Abstraction and Refinement
State-space explosion

In practice, model-checking encounters the problem of state-space explosion:

- due to data: var with $n$ bits $\rightarrow 2^n$ states
- due to concurrency: $n$ parallel components with $n!$ different orderings

Countermeasures:

- Compression: efficient representations (e.g. BDDs)
- Reduction: find a simpler, equivalent problem
- Abstraction: identify and ignore “unimportant” information
Example 1 (loop)

Consider the following program with three numeric variables $x, y, z$.

\begin{align*}
\ell_1: & \quad y = x+1; \\
\ell_2: & \quad z = 0; \\
\ell_3: & \quad \text{while (} z < 100 \text{) } z = z+1; \\
\ell_4: & \quad \text{if (} y < x \text{) error; }
\end{align*}

**Question:** Is the error location reachable?
Example 2 (Sorting)

Another program with three numeric variables \( x, y, z \).

\[ \ell_1: \text{if } x > y \text{ then swap } x, y \text{ else skip;} \]
\[ \ell_2: \text{if } y > z \text{ then swap } y, z \text{ else skip;} \]
\[ \ell_3: \text{if } x > y \text{ then swap } x, y \text{ else skip;} \]
\[ \ell_4: \text{skip} \]

**Assumption:** initially, \( x, y, z \) are all different

**Question:** Are \( x, y, z \) sorted in ascending order when reaching \( \ell_4 \)?
Example 3 (Device driver)

C code for Windows device driver

Operations on a semaphore: lock, release

Lock and release must be used alternatingly
Abstraction

Idea: throw away (abstract from) “unimportant” information

Handling *infinite* state spaces

Reduce (large) finite problems to smaller ones

Alternative point of view: merge “equivalent” states
Example 1

Omit concrete values of $x, y, z$; retain only the following information: program counter, predicate $y < x$

Resulting (abstract) Kripke structure:

Result: $\ell_4$ is reachable only with $y \geq x$; the error will not happen.
Example 2

Omit concrete values of $x, y, z$; retain only program counter and permutation of $x, y, z$

Result: $\ell_4$ is reachable only with $xyz$; no error.
Questions: What is the logical relation between the original programs and their abstract versions? What do the abstract versions really say about the original programs?

In Example 1, the error is unreachable in both the original and the abstract version.

However, in Example 1, the original structure terminates but the abstract version does not.

Which conditions must hold for the abstract structure in order to draw meaningful conclusions about the original structure?
Let $\mathcal{K}_1 = (S, \rightarrow_1, s_0, AP, \nu)$ and $\mathcal{K}_2 = (T, \rightarrow_2, t_0, AP, \mu)$ be two Kripke structures ($S, T$ are possibly infinite), and let $H \subseteq S \times T$ be a relation.

$H$ is called a simulation from $\mathcal{K}_1$ to $\mathcal{K}_2$ iff

(i) $(s_0, t_0) \in H$;

(ii) for all $(s, t) \in H$ we have: $\nu(s) = \mu(t)$;

(iii) if $(s, t) \in H$ and $s \rightarrow_1 s'$, then there exists $t'$ such that $t \rightarrow_2 t'$ and $(s', t') \in H$.

We say: $\mathcal{K}_2$ simulates $\mathcal{K}_1$ (written $\mathcal{K}_1 \leq \mathcal{K}_2$) if such a simulation $H$ exists.
Intuition: $\mathcal{K}_2$ can do anything that is possible in $\mathcal{K}_1$.

$\mathcal{K}_2$ simulates $\mathcal{K}_1$ (with $H = \{(a, f), (b, g), (c, g), (d, i), (e, k)\}$).

However: $\mathcal{K}_1$ does not simulate $\mathcal{K}_2$!
Bisimulation

A relation $H$ is called a bisimulation between $\mathcal{K}_1$ and $\mathcal{K}_2$ iff $H$ is a simulation from $\mathcal{K}_1$ to $\mathcal{K}_2$ and $\{(t, s) \mid (s, t) \in H\}$ is a simulation from $\mathcal{K}_2$ to $\mathcal{K}_1$.

We say: $\mathcal{K}_1$ and $\mathcal{K}_2$ are bisimilar (written $\mathcal{K}_1 \equiv \mathcal{K}_2$) iff such a relation $H$ exists.
Careful: In general, $\mathcal{K}_1 \leq \mathcal{K}_2$ and $\mathcal{K}_2 \leq \mathcal{K}_1$ do not imply $\mathcal{K}_1 \equiv \mathcal{K}_2$!
Let $\mathcal{K}_1 \leq \mathcal{K}_2$ and $\phi$ an LTL formula. Then:

$\mathcal{K}_2 \models \phi$ implies $\mathcal{K}_1 \models \phi$ (for universal model checking).

The other direction is not guaranteed!
Existential abstraction

Let $\mathcal{K} = (S, \rightarrow, r, AP, \nu)$ be a Kripke structure (concrete structure).

Let $\approx$ be an equivalence relation on $S$ such that for all $s \approx t$ we have $\nu(s) = \nu(t)$ (we say: $\approx$ respects $\nu$).

