Pushdown systems

Example 1

A small program (where $n \geq 1$):

bool g=true;

void main() {

  level$_1$();

  level$_1$();

  assume(g);

}

void level$_n$() {

  g:=not g;

}

void level$_i$() {

  level$_{i+1}$();

  level$_{i+1}$();

}

Question: Will g be true when the program terminates?

Example 1 has got *finitely* many states.
(The call stack is bounded by n.)

Can be treated by “inlining” (replace procedure calls by a copy of the callee).

Inlining causes an exponential state-space explosion.

Inlining is inefficient: every copy of each procedure will be investigated separately.

Inlining not applicable for recursive procedure calls.

Example 2: Recursive program

procedure p;
p_0: **if** ? **then**
p_1:   **call** s;
p_2:   **if** ? **then call** p; **end if**;
    else
p_3:   **call** p;
    end if
p_4: **return**

procedure s;
s_0: **if** ? **then return**; **end if**;
s_1: **call** p;
s_2: **return**;

procedure $main$;
m_0: **call** s;
m_1: **return**;

$S = \{p_0, \ldots, p_4, s_0, \ldots, s_2, m_0, m_1\}^*$,   initial state m_0

$$
\begin{tikzcd}
 & & & \textbf{m1} \arrow[r] & \varepsilon & & & \\
 {} \arrow[r] & \textbf{m0} \arrow[r] & \textbf{s0 m1} \arrow[ur] \arrow[dr] & & & \textbf{p1 s2 m1} \arrow[r] & \textbf{s0 p2 s2 m1} \arrow[r] & \textbf{...} \\
 & & & \textbf{s1 m1} \arrow[r] & \textbf{p0 s2 m1} \arrow[ur] \arrow[dr] & & & \\
 & & & & & \textbf{p3 s2 m1} \arrow[r] & \textbf{p0 p4 s2 m1} \arrow[r] & \textbf{...}
\end{tikzcd}
$$

Example 2 has got infinitely many states.

Inlining not applicable!

Cannot be analyzed by naïvely searching all reachable states.

We shall require a *finite* representation of infinitely many states.

Example 3: Quicksort

```
void quicksort (int left, int right) {
   int lo,hi,piv;
   if (left >= right) return;
   piv = a[right]; lo = left; hi = right;
   while (lo <= hi) {
     if (a[hi]>piv) {
       hi = hi - 1;
     } else {
       swap a[lo],a[hi];
       lo = lo + 1;
     }
   }
   quicksort(left,hi);
   quicksort(lo,right);
}
```

Question: Does Example 3 sort correctly? Is termination guaranteed?

The mere structure of Example 3 does not tell us whether there are infinitely many reachable states:

finitely many if the program terminates

infinitely many if it fails to terminate

Termination can only be checked by directly dealing with infinite state sets.

A computation model for procedural programs

Control flow:

- sequential program (no multithreading)
- procedures
- mutual procedure calls (possibly recursive)

Data:

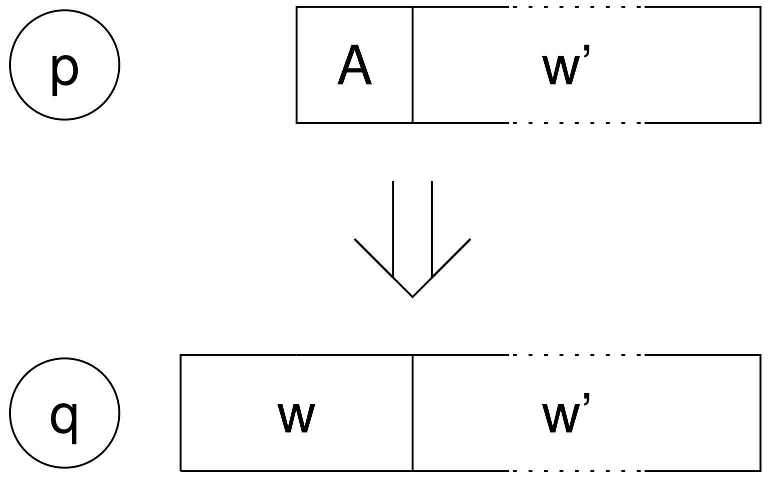
- global variables (restriction: only finite memory)
- local variables in each procedure (one copy per call)

Pushdown systems

A pushdown system (PDS) is a triple (P, Γ, Δ), where

- P is a finite set of control states;
- Γ is a finite stack alphabet;
- Δ is a finite set of rules.

Rules have the form $pA \hookrightarrow qw$, where $p, q \in P$, $A \in \Gamma$, $w \in \Gamma^*$.



