Logic, descriptive complexity and theory of databases.
Lecture 4

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1 Locality

Let $\sigma$ be a purely relational vocabulary (without function symbols) and let $I$ be a structure over $\sigma$.

Recall the following definitions (from the previous lecture):

- The Gaifman graph of $I$, $G(I)$.
- The distance between any two elements in $I$, $d_I(a,b) = d_{G(I)}(a,b)$.
- The $k$-neighborhood of $a$ in $I$, $N^I_k(a)$.
- Given two $\sigma$-structures $I, J$, we say that $N^I_k(a) \sim N^J_k(b)$ if there exists a bijection, $h : N^I_k(a) \rightarrow N^J_k(b)$
- Given two $\sigma$-structures $I, J$, and $a \in I, b \in J$, we say that $(I, a) \equiv^k (J, b)$ if $\exists h : I \rightarrow J$ such that $\forall c \in I, N^I_k(ac) \sim N^J_k(bh(c))$

**Lemma 1.** If $I \equiv^k J$ and $N^I_{sk+1}(\bar{a}) \sim N^J_{sk+1}(\bar{b})$ then $(I, \bar{a}) \equiv^k (J, \bar{b})$.

**Proof.** Assume that we have the following isomorphism:

$g : N^I_{sk+1}(\bar{a}) \rightarrow N^J_{sk+1}(\bar{b})$

Assume also we have a bijection $h : I \rightarrow J$ sending $\bar{a}$ to $\bar{b}$ and such that $\forall c \in I, N^I_k(\bar{ac}) \sim N^J_k(\bar{bh(c)})$

Now, we will create a new isomorphism $f$ from $g$ and $h$,

$f : I \rightarrow J$

which ensures that $(I, \bar{a}) \equiv^k (J, \bar{b})$.

The function $f$ is obtained as follows :-
• \( \forall c \in N_{2k+1}(\pi), N_k(\pi c) \subseteq N_{3k+1}(\pi) \).

Hence, we set \( f(c) = g(c) \) and we have that \( N_k^I(\pi c) \sim N_k^I(\bar{\pi} c) \).

• If \( c \notin N_{2k+1}(\pi) \), then,

\[
N_k^I(c) \cap N_k^I(\pi) = \emptyset
\]

\[
N_k^I(\pi c) = N_k^I(c) \cup N_k^I(\pi)
\]

We are looking for \( f(c) \) in \( J \) such that \( N_k^I(c) \sim N_k^I(f(c)) \) and \( N_k^I(f(c)) \cap N_k^I(\bar{\pi}) = \emptyset \). If we succeed we are done because then

\[
N_k^I(\bar{\pi} f(c)) \sim N_k^I(f(c)) \sqcup N_k^I(\bar{\pi}) \sim N_k^I(c) \sqcup N_k^I(\pi) \sim N_k^I(\pi c).
\]

Let \( \tau \) be the isomorphism “type” of \( N_k^I(c) \).

i) \( \tau \notin N_{2k+1}^I(c) \), then we can set \( f(c) = h(c) \) and we are done.

ii) If \( \tau \in N_{2k+1}^I(\pi) \), then

\[
\#\tau \in I = (\#\tau \in N_{2k+1}^I(\pi)) + (\#\tau \notin N_{2k+1}^I(\pi))
\]

But,

\[
(\#\tau \notin N_{2k+1}^I(\pi)) = (\#\tau \notin N_{2k+1}^I(\bar{\pi}))
\]

Hence we have

\[
(\#\tau \notin N_{2k+1}^I(\pi)) = (\#\tau \notin N_{2k+1}^I(\bar{\pi}))
\]

Hence, we can set \( f(c) \) to be any element of type \( \tau \) not in \( N_{2k+1}^I(\bar{\pi}) \) and not already in the image of \( f \) and we are done.

\[ \square \]

**Theorem 1** (Hanf locality). \( \forall \phi \in FO, \exists k, \text{ such that } \forall I, J, \)

\[
(I, \pi) \models_k (J, \bar{\pi}) \implies \models I \models \phi(\pi) \iff J \models \phi(\bar{\pi})
\]

**Proof.** Let \( \phi \in FO \) and \( m = qr(\phi) \). The proof is by induction on \( m \).

When \( m = 0 \), taking \( k = 0 \), yields \( I \models \phi(\pi) \iff J \models \phi(\bar{\pi}) \).

For any \( m > 0 \), and for any \( \phi_1, \phi_2 \in FO \) which satisfy Hanf’s theorem with values \( k_1 \) and \( k_2 \) respectively,

- If \( \phi = \neg \phi_1 \), take \( k = k_1 \).