Let $[s] := \{ t \mid s \approx t \}$ denote the equivalence class of $s$; $[S]$ denotes the set of all equivalence classes.

The abstraction of $S$ w.r.t. $\approx$ denotes the structure $\mathcal{K}' = ([S], \rightarrow', [r], AP, \nu')$, where

\[ [s] \rightarrow' [t] \text{ for all } s \rightarrow t; \]
\[ \nu'( [s] ) = \nu(s) \text{ (this is well-defined!)}. \]
Consider the Kripke structure below:
States partitioned into equivalence classes:
Abstract structure obtained by quotienting:
Let $\mathcal{K}'$ be a structure obtained by abstraction from $\mathcal{K}$.

Then $\mathcal{K} \leq \mathcal{K}'$ holds (simulation relation: $\{ (s, [s]) \mid s \in S \}$)

Thus, if $\mathcal{K}'$ satisfies some LTL formula, so does $\mathcal{K}$. 
What happens if $\approx$ does not respect $\nu$?

Then $K \not\preceq K'$ does not hold.

Example: The abstraction satisfies $Gp$, the concrete system does not.
Abstraction gives rise to additional paths in the system:

Every concrete run has got a corresponding run in the abstraction . . .
Abstraction gives rise to additional paths in the system:

... but not every abstract run has got a corresponding run in the concrete system.
Suppose that $K' \not\models \phi$, where $\rho$ is a counterexample.

Check whether there is a run in $K$ that “corresponds” to $\rho$.

If yes, then $K \not\models \phi$.

If no, then we can use $\rho$ to refine the abstraction; i.e. we remove some equivalences from the relation $H$, introducing additional distinct states in $K'$ so that $\rho$ disappears.

The refinement can be repeated until a definite answer for $K \models \phi$ (positive or negative) can be determined. This technique is called counterexample-guided abstraction refinement (CEGAR) [Clarke et al, 1994].
The abstraction-refinement cycle

Input: $\mathcal{K}, \phi$

Determine $\approx$

Compute $K'$

$K' \models \phi$?

$K \models \phi$

Refine $\approx$

$\rho$ realizable in $K$?

no, counterexample $\rho$

yes

$K \not\models \phi$
Simulation of $\rho$

**Problem:** Given a counterexample $\rho$, is there a run corresponding to $\rho$ in $\mathcal{K}$?

**Solution:** “Simulate” $\rho$ on $\mathcal{K}$.

**Remark:** Any counterexample $\rho$ can be partitioned into a finite stem and a finite loop, i.e. $\rho = w_S w_L^\omega$ for suitable $w_S, w_L$.

**Case distinction:** The simulation may fail in the stem or in the loop.
Example 1: $G \rightarrow \text{black}$

Abstraction yields a counterexample with stem $a_1a_2a_3a_4$ and loop $a_4$. 
Simulating the stem

Let \( w_S = b_0 \cdots b_k \).

Start with \( S_0 = \{ r \} \). (We have \( b_0 = [r] \).)

For \( i = 1, \ldots, k \), compute \( S_i = \{ t \mid t \in b_i \land \exists s \in S_{i-1} : s \rightarrow t \} \).

If \( S_k \neq \emptyset \), then there is a concrete correspondence for \( w_S \).

If \( S_k = \emptyset \): Find the smallest index \( \ell \) with \( S_{\ell} = \emptyset \): The refinement should distinguish the states in \( S_{\ell-1} \) and those \( b_{\ell-1} \)-states that have immediate successors in \( b_{\ell} \).
Example: $w_S = a_1a_2a_3a_4$

$S_0 = \{s_2\}$, $S_1 = \{s_4\}$, $S_2 = \{s_5\}$, $S_3 = \emptyset$.

In the next refinement, $s_5$ and $s_7$ must be distinguished.

Possible new equivalence classes: $\{s_5, s_6\}$, $\{s_7\}$ or $\{s_5\}$, $\{s_6, s_7\}$. 
Next try: $G \neg black$ with refinement

The new abstraction does not yield any counterexample; therefore, $G \neg black$ also holds in the concrete system.
Example 2: $FG\ \text{red}$

The abstraction yields a counterexample with stem $a_1a_2$ and loop $a_3a_2$. 
Simulating a loop

Assume \( w_S = b_0 \cdots b_k \), \( w_L = c_1 \cdots c_\ell \)

\( w_S \) is simulated as before, however \( w_L \) may have to be simulated multiple times.

Let \( m \) be the size of the smallest equivalence class in \( w_L \):

\[
m = \min_{i=1,\ldots,\ell} |c_i|
\]

Then we simulate the path \( w_S w_L^{m+1} \); doing so, either the simulation will fail, or we will discover a real counterexample.

Refinement: same as before.
Example: $w_S = a_1 a_2$, $w_L = a_3 a_2$, $m = 2$

The simulation succeeds because there is a loop around $s_4$. Thus, there is a real counterexample, so $\mathcal{K} \not\models \phi$. 