Like acceptors for context-free language, but without any input!

Behaviour of a PDS

Let $\mathcal{P} = (P, \Gamma, \Delta)$ be a PDS and $c_0 \in P \times \Gamma^*$.

With $\mathcal{P}$ we associate a transition system $\mathcal{T}_{\mathcal{P}} = (S, \rightarrow, r)$ as follows:

- $S = P \times \Gamma^*$ are the states (which we call configurations);
- we have $pAw' \rightarrow qww'$ for all $w' \in \Gamma^*$ iff $pA \hookrightarrow qw \in \Delta$;
- $r = c_0$ is the initial configuration.

Transition system of a PDS

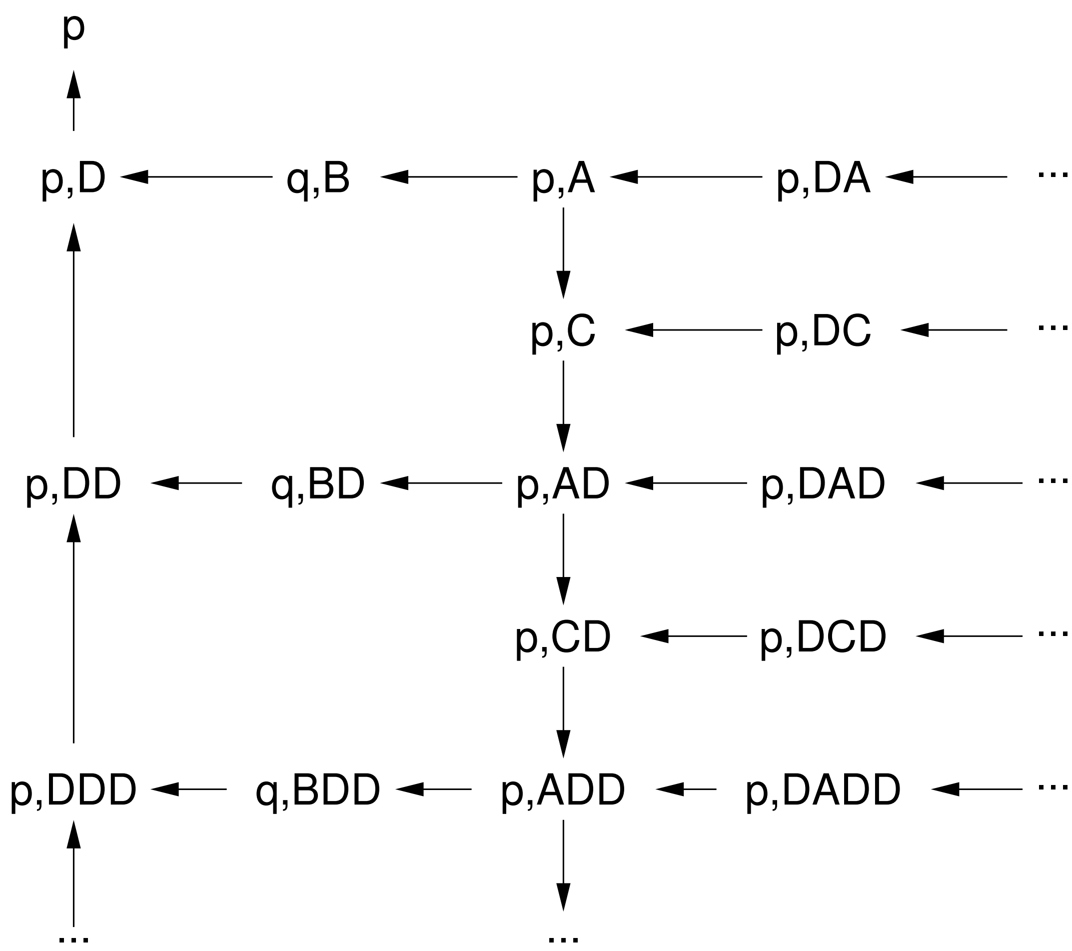
$pA \hookrightarrow qB$

$pA \hookrightarrow pC$

$qB \hookrightarrow pD$

$pC \hookrightarrow pAD$

$pD \hookrightarrow p\varepsilon$



Procedural programs and PDSs

P may represent the valuations of global variables.

Γ may contain tuples of the form (*program counter*, *local valuations*)

Interpretation of a configuration pAw:

global values in p, current procedure with local variables in A

"suspended" procedures in w

Rules:

$pA \hookrightarrow qB \mathrel{\widehat{=}}$ statement within a procedure

$pA \hookrightarrow qBC \mathrel{\widehat{=}}$ procedure call

$pA \hookrightarrow q\varepsilon \mathrel{\widehat{=}}$ return from a procedure

Reachability in PDS

Let $\mathcal{P}$ be a PDS and c, c' two of its configurations.