- If \( \phi = \phi_1 \land \phi_2 \), we can take \( k = max(k_1, k_2) \) and this is sufficient as if \( I \models_k J \) then for all \( k' < k \), we have that \( I \models_k' J \).

- \( \phi = \phi_1 \lor \phi_2 \), is similar to the above case.

- The only interesting case is when \( \phi(\pi) = \exists y(\phi(\pi y)) \). Inductively, let \( k' \) be the value obtained for \( \psi(\pi y) \). Then, \( k = 3 \cdot k' + 1 \) is sufficient. In order to prove this, let’s assume that

\[
I, \pi \models_{3 \cdot k' + 1} J, \bar{\pi}
\]

Note that the above also implies that \( I, \pi \models_k J, \bar{\pi} \).

Now, by definition of \( \models_{3 \cdot k' + 1} \), we have a bijection \( f \),

\[
f : I \to J
\]
such that,
\[ \forall c, N_{3k'+1}^I(p;c) \sim N_{3k'+1}^J (\overline{b}f(c)) \]
From Lemma 1, we have,
\[ I, \overline{pc} \models_k J, \overline{bf}(c) \]
Now,
\[ I \models \phi(\overline{p}) \]
Hence there is a \( c \) in \( I \) such that
\[ I \models \psi(\overline{pc}) \]
But, by our choice of \( k' \) and by induction we get,
\[ I \models \psi(\overline{pc}) \iff J \models \psi(\overline{bf}(c)) \]
Hence
\[ J \models \phi(\overline{b}) \]

**Theorem 2** (Gaifman Locality). \( \forall \) formulae \( \phi(\overline{x}) \in FO, \exists k \in \mathbb{N} \text{ such that } \forall I, \text{ if } N_{k}^I(\overline{p}) \sim N_{k}^I(\overline{b}), \text{ then, } I \models \phi(\overline{p}) \iff I \models \phi(\overline{b}) \).

*Proof.* Given a formula, \( \phi(\overline{x}) \in FO \), we have a \( k' \in \mathbb{N} \) that satisfies Hanf’s theorem. Let \( k = 3 \cdot k' + 1 \).

Of course, \( I \models_{k'} I \). Now, if we have
\[ N_{3k'+1}^I(\overline{p}) \sim N_{3k'+1}^J(\overline{b}) \]
Then, from Lemma 1,
\[ I, \overline{p} \models_{k'} I, \overline{b} \]
and, by Hanf’s locality, we have,
\[ I \models \phi(\overline{p}) \iff I \models \phi(\overline{b}) \]

### 2 Fixpoint Logics

Let \( \sigma \) be a fixed relational vocabulary.

In this section, we consider formulae of the form, \( \phi(\overline{x}) \in FO(\sigma \cup \{ R \}) \) where \( R \) and \( \overline{x} \) are of arity \( k \) and \( R \notin \sigma \). We denote such formulae by, \( \phi(R, \overline{x}) \).

Given any \( \phi(R, \overline{x}) \), we construct a new relation \( \mu_{R, \overline{x}}[\phi(R, \overline{x})] \) of arity \( k \). We now consider two fixpoint logics, Inflationary Fixed Point Logic (IFP) and Partial fixed Point Logic (PFP). The syntax of formulae in IFP as well as PFP is \( FO + \mu \).

#### 2.1 Semantics

Let \( I \) be a \( \sigma \)-model.
2.1.1 PFP

\[ R = \emptyset \] (empty relation)

\[ R_1 = \{ \pi \mid (I, \emptyset) \models \phi(R, \pi) \} \]

\[ R_2 = \{ \pi \mid (I, R_1) \models \phi(R, \pi) \} \]

\[ R_3 = \{ \pi \mid (I, R_2) \models \phi(R, \pi) \} \]

\[ \vdots \]

If \( R_1, R_2, R_3 \ldots \) converges, \( \mu_{R,\pi}(\phi(R, \pi)) \) is the limit else \( \mu_{R,\pi}(\phi(R, \pi)) = \emptyset \).

2.1.2 IFP

\[ R = \emptyset \] empty relation

\[ R_1 = \{ \pi \mid (I, \emptyset) \models \phi(R, \pi) \} \]

\[ R_2 = R_1 \cup \{ \pi \mid (I, R_1) \models \phi(R, \pi) \} \]

\[ R_3 = R_2 \cup \{ \pi \mid (I, R_2) \models \phi(R, \pi) \} \]

\[ \vdots \]

\[ R_\infty \]

There is always a limit, denoted above by \( R_\infty \), and we set \( \mu_{R,\pi}(\phi(R, \pi)) = R_\infty \).

