneous system
   ▶ in \( \text{NPTime} \)

2. up/down pumpable paths
   ▶ place boundedness problem, in \( \text{ExpSpace} \) [Demri ‘13, Blockelet and S. ‘11]
Imperfect Marked Witness Graphs

When imperfect: decompose in a set of sequences of graphs
Imperfect Marked Witness Graphs

0. solution of characteristic system
   ▶ otherwise, no run; empty decomposition
Imperfect Marked Witness Graphs

1. solution of homogeneous system
   a. otherwise, bound on number of uses of a transition
   b. or bound on an “ω” in input/output constraint
**Imperfect Marked Witness Graphs**

1. solution of homogeneous system
   
   a. otherwise, bound on number of uses of a transition

   Here, \( t_3 = 1 \) and \( t_5 = 0 \) in the characteristic system

   b. or bound on an “\( \omega \)” in input/output constraint
**Imperfect Marked Witness Graphs**

1. solution of homogeneous system
   a. otherwise, bound on number of uses of a transition

   ![Diagram](image)

   - here, \( t_3 = 1 \) and \( t_5 = 0 \) in the characteristic system

b. or bound on an “\( \omega \)” in input/output constraint
**Imperfect Marked Witness Graphs**

1. solution of homogeneous system
   a. otherwise, bound on number of uses of a transition

   ![Diagram of Vector Addition Systems]

   ▶ here, $t_3 = 1$ and $t_5 = 0$ in the characteristic system; decompose as

   ![Decomposed diagram of Vector Addition Systems]

   b. or bound on an “ω” in input/output constraint
**Imperfect Marked Witness Graphs**

2. up/down pumpable paths
   - otherwise, bound on some reachable/co-reachable coordinates
Improper Marked Witness Graphs

2. up/down pumpable paths
   ▶ otherwise, bound on some reachable/co-reachable coordinates

\[
\begin{align*}
q_0, 1, 0, 1 & \rightarrow q_0, \omega, \omega, \omega \\
(1, 1, -1) & \rightarrow q_0, \omega, \omega, \omega \\
(1, 0, 0) & \rightarrow q_1, \omega, \omega, \omega \\
(0, -1, 0) & \rightarrow q_1, \omega, \omega, \omega \\
(0, -1, 0) & \rightarrow q_1, 2, 2, 1
\end{align*}
\]
Imperfect Marked Witness Graphs

2. up/down pumpable paths
   ▶ otherwise, bound on some reachable/co-reachable coordinates

\[(1,1,-1)\]
\[q_0,1,0,1 \rightarrow q_0,\omega,\omega,\omega \rightarrow q_0,\omega,\omega \rightarrow (1,0,0) \rightarrow q_1,\omega,\omega,\omega \rightarrow q_1,2,2,1\]
\[(-1,0,1)\]

▶ here decompose as

\[q_0,2,\omega,0\]
\[(-1,0,1) \rightarrow (1,1,-1)\]
\[q_0,1,0,1 \rightarrow q_0,1,\omega,1 \rightarrow q_0,1,\omega,1 \rightarrow (1,0,0) \rightarrow q_1,2,\omega,1 \rightarrow q_1,2,2,1\]
\[(-1,0,1) \rightarrow (1,1,-1)\]
\[q_0,0,\omega,2\]
Vector Addition Systems
Reachability
Termination
Complexity

**DECOMPOSITION ALGORITHM**

*[Mayr'81, Kosaraju'82, Lambert'92]*

**STRUCTURES: Sequences of Marked Witness Graphs**

\[ \sigma = \begin{array}{c}
M_0 \\
M_1 \\
M_k
\end{array} \]

**Algorithm**

\[ S_0, S_1, \ldots: \text{finite sets of sequences of marked witness graphs} \]

**init** \( S_0 \)

\[ \forall n : \text{if } S_n = \{ \sigma \} \cup S \text{ and } \text{perfect}(\sigma) \]

\[ S_{n+1} = S \cup \text{decompose}(\sigma) \]

\[ \text{otherwise stop: src} \rightarrow^* \text{tgt iff } S_n = \{ \} \]
**Decomposition Algorithm**

[Mayr’81, Kosaraju’82, Lambert’92]

**Structures:** Sequences of Marked Witness Graphs

\[ \sigma = \text{src} \xrightarrow{a_1} \text{out}_0 \xrightarrow{\text{in}} \text{in}_1 \xrightarrow{\text{out}} \cdots \xrightarrow{\text{out}} \text{tgt} \]