Problem: Does $c \rightarrow^* c'$ hold in $\mathcal{T}_{\mathcal{P}}$?

Note: $\mathcal{T}_{\mathcal{P}}$ has got infinitely many (reachable) states.

Nonetheless, the problem is decidable!

Finite automata

To represent (infinite) sets of configurations, we shall employ finite automata.

Let $\mathcal{P} = (P, \Gamma, \Delta)$ be a PDS. We call $\mathcal{A} = (Q, \Gamma, P, T, F)$ a $\mathcal{P}$-automaton.

The alphabet of $\mathcal{A}$ is the stack alphabet Γ.

The initial states of $\mathcal{A}$ are the control states P.

We say that $\mathcal{A}$ accepts the configuration pw if $\mathcal{A}$ has got a path labelled by input w starting at p and ending at some final state.

Let $\mathcal{L}(\mathcal{A})$ be the set of configurations accepted by $\mathcal{A}$.

A set C of configurations is called regular iff there is some $\mathcal{P}$-automaton $\mathcal{A}$ with $\mathcal{L}(\mathcal{A}) = C$.

An automaton is normalized if there are no transitions leading into initial states.

Remark: In the following, we shall use the following notation:

$pw \Rightarrow p'w'$ (in the PDS $\mathcal{P}$) and $p \xrightarrow{w} q$ (in $\mathcal{P}$-automata)

Reachability in PDS

Let $pre^*(C) = \{\, c' \mid \exists c \in C\colon c' \Rightarrow c \,\}$ denote the predecessors of C, and let $post^*(C) = \{\, c' \mid \exists c \in C\colon c \Rightarrow c' \,\}$ the successors.

The following result is due to Büchi (1964):

> Let C be a regular set and $\mathcal{A}$ be a *normalized* $\mathcal{P}$-automaton accepting C.
>
> If C is regular, then so are $pre^*(C)$ and $post^*(C)$.
>
> Moreover, $\mathcal{A}$ can be transformed into an automaton accepting $pre^*(C)$ resp. $post^*(C)$.

The basic idea (for pre)

Saturation rule: Add new transitions to $\mathcal{A}$ as follows:

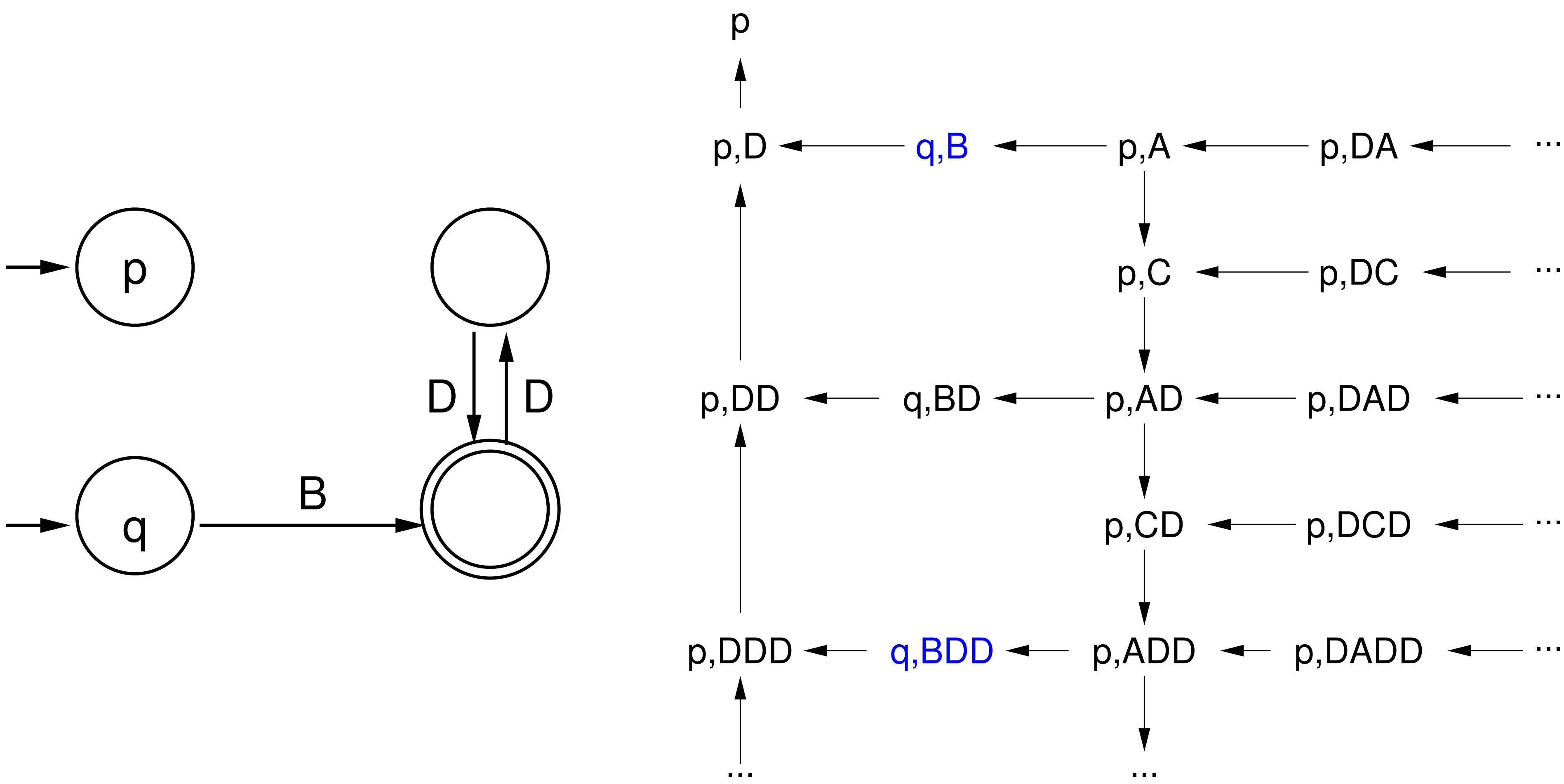
> If $q \xrightarrow{w} r$ currently holds in $\mathcal{A}$ and $pA \hookrightarrow qw$ is a rule, then add the transition (p, A, r) to $\mathcal{A}$.