Example 1. Let \( \sigma = \{ E \} \) and

\[ \phi(R, xy) = (E(x, y) \lor \exists z(E(x, z) \land R(z, y))) \]

Then, the fixed point formula

\[ \forall z_1, z_2(\mu_{R,xy}[\phi(R, xy)](z_1, z_2) \lor (z_1 = z_2)) \]

checks if the graph is connected in both IFP as well as PFP because for both IFP and PFP we have:

\[ \emptyset \]

\[ R_1 = E \]

\[ R_2 = E \cup (E \circ E) \]

\[ R_3 = E \cup (E \circ E) \cup (E \circ E \circ E) \]

\[ \vdots \]

\[ R_\infty = E^+ \]

Example 2. Let \( \sigma = (\prec, P_n) \) be the relational signature for words.
Let \( \text{min}, \text{max}, +1 \) denote respectively the minimum element, the maximum element, and the successor function for the word. Given the linear order all the above can be encoded in FO.

Now, what does the following formulae compute? (It computes the same relation for both IFP and PFP)

\[ \phi(R, xyz) = ((y = \text{min}) \land (z = x)) \lor \]

\[ (\exists u v R(xuv) \land y = u + 1 \land z = v + 1) \]
The value of $\mu_{R}(\phi)$ is $z=x+y$.

Now assume $\phi$ is the same formula as above. What does the following formula compute?

$$\phi'(S,xyz) = (y = \min \land z = \min) \lor \exists u \forall R(xuv) \land y = u + 1 \land \mu_{R}(\phi)(x,v,z)$$

The result of $\mu_{S}(\phi')$ is $z=x \times y$

Note that in the above example we had a nesting of $\mu$ relations but, it turns out that the definition of $\mu$ is very robust and many variants of the logic can be simulated with a single outermost $\mu$ quantifier.

**Example 3.** Now for something nontrivial. It is an encoding of a question on the Game of Life into PFP.

Let $\sigma = \{E,T\}$, where $E$ is a binary predicate denoting the edge relation while $T$ is unary indicating the set of nodes alive at any point of time. The rules of the game are as follows :-

1. A dead vertex becomes alive if $\geq 2$ neighbors are alive.
2. A living vertex dies if $\geq 3$ neighbors are alive.

$I$ is the initial situation. The question: Is there a point in the limit?

The above question can be encoded by a formula given below which gives the correct answer in PFP.

$$\phi(S,x) = T(x) \lor (S(x) \lor A(x) \land \neg R(x))$$

where

$$A(X) : \exists z_1 z_2 (z_1 \neq z_2) \land E(x, z_1) \land E(x, z_2) \land S(z_1) \land S(z_2) \land \neg S(x)$$

and

$$R(x) : \exists z_1 z_2 z_3 ((z_1 \neq z_2) \land (z_2 \neq z_3) \land (z_1 \neq z_3) \land E(x, z_1) \land E(x, z_2) \land E(x, z_3) \land S(z_1) \land S(z_2) \land S(z_3) \land S(x))$$

### 2.2 Parameters

In this section, we see the robustness of the definition of $\mu$. In particular we see how variations to the definition of $\mu$ could all be simulated by a formula as per the definition described at the beginning of this section.

**2.2.1 Simulating multiple relations**

If, in addition to the relation $R$, of arity $k$, and the $k$-tuple of free variables $\pi$, we had other relations in our formulae outside $\sigma$ and $\forall \pi$ were the set of free variables not in $\pi$, then, we can still eliminate these additional relations and free variables as the fixed points of such formulae can be simulated by a single outermost $\mu$ relation and a single relation with greater arity.

Consider the formula, $\phi(R,\pi,S,\overline{\pi})$. The fixpoint of such a formula given by $\mu_{R,\pi}(\phi(R,\pi,S,\overline{\pi}))$, $\forall S$ and $\forall \overline{\pi}$ is a relation of arity $k$. There is a formula of the form $\psi(T,\pi)$ where $T$ and $\pi$ are of arity $(k + |\overline{\pi}| + \text{arity}(S))$ such that

$$\mu_{T,\pi}(\psi(T,\pi)) = \mu_{T,\pi}(\phi) \times S \times \overline{\pi}$$

And $\psi(T,\pi)$ is $\phi(R,\pi,S,\overline{\pi})$ wherein the following replacements are made

$$R(\pi) : \exists \overline{\pi} \exists \pi T(\pi, \overline{\pi} \overline{\pi})$$

$$S(\pi) : \exists \overline{\pi} \exists \pi T(\pi, \overline{\pi} \overline{\pi})$$
2.2.2 Simultaneous fixpoints

