TD 4

**Exercise 1.** Given $p \in \text{AP}$, we are only interested in runs visiting finitely often states satisfying $p$. Hence, we define path quantifiers $E_p$ and $A_p$ as

$$E_p \varphi = E(\text{FG} \neg p \land \varphi)$$

$$A_p \varphi = A((\text{FG} \neg p) \Rightarrow \varphi)$$

We consider $\text{CTL}_p(\text{AP},X,U)$ defined by the syntax

$$\varphi ::= T | p \in \text{AP} | \neg \varphi \lor \varphi | E_p X \varphi | E_p U \varphi | E_p G \varphi$$

where $q \in \text{AP}$.

1. Show that we can add $A_p X \varphi$ and $A_p \varphi U \varphi$ to the syntax without changing the expressive power of $\text{CTL}_p(\text{AP},X,U)$.

2. Prove that $\text{CTL}_p(\text{AP},X,U)$ formulae can be expressed in $\text{CTL}(\text{AP},X,U)$.

3. Prove that $\text{CTL}(\text{AP},X,U)$ is strictly more expressive than $\text{CTL}_p(\text{AP},X,U)$.

**Exercise 2.** Given a family of sets of states $(F_i)_{1 \leq i \leq n}$, we define fair runs as runs that visit infinitely often each $F_i$. Formally, $\sigma$ is fair if $\sigma \models \bigwedge_{1 \leq i \leq n} \text{GF} F_i$ where $s \models F_i$ iff $s \in F_i$. We define path quantifiers $E_f$ and $A_f$ as

$$E_f(\varphi) = E(\varphi \land \bigwedge_{1 \leq i \leq n} \text{GF} F_i)$$

$$A_f(\varphi) = A((\bigwedge_{1 \leq i \leq n} \text{GF} F_i) \Rightarrow \varphi)$$

We consider $\text{CTL}_f(\text{AP},X,U)$ defined by the syntax

$$\varphi ::= T | p \in \text{AP} | \neg \varphi \lor \varphi | E_f X \varphi | E_f U \varphi | A_f \varphi U \varphi$$

Show that $\text{CTL}_f(\text{AP},X,U)$ cannot be expressed in $\text{CTL}$.

**Exercise 3.** We consider the logic $\text{CTL-Sync}$ defined by the following syntax:

$$\varphi ::= T | p \in \text{AP} | \neg \varphi \lor \varphi | E X \varphi | E \varphi U \varphi | A X \varphi | A \varphi U \varphi | \varphi \lor \varphi | \varphi U \varphi | \varphi UA \varphi$$

where:

- $s \models \varphi_1 \text{UE} \varphi_2$ if there exists $k \geq 0$ such that for all $j, 0 \leq j < k$ there exists a path $s_0s_1...s_k$ with $s_0 = s$ such that $s_j \models \varphi_1$ and $s_k \models \varphi_2$
- $s \models \varphi_1 \text{UA} \varphi_2$ if there exists $k \geq 0$ such that for all $j, 0 \leq j < k$ and for all paths $s_0s_1...s_k$ with $s_0 = s$ we have $s_j \models \varphi_1$ and $s_k \models \varphi_2$
1. For each pair of formula, decide whether they are equivalent (in finite structures)
   - $\text{FE} p$ and $\text{EF} p$
   - $\text{FA} p$ and $\text{AF} p$
   - $\text{GE} p$ and $\text{EG} p$
   - $\text{GE EF} p$ and $\text{EG EF} p$

2. Show that CTL-Sync is strictly more expressive than CTL.