Repeat this until no other transition can be added.

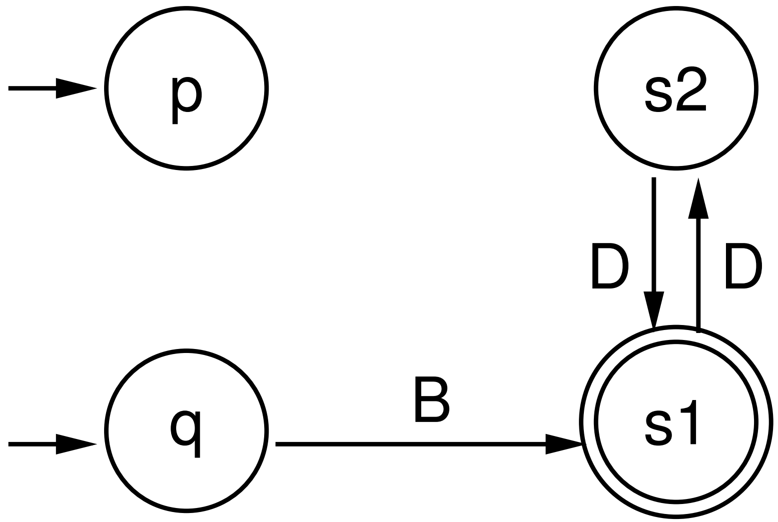
At the end, the resulting automaton accepts $pre^*(C)$.

For $post^*(C)$: similar procedure.

