Exam

29 November 2010

Exam duration: 3 hours. Lecture notes are authorized. All exercises are independent. It is strongly advised (and I will take this into account while grading) that you do completely one exercise rather than treating many exercise superficially.
Write clearly and justify all your claims.

You can answer either in English or in French.

Exercise 1

In this exercise we use the signature for words.
Recall that LFP is the least fixpoint logic allowing fixpoint formulas that contain only positive atoms using a second order predicate.
We define mLFP as the monadic fragment of LFP: only unary second order predicates can be used in the formulas. In other words, we can only compute relations of arity 1 using mLFP fixpoints. Recall that fixpoints can be nested: an mLFP formula may look like this:

$$\mu S,x[a(x) \lor \mu T,y[b(y) \lor \forall z(T(z) \leftrightarrow S(z) \lor x = y)](x)]$$

1. Give a mLFP formula defining the language \((aa)^*\).
2. Give a mLFP formula defining the language \(((aa)^+b^+(aa)^+b^+)^*\).
3. Show that mLFP=REG
   Hint: for one direction, show that mLFP is included in MSO. For the other direction try to generalize your construction for 1. and 2. and code the run of an arbitrary automata
4. Give an LFP formula defining the language \(\{a^n b^n \mid n \in \mathbb{N}\}\).
   Hint: Careful we are now back with the full LFP, not anymore mLFP. The previous result shows that you need to compute at least one relation of arity at least 2 using fixpoint
5. Show that any Context-Free language is definable in LFP.
Recall that IFP is the inflationary fixpoint logic, allowing arbitrary formulas but having an inflationary semantic \((R_{i+1} = R_i \cup \phi(R_i))\). As above we can define mIFP by restricting IFP to its monadic fragment.
6. Show that mLFP is included in mIFP.
7.* Show that mLFP is strictly included in mIFP
   Hint: Let \(L = \{a^n b^m \mid n \leq m\}\). Show that L is definable in mIFP and conclude with 3.
   Because of 3. we know that the formula must use negated second order atoms. Don’t spend too much time on this item, it's tricky.
Exercise 2

We now work on graphs. The schema is $\sigma = \{E\}$ where $E$ is a binary relation.

Recall that a graph is planar if it can be drawn on the plane with no crossing edges. Kuratowski’s Theorem states that a graph is planar iff $K_5$ or $K_{3,3}$ (depicted below) are not minor of this graph.

A minor of a graph is obtained from the initial graph by removing edges or removing vertices or by contracting an edge (i.e. removing the edge and identifying its two endpoints).

![Figure 1: $K_5$ and $K_{3,3}$](image)

1. Show that PLANARITY is definable in MSO.
   
   *Hint:* express in MSO that $K_5$ or $K_{3,3}$ is a minor.

2. Show that $\mu(\text{PLANARITY}) = 0$.
   
   *Hint:* Show that having $K_5$ as a subgraph has probability 1 using the appropriate extension axioms.

3. Show that PLANARITY is not definable in FO.
   
   *Hint:* use Hanf locality of FO. For every $n$, exhibit two structures that have the same neighborhoods up to distance $n$, one being planar the other one not.
Exercise 3

The schema is now $\sigma = \{P_1, \cdots, P_m\}$ where each $P_i$ is unary. The goal of this exercise is to show that in this case PFP = IFP = FO.

Let $\varphi$ be a formula of PFP, we are looking for a first order equivalent formula.

Let’s consider first the case where $\varphi$ has no free variable: it is a sentence.

Fix $k \in \mathbb{N}$ and recall the definition of $k$-pebble Ehrenfeucht-Fraïssé-games. Recall that we denote by $I \equiv^k J$ the fact that Duplicator has a winning strategy in the $k$-pebble game played on $I, J$ (ie she can play forever while maintaining partial isomorphism). Recall finally that an atom is an expression of the form $R(\bar{x})$ or $\neg R(\bar{x})$.

An atomic type of arity 1 is a maximal set of consistent atomic formulas having $x$ as unique variable. In other words an atomic type is a set $\tau_{M,a}$ of the form $\tau_{M,a} = \{\phi(x) : M \models \phi(a)\}$, for some instance $M$ and some element $a$ of $M$.

Let $T$ be the set of atomic types.

(1) Show that for each atomic type $\tau \in T$, there is a quantifier free formula $\alpha_\tau(x)$ such that $M \models \alpha_\tau(a)$ iff $\tau = \tau_{M,a}$.

(2) On such a schema $\sigma$, what is the cardinality of $T$?

Given $\tau \in T$ and an instance $I$ over $\sigma$, we say that $\tau$ occur in $I$ if there is an element $a$ of $I$ such that $I \models \alpha_\tau(a)$. We denote by $f_k(I)$ the number of occurrences in $I$ of each atomic type of $T$ up to threshold $k$. More precisely, $f_k(I)$ is the function associating to each $\tau \in T$, the minimum number between $k$ and the number of occurrences of $\tau$ in $I$: $f_k(I)(\tau) = \min\{k, |\{a \in I \mid I \models \alpha_\tau(a)\}|\}$.

(3) Let $I$ and $J$ be two instances over $\sigma$. Show that $f_k(I) = f_k(J)$ implies $I \equiv^k J$.

Hint: in other words you are ask for a winning strategy for the Duplicator assuming $f_k(I) = f_k(J)$.

An instance $I$ over $\sigma$ is said to be “$k$-primitive” if for each $\tau \in T$, $I$ has at most $k$ occurrences of $\tau$. A formula $\psi$ is said to describe an instance $I$ if $\forall J J \models \psi$ iff $I = J$.

(4) Show that for all $k$, there exists only a finite number of $k$-primitive instances and that each such primitive instance can be described by a formula of FO.

(5) Show that $\varphi$ is expressible in FO.

Hint: This is the difficult question. Please restate all the results from the lecture that you are using in your argument, in particular concerning the links between PFP and $\equiv^k$. Of course you are expected to build on 1., 2. and 3.

(5) Modify your argument in order to include the case where $\varphi$ has free variables.