Automata, as proof systems

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I. Decidability

A weakness or a strength?

An easy decidability proof method

When each rule

$$\frac{s_1 \ \dots \ s_n}{s'}$$

is such that $s_1 \prec s'$, ..., $s_n \prec s'$, for some well-founded order $\prec$

An example

$$\frac{even(x)}{odd(a(x))} \qquad \frac{odd(x)}{even(a(x))} \qquad \frac{}{even(\varepsilon)}$$

$odd(a(a(a(\varepsilon))))$ provable
$even(a(a(a(\varepsilon))))$ not provable

An already existing notion

$$\frac{even(x)}{odd(a(x))} \qquad \frac{odd(x)}{even(a(x))} \qquad \frac{}{even(\varepsilon)}$$

$$odd \xrightarrow{a} even \qquad even \xrightarrow{a} odd \qquad even \text{ final}$$

An automaton:
$odd(a(a(a(\varepsilon))))$ provable: *aaa* recognized in *odd*
$even(a(a(a(\varepsilon))))$ not provable: *aaa* not recognized in *even*

But also... an already existing notion

$$\frac{A \quad B}{A \wedge B}\ \wedge\text{-intro}$$

$$\frac{A \wedge B}{A}\ \wedge\text{-elim}$$

Introduction rule

Generalizations

Given a well-founded order $\prec$

An introduction rule:

$$\frac{s_1 \; \ldots \; s_n}{s'}$$

if $s_1 \prec s'$, ..., $s_n \prec s'$

An automaton: a (finite in conclusions) inference system containing introduction rules only

Provability decidable: finite search space

Alternation

Not all rules are unary

$$\frac{P(x) \quad Q(x)}{P(a(x))}$$

$$\frac{R(x) \quad S(x)}{P(a(x))}$$

$$\ldots$$

All inference systems are alternating:
may chose any rule
must prove all the premises

II. Curry-(de Bruijn)-Howard correspondence and the representation of finite state automata

A question

$$odd \xrightarrow{a} even \qquad even \xrightarrow{a} odd \qquad even \text{ final}$$

$$\frac{even(x)}{odd(a(x))} \qquad \frac{odd(x)}{even(a(x))} \qquad \frac{}{even(\varepsilon)}$$

$$\frac{even}{odd}\, a \qquad \frac{odd}{even}\, a \qquad \frac{}{even}\, \varepsilon$$

Proofs labeled with propositions, rules names, and both

$$\dfrac{\dfrac{\dfrac{\dfrac{}{even}}{odd}}{even}}{odd}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{}\,\varepsilon}{}\,a}{}\,a}{}\,a$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{even}\,\varepsilon}{odd}\,a}{even}\,a}{odd}\,a$$

proof-checking still decidable

conclusion still computable (*)

(*) determinism: otherwise being a conclusion decidable

A linear notation for proofs labeled with propositions

$$\frac{\frac{\frac{\overline{\quad}\ \varepsilon}{}\ a}{}\ a}{}\ a$$

$a(a(a(\varepsilon)))$
Proof-term
Being a conclusion decidable: $a(a(a(\varepsilon))) : odd$ decidable

The set of pairs $\pi : A$ such that π ~~has type A~~ is a linear representation of a proof of A decidable

Why?

Transform each rule

$$\frac{s_1 \; ... \; s_n}{s'} \, f$$

into

$$\frac{\pi_1 : s_1 \; ... \; \pi_n : s_n}{f(\pi_1, ..., \pi_n) : s'}$$

A automaton that proves exactly the pairs $\pi : A$ such that π is a linear representation of a proof of A

This automaton: type-checking algorithm of the inference system

Projection of a decidable set

An example

$$\frac{even}{odd}\ a \qquad \frac{odd}{even}\ a \qquad \frac{}{even}\ \varepsilon$$

$$\frac{x : even}{a(x) : odd} \qquad \frac{x : odd}{a(x) : even} \qquad \frac{}{\varepsilon : even}$$

$$\frac{even(x)}{odd(a(x))} \qquad \frac{odd(x)}{even(a(x))} \qquad \frac{}{even(\varepsilon)}$$