Automaton $\mathcal{A}$ for C

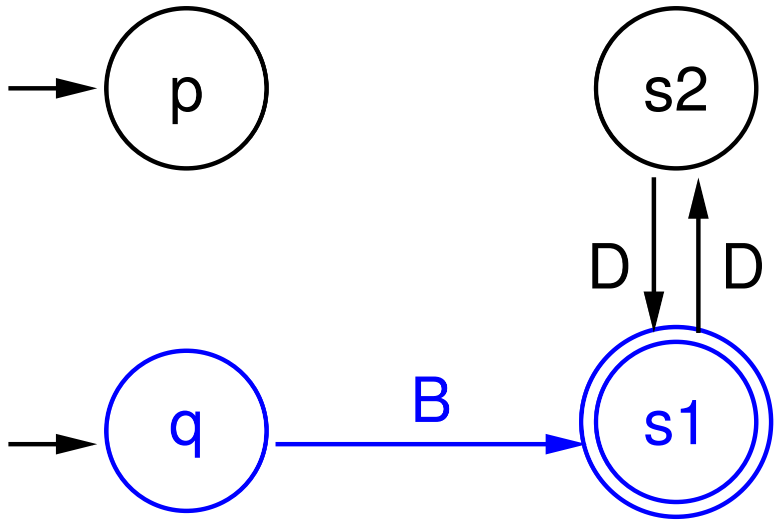


Extending $\mathcal{A}$



Extending $\mathcal{A}$

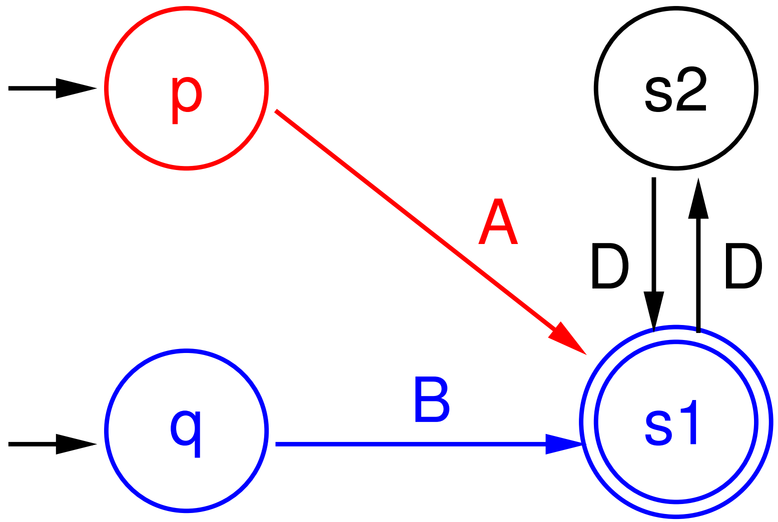
If the right-hand side of a rule can be read,



Rule: $pA \hookrightarrow qB$ Path: $q \xrightarrow{B} s_1$

Extending $\mathcal{A}$

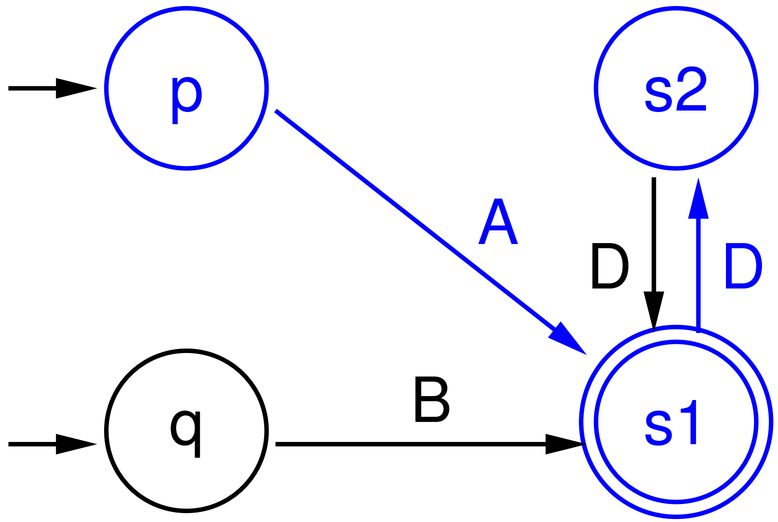
If the right-hand side of a rule can be read, add the left-hand side.



Rule: $pA \hookrightarrow qB$ Path: $q \xrightarrow{B} s_1$ New path: $p \xrightarrow{A} s_1$

Extending $\mathcal{A}$

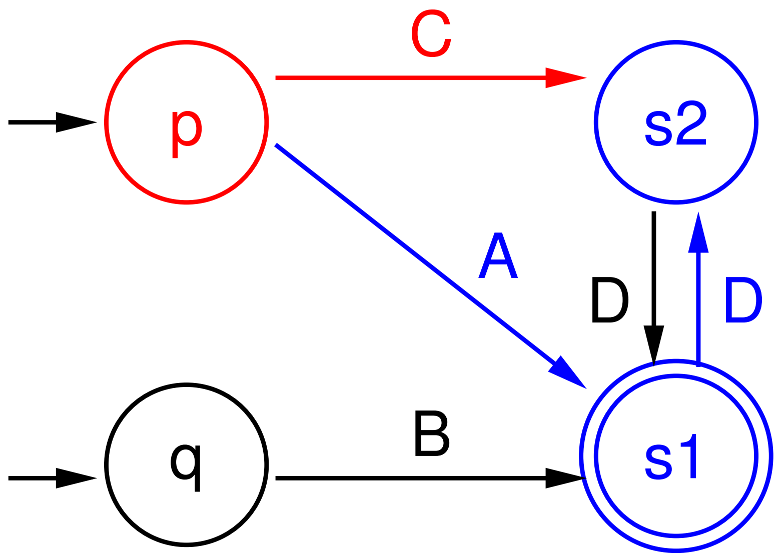
If the right-hand side of a rule can be read,



