TD 8: Partial-Order Reduction

Reminder:

(C0) \( \text{red}(s) = \emptyset \) iff \( \text{en}(s) = \emptyset \).

(C1) For every path \( s \xrightarrow{a_1} s_1 \xrightarrow{a_2} \cdots \xrightarrow{a_n} s_n \xrightarrow{a} t \) in \( \mathcal{K} \) (for any \( n \geq 0 \)), if \( a \notin \text{red}(s) \) and \( a \) depends on some action in \( \text{red}(s) \) (i.e. there exists \( b \in \text{red}(s) \) such that \( (a, b) \notin I \)), then there exists \( 1 \leq i \leq n \) such that \( a_i \in \text{red}(s) \).

(C2) If \( \text{red}(s) \neq \emptyset \), then all actions in \( \text{red}(s) \) are invisible.

(C3) For all cycles in the reduced system \( \mathcal{K}' \), the following holds: if \( a \in \text{en}(s) \) for some state \( s \) in the cycle, then \( a \in \text{red}(s') \) for some (possibly other) state \( s' \) in the cycle.

Exercise 1. Consider the following transition system with state set \( S = \{ s_0, \ldots, s_7 \} \) and transition alphabet \( \Delta = \{ a, b, c, d \} \):

1. Compute the independance set \( I \) and the set of invisible actions \( U \).

2. Propose an assignment \( \text{red} : S \to 2^\Delta \) of ample sets satisfying conditions \( C_0 - C_3 \) of the lecture notes.
Exercise 2. Consider the condition \((C'_1)\): for any \(s\) with \(\text{red}(s) \neq \text{en}(s)\), any \(a\) in \(\text{red}(s)\) is independent from every \(b\) in \(\text{en}(s) \setminus \text{red}(s)\).

1. Show that \((C'_1)\) implies \((C'_2)\).

2. Show that \((C_0), (C'_1), (C_2), (C_3)\) are not sufficient to ensure stuttering equivalence, i.e., that there exists a Kripke structure \(\mathcal{K}\) and an assignment \(\text{red}\) satisfying conditions \((C_0), (C'_1), (C_2), (C_3)\) but such that the reduced system \(\mathcal{K}'\) induced by \(\text{red}\) is not stuttering equivalent to \(\mathcal{K}\).

Exercise 3. Consider the following system with \(A = \{a, b, c, d\}\):

\[
\begin{array}{cccccccc}
  & s_1 & b & s_0 & c & s_2 \\
\downarrow & a & & a & & \\
  & s_3 & b & s_4 & c & s_5 \\
\downarrow & a & & a & & \\
  & s_6 & b & s_7 & c & s_8 \\
\downarrow & d & & d & & \\
  & s_9 & d & s_10 & q \\
\end{array}
\]

1. Let \(\text{red}(s_0) = \{b, c\}\) and \(\text{red}(s) = \text{en}(s)\) for \(s \neq s_0\); show that this ample set assignment is compatible with \(C_0 - C_3\).

2. Exhibit a CTL\((U)\) formula that distinguishes between the original system and its reduction.

3. Can you propose an assignment that also complies with \(C_4\): if \(\text{red}(s) \neq \text{en}(s)\), then \(|\text{red}(s)| = 1|\)?

Exercise 4. Show that \((C_0) - (C_2)\) is not sufficient to ensure stuttering equivalence.

Exercise 5. Let \(\varphi\) be an LTL formula. We define the X-depth \(d_X(\varphi)\) and the U-depth \(d_U(\varphi)\) of \(\varphi\) as the maximal nesting of X- or U-operators in \(\varphi\):
\[
\begin{align*}
    d_X(p) &= 0 \\
    d_X(\neg \varphi) &= d_X(\varphi) \\
    d_X(\varphi \land \psi) &= \max(d_X(\varphi), d_X(\psi)) \\
    d_X(X \varphi) &= 1 + d_X(\varphi) \\
    d_X(\varphi U \psi) &= \max(d_X(\varphi), d_X(\psi)) \\
    d_U(p) &= 0 \\
    d_U(\neg \varphi) &= d_U(\varphi) \\
    d_U(\varphi \land \psi) &= \max(d_U(\varphi), d_U(\psi)) \\
    d_U(X \varphi) &= d_U(\varphi) \\
    d_U(\varphi U \psi) &= 1 + \max(d_U(\varphi), d_U(\psi))
\end{align*}
\]

We denote by LTL(\(U^m, X^n\)) the set of LTL formulas \(\varphi\) with 
\(d_X(\varphi) \leq n\) and \(d_U(\varphi) \leq m\), where \(n = \infty\) or \(m = \infty\) indicates no restriction of the operator in question.

1. We say that two words \(w, w' \in \Sigma^\omega\) are \(n\)-stutter-equivalent if there exists letters \(a_0, a_1, \ldots \in \Sigma\) and \(f, g : \mathbb{N} \rightarrow \mathbb{N}^*\) such that \(w = a_0^{f(0)} a_1^{f(1)} \ldots, w' = a_0^{g(0)} a_1^{g(1)} \ldots\), and for all \(i \geq 0\), \(a_i = a_{i+1}\) implies \(a_i = a_j\) for all \(j > i\), and \(f(i) < n + 1\) or \(g(i) < n + 1\) implies \(f(i) = g(i)\). Show that for all \(n \geq 0\) and \(\varphi \in \text{LTL}(U^{\infty}, X^n)\), \(L(\varphi)\) is closed under \(n\)-stutter-equivalence.

2. A similar principle can be formulated when the U-depth is restricted, by considering stuttering of factors instead of letters. Show that for all \(m \geq 1\) and \(\varphi \in \text{LTL}(U^m, X^0)\), for all \(u, v \in \Sigma^*\) and \(w \in \Sigma^\omega\), we have \(uv^m w \in L(\varphi)\) iff \(uv^{m+1} w \in L(\varphi)\).