Let $\phi_1(R_1, \ldots, R_k, \overline{x_1}), \phi_2(R_1, \ldots, R_k, \overline{x_2}), \ldots, \phi_k(R_1, \ldots, R_k, \overline{x_k})$ be a series of fixpoint formulae over the vocabulary $\sigma \cup \{R_1, \ldots, R_k\}$ and $\forall i$, the arity of $\overline{x_i}$ is $k_i$. The semantics of such a sequence of formulae is that for all $i \in [1, k]$, we initialise $R_i^0 = \phi$ and then, $\forall j > 0$, the formulae $\phi_i$ computes $R_i^{j+1}$ using the IFP or PFP semantics.

The simultaneous fixpoint of $\phi_1, \ldots, \phi_k$, denoted by $\mu_{R_1, \ldots, R_k}(\phi_1, \ldots, \phi_k)$ is the fixpoint of the relation $R_1 \times R_2 \times \cdots \times R_k$ of arity $k_1 \times k_2 \times \cdots \times k_k$ such that for all $R_i$ the fixpoint is reached by all formula $\phi_i$. For all such simultaneous formulae, $\exists \psi(T, \overline{x})$ such that $\mu_T, \overline{x}(\psi(T, \overline{x})) = \mu_{R_1, \ldots, R_k}(\phi_1, \ldots, \phi_k)$.

We will prove something much simpler but, the basic idea carries over to the general case. We will assume two formulae $\phi_1(R, S, \overline{x})$ and $\phi_2(R, S, \overline{y})$ and also assume that the arity of both $R$ and $S$ is $k$. Also, the semantics for evaluation is IFP.

Now we construct the formula $\psi(T, \overline{x}, u, v)$ where $\overline{x} = \overline{z}^k_{z_1}$ and $T$ are of arity $k + 1$ for the IFP semantics as follows:

$$\exists u, v(u \neq v)\mu_T(\psi(T, \overline{x}, u, v))$$

and $\psi(T, \overline{x}, u, v)$ is given by,

$$\psi(T, \overline{x}, u, v) = ((z_5 = u) \land \phi_1(R(\overline{w}))/T(\overline{x}, u), S(\overline{w})/T(\overline{x}, u), \overline{z})$$

$$\lor$$

$$((z_5 = v) \land \phi_2(R(\overline{w}))/T(\overline{x}, u), S(\overline{w})/T(\overline{x}, u), \overline{z})$$

Finally,

$$\exists u, v(u \neq v)\mu_T(\psi(T, \overline{x})) = \mu_{R, S}(\phi_1, \phi_2) \times \{u\} \lor \mu_{R, S}(\phi_1, \phi_2) \times \{v\}$$

Note that the formula above has two parameters, $u$ and $v$, but we have already seen that those could be eliminated.

2.3 Least Fixed Point (LFP)

We now turn to another fixed point logic, called the least fixed point (LFP) which is important for historical reasons and is also much simpler.

A formula in LFP is denoted by $\phi(R, \overline{x})$ and it is assumed that $\phi$ is monotone. A formula is said to be monotone if the following holds

$$R_1 \subseteq R_2 \Rightarrow \phi(R_1) \subseteq \phi(R_2)$$

In LFP, given that $\phi$ is monotone, $\mu_R(\phi)$ is the least fixpoint of $R \rightarrow \phi(R)$.

2.4 Propositions

Finally, we end with some interesting propositions, some of which are quite deep and others are exercise problems.

1. If $\phi$ is monotone then $\mu(\phi)$ is same for LFP, IFP and PFP. (exercise)

2. In terms of expressive power, LFP “=” IFP “$\subseteq$” PFP.

The inclusion of IFP into LFP is a difficult result proved by Gurevich and Shelah. The other inclusions are simple.

Exercise 1. Prove that the following problem is undecidable.

Input: $\phi(R, \overline{x}) \in FO$

Output: Is $R \rightarrow \phi(R)$ monotone or not?
2.4.1 Positive LFP

$\phi(R, \pi)$ is positive in R if R is never negated.

1. If $\phi$ is positive $\Rightarrow \phi$ is monotone. (exercise)

2. At the logical level, posLFP = LFP.
   (difficult — proved by Ajtai and Gurevich)