Rule: $pC \hookrightarrow pAD$ Path: $p \xrightarrow{A} s_1 \xrightarrow{D} s_2$

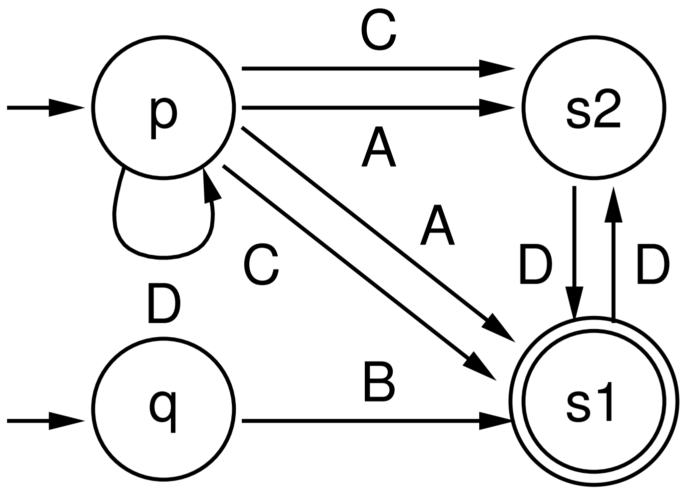
Extending $\mathcal{A}$

If the right-hand side of a rule can be read, add the left-hand side.

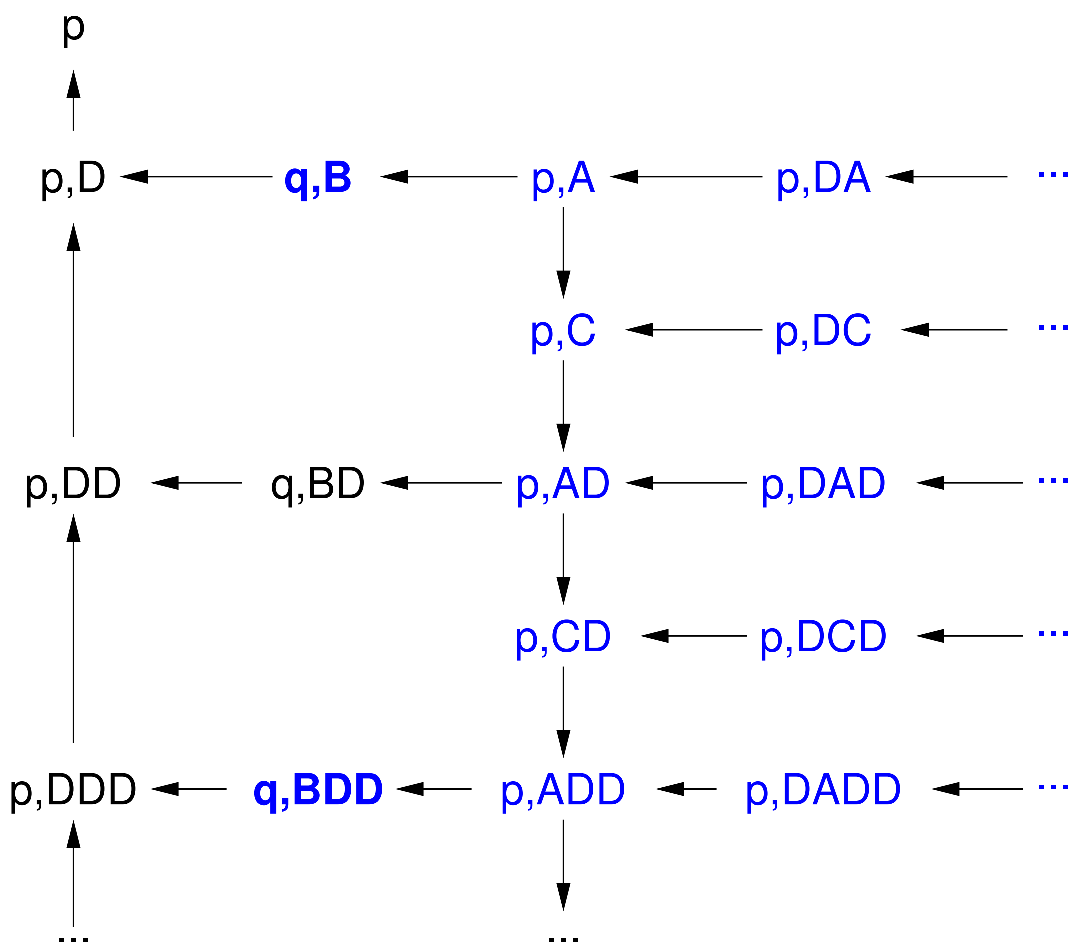


Rule: $pC \hookrightarrow pAD$ Path: $p \xrightarrow{A} s_1 \xrightarrow{D} s_2$ New path: $p \xrightarrow{C} s_2$

Final result



Complexity:
$\mathcal{O}(|Q|^2 \cdot |\Delta|)$ time.