**Algorithm**

\[ S_0, S_1, \ldots : \text{finite sets of sequences of marked witness graphs} \]

\[
\begin{align*}
\text{init} & \quad S_0 \\
\forall n & \quad \text{if } S_n = \{\sigma\} \cup S \text{ and } \neg \text{perfect}(\sigma) \\
S_{n+1} & \overset{\text{def}}{=} S \cup (\text{decompose}(\sigma)) \\
\text{otherwise stop: } \text{src} \rightarrow^* \text{tgt} \text{ iff } S_n \neq \emptyset
\end{align*}
\]
Decomposition Algorithm

[Mayr’81, Kosaraju’82, Lambert’92]

Structures: Sequences of Marked Witness Graphs

$\sigma = \langle M_0, a_1, M_1, \ldots, M_k \rangle$

Algorithm

$S_0, S_1, \ldots$: finite sets of sequences of marked witness graphs

init $S_0$

$\forall n \quad \text{if } S_n = \{\sigma\} \cup S \text{ and } \neg \text{perfect} (\sigma)$

$S_{n+1} \overset{\text{def}}{=} S \cup (\text{decompose} (\sigma))$

• otherwise stop: $\text{src} \rightarrow^* \text{tgt}$ iff $S_n \neq \emptyset$
"Finally the checker has to verify that the process comes to an end. Here again he should be assisted by the programmer giving a further definite assertion to be verified. This may take the form of a quantity which is asserted to decrease continually and vanish when the machine stops."

[Turing’49]
“Finally the checker has to verify that the process comes to an end. Here again he should be assisted by the programmer giving a further definite assertion to be verified. This may take the form of a quantity which is asserted to decrease continually and vanish when the machine stops. To the pure mathematician it is natural to give an ordinal number.”

[Turing’49]
**Termination**

**Ordinals**
isomorphism classes of well-orders

**Ranking Functions**
\[ f : \text{Conf} \to \alpha \text{ s.t. } c \to c' \text{ implies } f(c) > f(c') \]

**For the Decomposition Algorithm**
- into \( \omega^3 \)
- \( \sigma' \in \text{decompose}(\sigma) \text{ implies } f(\sigma) > f(\sigma') \)
TERMINATION

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FOR THE DECOMPOSITION ALGORITHM

- into \( \omega^3 \)
- \( \sigma' \in \text{decompose}(\sigma) \) implies \( f(\sigma) > f(\sigma') \)
The Length of Decreasing Sequences

- no upper bound in general
  e.g. in $\omega$:

  \[
  N > N - 1 > N - 2 > \cdots > 1 > 0
  \]

- even with a fixed initial element
  e.g. in $\omega + 2$:

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  \omega + 1 > \omega > N > \cdots > 1 > 0
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The Length of Decreasing Sequences

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  \[ \omega + 1 > \omega > N > \cdots > 1 > 0 \]
**Controlled Descending Sequences**

- $\alpha$ an **ordinal**
- $\mathbb{N}: \alpha \rightarrow \mathbb{N}$ an **ordinal norm**
- $g: \mathbb{N} \rightarrow \mathbb{N}$ monotone a **control function**
- $n_0 \in \mathbb{N}$ an **initial norm**

- a sequence $\beta_0, \beta_1, \ldots$ over $\alpha$ is $(g, n_0)$-controlled if
  \[ \forall i. N(\beta_i) \leq g^i(n_0) \]

- $(g, n_0)$-controlled descending sequences $\beta_0 > \beta_1 > \ldots$ over $\alpha$ have a maximal length $L_{\alpha, g}(n_0)$
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**Length Function Theorem (for Ordinals, S., '14)**

Descending sequences over $\omega^\omega^3$ controlled by Ackermannian functions are of at most cubic Ackermannian length.
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CONTROLLING DECOMPOSITIONS

0. no solution to characteristic system
   ▶ empty decomposition

1. no solution to homogeneous system
   a. bound on number of uses of a transition
   b. or bound on an “ω” in input/output constraint
      ▶ exponential blow-up

2. no up/down pumpable paths
   ▶ Ackermannian blow-up
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**Coverability Trees**

[Karp and Miller '69]

**Pumping up?**

\[ q_0, 1, 0, 1 \rightarrow q_0, \omega, \omega, \omega \]

\[ (1, 1, -1) \]

\[ (-1, 0, 1) \]
Coverability Trees

[Karp and Miller ‘69]

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Termination
\[ Q \times N \]
\[ \text{is a wqo:} \]
\[ r\text{-bad sequence} \]
\[ c_0, c_1, \ldots : \]
\[ \not\exists i_0 < i_1 < \cdots < i_r \text{ s.t.} \]
\[ c_{i_0} \leq c_{i_1} \leq \cdots \leq c_{i_r} \]

\[ r\text{-bad sequences are finite} \]
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\[ q_{0,1,0,1} \rightarrow q_{0,\omega,\omega,\omega} \]

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Termination

\[ Q \times N^d \] is a wqo:

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Coverability Trees

[Karp and Miller '69]

**Pumping up?**

$q_{0,1,0,1} \rightarrow q_{\omega,\omega,\omega}$

$q_{0,1,0,1} \rightarrow (1,1,-1)$

$q_{0,2,1,0}$

$q_{0,0,0,2}$

$q_{0,1,\omega,1}$

$q_{0,1,0,1}$

$(1,1,-1)$

$(-1,0,1)$

$(-1,0,1)$

$q_{0,0,0,2}$

$13/17$
**Coverability Trees**

[Karp and Miller ’69]

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Coverability Graphs

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COVERABILITY GRAPHS

(Karp and Miller '69)

PUMPING UP?