Another example

$$\frac{A \wedge B}{A}\,fst \qquad \frac{A \wedge B}{B}\,snd \qquad \frac{A \Rightarrow B \quad A}{B}\,app \qquad \frac{}{P \wedge (P \Rightarrow Q)}\,c$$

$$\frac{\pi : A \wedge B}{fst(\pi) : A} \qquad \frac{\pi : A \wedge B}{snd(\pi) : B} \qquad \frac{\pi_1 : A \Rightarrow B \quad \pi_2 : A}{app(\pi_1, \pi_2) : B} \qquad \frac{}{c : P \wedge (P \Rightarrow Q)}$$

$$\dfrac{\dfrac{\dfrac{}{P \wedge (P \Rightarrow Q)}\,c}{P \Rightarrow Q}\,\pi_2 \qquad \dfrac{\dfrac{}{P \wedge (P \Rightarrow Q)}\,c}{P}\,\pi_1}{Q}\,app$$

$$app(snd(c), fst(c)) : Q$$

Curry-(de Bruijn)-Howard correspondence

(minus trivial details about bound variables specific to Natural deduction)

III. Transforming an inference system into an automaton

Eliminating the non-introduction rules

$$\dfrac{\dfrac{\pi_1}{s_1} \qquad \cdots \qquad \dfrac{\pi_n}{s_n}}{s'} \; R' \text{ (non-intro)}$$

Eliminating the non-introduction rules

$$\dfrac{\dfrac{\cdots}{s_1}\ R_1\ \text{(intro)} \quad \ldots \quad \dfrac{\cdots}{s_n}\ R_n\ \text{(intro)}}{s'}\ R'\ \text{(non-intro)}$$

(General) cut - Cut-elimination

$$\dfrac{\dfrac{\cdots}{s_1}\ R_1\ \text{(intro)} \quad \ldots \quad \dfrac{\cdots}{s_n}\ R_n\ \text{(intro)}}{s'}\ R'\ \text{(non-intro)}$$

π cut-free: no cuts in π

An inference system has the cut-elimination property if every proof can be transformed into a cut-free proof

A theorem

A cut-free proof has introduction rules only

Induction over proof structure

$$\dfrac{\dfrac{\pi_1}{s_1} \quad \dots \quad \dfrac{\pi_n}{s_n}}{s'} \, R'$$

π_1, ..., π_n introduction rules only, cut-free: R' introduction rule

Corollary: in an inference system that has the cut-elimination property, drop the non-introduction rules, preserving provability

Corollary: cut-elimination $\longrightarrow$ automaton $\longrightarrow$ decidability

When an inference system does not have the cut-elimination property

Transform it in such a way it does

Saturation

A word about Natural deduction

Undecidable

Can it have the cut elimination property?

No

A word about Natural deduction

Introduction rules: $\Rightarrow$-intro, $\forall$-intro, $\wedge$-intro... and axiom

$$\overline{\Gamma, A \vdash A}\text{ axiom}$$

Non-introduction rules: $\Rightarrow$-elim, $\forall$-elim, $\wedge$-elim...

$$\dfrac{\overline{P \wedge Q \vdash P \wedge Q}\text{ axiom}}{P \wedge Q \vdash P}\wedge\text{-elim}$$

is a (general) cut but not a (specific) cut
Saturation leads to include more introduction rules so in such as way that elimination rules can be dropped: various sequent calculi

Conclusion: why teaching is important

Introduction rule, cut, Curry-(de Bruijn)-Howard correspondence...
(also) ICALP A notions
Automaton, saturation... (also) ICALP B notions

But they need to be (slightly) generalized

π recognized in $A = \pi$ is a proof of A

What we learn from our (undergrad) students: informatics is scattered into too many subdomains that have nothing in common

Some work needs to be done to unify our science
This ~~can~~ must be done