Proof of correctness

We shall show:

Let $\mathcal{B}$ be the $\mathcal{P}$-automaton arising from $\mathcal{A}$ by applying the saturation rule.
Then $\mathcal{L}(\mathcal{B}) = pre^*(C)$.

Part 1: Termination

The saturation rule can only be applied finitely many times because no states are added and there are only finitely many possible transitions.

Part 2: $pre^*(C) \subseteq \mathcal{L}(\mathcal{B})$

Let $c \in pre^*(C)$ and $c' \in C$ such that c' is reachable from c in k steps. We proceed by induction on k (simple).

Part 3: $\mathcal{L}(\mathcal{B}) \subseteq pre^*(C)$

Let $\xrightarrow[i]{}$ denote the transition relation of the automaton after the saturation rule has been applied i times.

We show the following, more general property: If $p \xrightarrow[i]{w} q$, then there exist $p'w'$ with $p' \xrightarrow[0]{w'} q$ and $pw \Rightarrow p'w'$; if $q \in P$, then additionally $w' = \varepsilon$.

Proof by induction over i: The base case $i = 0$ is trivial.

Induction step: Let $t = (p_1, A, q')$ be the transition added in the i-th application and k the number of times t occurs in the path $p \xrightarrow[i]{w} q$.

Induction over k: Trivial for $k = 0$. So let $k > 0$.

There exist p_2, p', u, v, w', w_2 with the following properties:

(1) $p \xrightarrow[i-1]{u} p_1 \xrightarrow[i]{A} q' \xrightarrow[i]{v} q$ (splitting the path $p \xrightarrow[i]{w} q$)

(2) $p_1 A \hookrightarrow p_2 w_2$ (pre-condition for saturation rule)

(3) $p_2 \xrightarrow[i-1]{w_2} q'$ (pre-condition for saturation rule)

(4) $pu \Rightarrow p_1 \varepsilon$ (ind.hyp. on i)

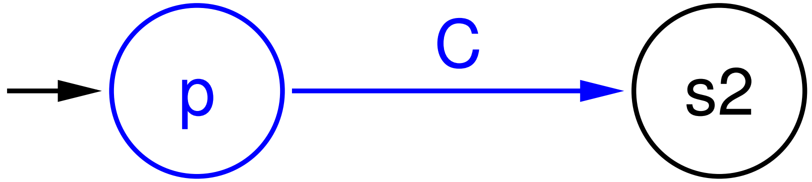
(5) $p_2 w_2 v \Rightarrow p' w'$ (ind.hyp. on k)

(6) $p' \xrightarrow[0]{w'} q$ (ind.hyp. on k)

The desired proof follows from (1), (4), (2), and (5).
If $q \in P$, then the second part follows from (6) and the fact that $\mathcal{A}$ is normalized.

Example: *post** (without proof)

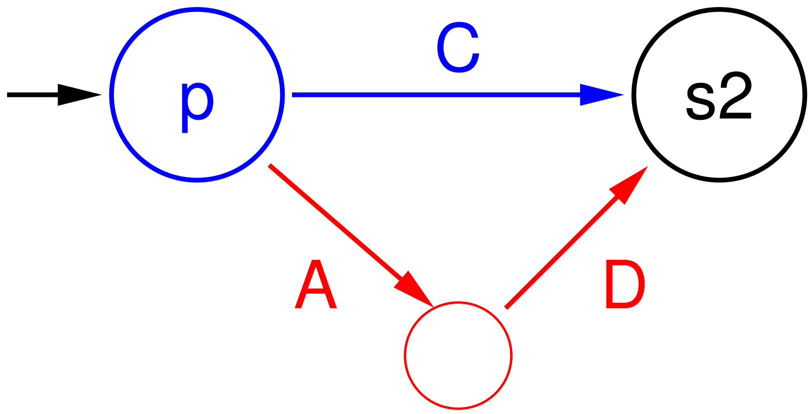
If the *left-hand side* of a rule can be read,   .