$q_0, 1, 0, 1 \rightarrow q_0, \omega, \omega, \omega$

$(1,1,-1)$

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Termination

$Q \times N^d$ is a wqo:

$r$-bad sequence $c_0, c_1, ...$

$\not\exists i_0 < i_1 < \cdots < i_r$ s.t.

$c_{i_0} \leq c_{i_1} \leq \cdots \leq c_{i_r}$

$r$-bad sequences are finite
**Coverability Graphs**

[Karp and Miller '69]

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$$q_0,1,0,1 \rightarrow q_0,\omega,\omega,\omega$$

$$(-1,0,1)$$

**Termination**

$$Q \times \mathbb{N}^d$$ is a wqo:

- r-bad sequence $$c_0, c_1, \ldots$$: $$\nexists i_0 < i_1 < \cdots < i_r$$ s.t.
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COVERABILITY GRAPHS

[Karp and Miller '69]

**Pumping up?**

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**Termination**

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- **r-bad sequences are finite**
**Controlled Bad Sequences**

- $(A, \leq)$ a wqo
- $\| \cdot \| : A \to \mathbb{N}$ a norm
- $g : \mathbb{N} \to \mathbb{N}$ monotone a control function
- $n_0 \in \mathbb{N}$ an initial norm

A sequence $x_0, x_1, \ldots$ over $A$ is $(g, n_0)$-controlled if

$$\forall i. \| x_i \| \leq g^i(n_0)$$

$(g, n_0)$-controlled $r$-bad sequences $x_0, x_1, \ldots$ over $A$ have a maximal length $L_{A, g, r}(n_0)$
**CONTROLLED BAD SEQUENCES**

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CONTROLLED BAD SEQUENCES

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- \((g, n_0)\)-controlled \(r\)-bad sequences \(x_0, x_1, \ldots\) over \(A\) have a maximal length \(L_{A, g, r}(n_0)\)

LENGTH FUNCTION THEOREM (FOR DICKSON’S LEMMA, FIGUEIRA ET AL., '11)

d-bad sequences over \(Q \times \mathbb{N}^d\) controlled by primitive recursive functions are of at most Ackermannian length.
Hitchhiker’s Guide to Galactic Complexity
HITCHHIKER’S GUIDE TO GALACTIC COMPLEXITY

- **Elementary**
  - F_3 = Tower
  - F_ω = Ackermann
  - F_{ω^3} = \bigcup e \text{ elementary } DTime(tower(e(n)))
Hitchhiker’s Guide to Galactic Complexity

\[ F_3 \overset{\text{def}}{=} \bigcup_{e \text{ elementary}} \text{DTime}\left(\text{tower}(e(n))\right) \]

- **Elementary**
- **Primitive Recursive**
- **Multiply Recursive**
- **Vector Addition Systems**
- **Reachability**
- **Termination**
- **Complexity**
Hitchhiker’s Guide to Galactic Complexity

\[ F_\omega \overset{\text{def}}{=} \bigcup_{p \text{ primitive recursive}} \text{DTIME}(\text{ackermann}(p(n))) \]

- **Elementary**
  - \( F_3 = \text{Tower} \)
  - \( F_\omega = \text{Ackermann} \)

- **Primitive Recursive**

- **Multiply Recursive**

---

Vector Addition Systems  Reachability  Termination  Complexity
Hitchhiker’s Guide to Galactic Complexity

Elementary

Primitive Recursive

Multiply Recursive

$F_{\omega} = \text{Ackermann}$

$F_{3} = \text{Tower}$

$F_{\omega^{3}} \overset{\text{def}}{=} \bigcup_{p \in \mathcal{F}_{< \omega^{3}}} \text{DTime}(F_{\omega^{3}}(p(n)))$
CONCLUDING REMARKS

- reachability problem: \( \text{ExpSpace} < ? < \mathcal{F}_{\omega^3} \)

- decomposition algorithm:
  - Ackermann lower bound on the algorithm
  - solves more than just reachability:
    e.g. inclusion problem between downward-closures of VAS languages is Ackermann-hard [Zetzsche ‘16]
Concluding Remarks

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- decomposition algorithm:
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- solves more than just reachability:
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Hot Topics?