TD 4

Exercise 1. Given $p \in \mathrm{AP}$, we are only interested in runs visiting finitely often states satisfying p. Hence, we define path quantifiers E_p and A_p as

$$\mathsf{E}_p\,\varphi = \mathsf{E}(\mathsf{FG}\,\neg p \wedge \varphi)$$

$$\mathsf{A}_p\,\varphi = \mathsf{A}((\mathsf{FG}\,\neg p) \Rightarrow \varphi)$$

We consider $\mathrm{CTL}_p(\mathrm{AP}, \mathsf{X}, \mathsf{U})$ defined by the syntax

$$\varphi ::= \top | q \in \mathrm{AP} | \neg\varphi | \varphi \vee \varphi | \,\mathsf{E}_p\,\mathsf{X}\,\varphi | \,\mathsf{E}_p\,\varphi\,\mathsf{U}\,\varphi | \,\mathsf{E}_p\,\mathsf{G}\,\varphi$$

where $q \in \mathrm{AP}$.

1. Show that we can add $\mathsf{A}_p\,X\varphi$ and $\mathsf{A}_p\,\varphi\,\mathsf{U}\,\varphi$ to the syntax without changing the expressive power of $\mathrm{CTL}_p(\mathrm{AP}, \mathsf{X}, \mathsf{U})$.
2. Prove that $\mathrm{CTL}_p(\mathrm{AP}, \mathsf{X}, \mathsf{U})$ formulæ can be expressed in $\mathrm{CTL}(\mathrm{AP}, \mathsf{X}, \mathsf{U})$.
3. Prove that $\mathrm{CTL}(\mathrm{AP}, \mathsf{X}, \mathsf{U})$ is strictly more expressive than $\mathrm{CTL}_p(\mathrm{AP}, \mathsf{X}, \mathsf{U})$.

Exercise 2. Given a family of sets of states $(F_i)_{1\leq i\leq n}$, we define fair runs as runs that visit infinitely often each F_i. Formally, σ is fair if $\sigma \models \bigwedge_{1\leq i\leq n} \mathsf{GF}\,F_i$ where $s \models F_i$ iff $s \in F_i$. We define path quantifiers E_f and A_f as

$$E_f(\varphi) = \mathsf{E}(\varphi \wedge \bigwedge_{1\leq i\leq n} \mathsf{GF}\,F_i)$$

$$A_f(\varphi) = \mathsf{A}((\bigwedge_{1\leq i\leq n} \mathsf{GF}\,F_i) \Rightarrow \varphi)$$

We consider $\mathrm{CTL}_f(\mathrm{AP}, \mathsf{X}, \mathsf{U})$ defined by the syntax

$$\varphi ::= \top | p \in \mathrm{AP} | \neg\varphi | \varphi \vee \varphi | \,\mathsf{E}_f\,\mathsf{X}\,\varphi | \,\mathsf{E}_f\,\varphi\,\mathsf{U}\,\varphi | \,\mathsf{A}_f\,\varphi\,\mathsf{U}\,\varphi$$

Show that $\mathrm{CTL}_f(\mathrm{AP}, \mathsf{X}, \mathsf{U})$ cannot be expressed in CTL.

Exercise 3. We consider the logic CTL-Sync defined by the following syntax:

$$\varphi ::= \top | p \in \mathrm{AP} | \neg\varphi | \varphi \vee \varphi | \,\mathsf{EX}\,\varphi | \,\mathsf{E}\,\varphi\,\mathsf{U}\,\varphi | \,\mathsf{AX}\,\varphi | \,\mathsf{A}\,\varphi\,\mathsf{U}\,\varphi | \varphi\,\mathsf{UE}\,\varphi | \varphi\,\mathsf{UA}\,\varphi$$

where:

- $s \models \varphi_1\,\mathsf{UE}\,\varphi_2$ if there exists $k \geq 0$ such that for all $j, 0 \leq j < k$ there exists a path $s_0 s_1 ... s_k$ with $s_0 = s$ such that $s_j \models \varphi_1$ and $s_k \models \varphi_2$
- $s \models \varphi_1\,\mathsf{UA}\,\varphi_2$ if there exists $k \geq 0$ such that for all $j, 0 \leq j < k$ and for all paths $s_0 s_1 ... s_k$ with $s_0 = s$ we have $s_j \models \varphi_1$ and $s_k \models \varphi_2$

1. For each pair of formula, decide whether they are equivalent (in finite structures)
   - $\mathsf{FE}\,p$ and $\mathsf{EF}\,p$
   - $\mathsf{FA}\,p$ and $\mathsf{AF}\,p$
   - $\mathsf{GE}\,p$ and $\mathsf{EG}\,p$
   - $\mathsf{GE}\,\mathsf{EF}\,p$ and $\mathsf{EG}\,\mathsf{EF}\,p$
2. Show that CTL-Sync is strictly more expressive than CTL.