Rule: $pC \hookrightarrow pAD$ Path: $p \xrightarrow{C} s_2$

Example: *post** (without proof)

If the *left-hand side* of a rule can be read, add the *right-hand side*.



Rule: $pC \hookrightarrow pAD$ Path: $p \xrightarrow{C} s_2$ New Path: $p \xrightarrow{AD} s_2$

LTL and Pushdown Systems

Let $\mathcal{P} = (P, \Gamma, \Delta)$ be a PDS with initial configuration c_0, let $\mathcal{T}_\mathcal{P}$ denote the corresponding transition system, AP a set of atomic propositions, and $\nu\colon P \times \Gamma^* \to 2^{AP}$. a valuation function.

$\mathcal{T}_\mathcal{P}$, AP, and ν form a Kripke structure $\mathcal{K}$; let ϕ be an LTL formula (over AP).

Problem: Does $\mathcal{K} \models \phi$?

Undecidable for arbitrary valuation functions!
(could encode undecidable decision problems in ν ...)

However, LTL model checking *is* decidable for certain “reasonable” restrictions of ν.

In the following, we consider “simple” valuation functions satisfying the following restriction:

$\nu(pAw) = \nu(pA)$, for all $p \in P$, $A \in \Gamma$, and $w \in \Gamma^*$.

In other words, the “head” of a configuration holds all information about atomic propositions.

LTL model checking is decidable for such “simple” valuations.

Approach

Same principle as for finite Kripke structures:

Translate $\neg\phi$ into a Büchi automaton $\mathcal{B}$.

Build the cross product of $\mathcal{K}$ and $\mathcal{B}$.

Test the cross product for emptiness.

Note that the cross product is not a Büchi automaton in this case, but another pushdown system (with a Büchi-style acceptance condition).

Büchi PDS

The cross product is a new pushdown system $\mathcal{Q}$, as follows:

Let $\mathcal{P} = (P, \Gamma, \Delta)$ be a PDS, $p_0 w_0$ the initial configuration, and AP, ν as usual.

Let $\mathcal{B} = (Q, 2^{AP}, q_0, T, F)$ be the Büchi automaton for $\neg\phi$.

Construction of $\mathcal{Q}$:

$\mathcal{Q} = (P \times Q, \Gamma, \Delta')$, where

$(p, q)A \hookrightarrow (p', q')w \in \Delta'$ iff

– $pA \hookrightarrow p'w \in \Delta$ and

– $(q, L, q') \in T$ such that $\nu(pA) = L$.

Initial configuration: $(p_0, q_0)w_0$

Let ρ be a run of $\mathcal{Q}$ with $\rho(i) = (p_i, q_i)w_i$.

We call ρ accepting if $q_i \in F$ for infinitely many values of i.

The following is easy to see:

$\mathcal{P}$ does *not* satisfy ϕ iff there exists an accepting run in $\mathcal{Q}$.

Characterization of accepting runs

Question: If there an accepting run starting at $(p_0, q_0)w_0$?

In the following, we shall consider the following, more general global model-checking problem:

> Compute *all* configurations c such that there exists an accepting run starting at c.

Lemma: There is an accepting run starting at c iff there exists $(p, q) \in P \times Q$, $A \in \Gamma$ with the following properties:

(1) $c \Rightarrow (p, q)Aw$ for some $w \in \Gamma^*$

(2) $(p, q)A \Rightarrow (p, q)Aw'$ for some $w' \in \Gamma^*$, where

the path from $(p, q)A$ to $(p, q)Aw'$ contains at least one step;

the path contains at least one accepting Büchi state.

Repeating heads

We call $(p,q)A$ a repeating head if $(p,q)A$ satisfies properties (1) and (2).

Strategy:

1. Compute all repeating heads.
   E.g., check for each pair $(p,q)A$ whether
   $(p,q)A \in pre^*(\{\, (p,q)Aw \mid w \in \Gamma^* \,\})$. Visiting an accepting state can be encoded into the control state. (This is a simple but naïve method, one can do better.)

2. Compute the set $pre^*(\{\, (p,q)Aw \mid (p,q)A \text{ is a repeating head}, w \in \Gamma^* \,\})$

Remarks

Other temporal logics for PDS are also decidable (sketch):

CTL: Translate formula into an *alternating* automaton, adapt *pre** algorithm to alternating automata, then apply a technique similar to LTL.

CTL*: Adapt the technique from finite-state systems: Find an E-free subformula ϕ, compute the (regular) set configurations C satisfying $E\phi$. Then encode the states of the automaton for C into the stack, replace $E\phi$ by a fresh atomic proposition p that is true whenever the modified stack tells us that we are in a configuration satisfying $E\phi$.