Initiation à la Vérification

Binary Decision Diagrams

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MPRI, 2014/15
November 18, 2014
Set representations

The solution of the model-checking problem for CTL and finite-state systems can be expressed by operations on sets:

states satisfying (sub)formulae: $[\psi]$  
subformulae combined by set operations: $\cap$, $\cup$, $\ldots$  
e.g., $[\text{EX } \psi]$ can be obtained by the operation  
$\text{pre} \quad (S) \quad := \quad \{ \quad s \mid \exists \ t : s \rightarrow t \land t \in S \}$  

$\text{EG}$ and $\text{EU}$ require fixed-point iterations on set equations  
Likewise, computing the reachable states is expressible with set operations:  
Start by setting $X := I$, the set of initial states;  
Iterate $X := X \cup \{ t \mid \exists s : s \rightarrow t \land s \in S \}$ until fixpoint.
Set representations

How can such sets be represented:

explicit list: \( S = \{ s_1, s_2, s_4, \ldots \} \)

symbolic representation: compact notation or data structure

Idea: Find a data structure that

- can compress the representation of large state sets
- permits efficient operations that match the set operations needed in CTL model checking
Caveat

Due to the pigeon-hole principle, no lossless compression method can compress all sets (or work efficiently for all).

The idea that we study (binary decision diagrams) usually works well for systems whose states can be represented as Boolean vectors, with logical operations between them. We assume:

\[ S = \{0, 1\}^m \quad \text{for some } m \geq 1 \]

Remark: In general, the elements of any finite set can be represented by Boolean vectors if \( m \) is chosen large enough. However, this may not be adequate in all situations.
Literature

Some pointers:

H.R. Andersen, *An Introduction to Binary Decision Diagrams*, Lecture notes, Department of Information Technology, IT University of Copenhagen


Tools:

CUDD library, including DDcal (“BDD calculator”)

URL: http://vlsi.colorado.edu/~fabio/

SMV (BDD-based model checker):
http://www.cs.cmu.edu/~modelcheck/smv.html
Example 1: Petri-net

Consider the following Petri net:

A state can be written as \((p_1, p_2, \ldots, p_6)\), where \(p_i\), \(1 \leq i \leq 6\) indicates whether there is a token on \(P_i\).

Initial state \((1, 0, 1, 0, 1, 0)\);
other reachable states are, e.g., \((0, 1, 1, 0, 1, 0)\) or \((1, 0, 0, 1, 0, 1)\).
Example 2: Circuit

Half-adder:

The circuit has got two inputs \((x_1, x_2)\) and two outputs \((\text{carry}, \text{sum})\). Their admissible combinations can be denoted by Boolean 4-tuples, e.g. \((1, 0, 0, 1)\) \((x_1 = 1, x_2 = 0, \text{carry} = 0, \text{sum} = 1)\) is a possible combination.
The admissible combinations in Example 2 correspond to the following formula of propositional logic:

\[ F \equiv (\text{carry} \leftrightarrow (x_1 \land x_2)) \land (\text{sum} \leftrightarrow (x_1 \lor x_2) \land \neg \text{carry}) \]

In the following, we shall treat

sets of states (i.e. sets of Boolean vectors)

and formulae of propositional logic

simply as different representations of the same objects.
Binary decision graphs

Let $V$ be a set of variables (atomic propositions) and $<$ a total order on $V$, e.g.

$$x_1 < x_2 < \text{carry} < \text{sum}$$

A binary decision graph (w.r.t. $<$) is a directed, connected, acyclic graph with the following properties:

- there is exactly one root, i.e. a node without incoming arcs;
- there are at most two leaves, labelled by 0 or 1;
- all non-leaves are labelled with variables from $V$;
- every non-leaf has two outgoing arcs labelled by 0 and 1;
- if there is an edge from an $x$-labelled node to a $y$-labelled node, then $x < y$. 


Example 2: Binary decision graph (here: a full tree)

Paths ending in 1 correspond to vectors whose entry in the truth table is 1.
A **binary decision diagram** (BDD) is a binary decision graph with two additional properties:

- no two subgraphs are isomorphic;
- there are no *redundant* nodes, where both outgoing edges lead to the same target node.

Optionally, we omit the $0$-node and the edges leading there.

Remarks: On the following slides, the **blue** edges are meant to be labelled by 1, the **red** edges by 0.
Example 2: Eliminate isomorphic subgraphs (1/3)
Example 2: Eliminate isomorphic subgraphs (2/3)

Merged the isomorphic *sum*-nodes (and the leaves).
Two carry nodes can be merged, but no others $\rightarrow$ done.
Example 2: Remove redundant nodes (1/2)

Both edges of the right $sum$-node point to 0.
Example 2: Remove redundant nodes (2/2)

No more redundant nodes → we are done.
Example 2: Omit 0-node

Optionally, we can remove the 0-node and edges leading to it, which makes the representation clearer (but still unambiguous).
In the following, we shall investigate operations on BDDs that are needed for CTL model checking.

Construction of a BDD (from a PL formula)

Equivalence check

Intersection, complement, union

Relations, computing predecessors
Propositional logic with constants

In the following, we will consider formulae of propositional logic (PL), extended with the constants 0 and 1, where:

0 is an unsatisfiable formula;

1 is a tautology.
Substitution

Let $F$ and $G$ be formulae of PL (possibly with constants), and let $x$ be an atomic proposition.

$F[x/G]$ denotes the PL formula obtained by replacing each occurrence of $x$ in $F$ by $G$.

In particular, we will consider formulae of the form $F[x/0]$ and $F[x/1]$.

Example: Let $F = x \land y$. Then $F[x/1] = 1 \land y \equiv y$ and $F[x/0] = 0 \land y \equiv 0$. 
If-then-else

Let us introduce a new, ternary PL operator. We shall call it $ite$ (if-then-else).

Note: $ite$ does not extend the expressiveness of PL, it is simply a convenient shorthand notation.

Let $F, G, H$ be PL formulae. We define

$$ite (F, G, H) := (F \land G) \lor (\neg F \land H).$$

The set of INF formulae (if-then-else normal form) is inductively defined as follows:

0 and 1 are INF formulae;

if $x$ is an atomic proposition and $G, H$ are INF formulae, then $ite (x, G, H)$ is an INF formula.
Shannon partitioning

Let $F$ be a PL formula and $x$ an atomic proposition. We have:

$$F \equiv \text{ite}(x, F[x/1], F[x/0])$$

**Proof:** In the following, $G$ denotes the right-hand side of the equivalence above. Let $\nu$ be a valuation s.t. $\nu \models F$. Either $\nu(x) = 1$, then $\nu$ is also a model of $F[x/1]$ and of $x$ and therefore also of $G$. The case $\nu(x) = 0$ is analogous. For the other direction, suppose $\nu \models G$. Then either $\nu(x) = 1$ and the “rest” of $\nu$ is a model of $F[x/1]$. Then, however, $\nu$ will be a model for any formula in which some of the ones in $F[x/1]$ are replaced by $x$, in particular also for $F$. The case $\nu(x) = 0$ is again analogous.

**Remark:** $G$ is called the Shannon partitioning of $F$.

**Corollary:** Every PL formula is equivalent to an INF formula.
(Proof: apply the above partitioning multiple times.)
Construction of BDDs

We can now solve our first BDD-related problem: Given a PL formula $F$ and some ordering of variables $<$, construct a BDD w.r.t. $<$ that represents $F$.

If $F$ does not contain any atomic propositions at all, then either $F \equiv 0$ or $F \equiv 1$, and the corresponding BDD is simply the corresponding leaf node.

Otherwise, let $x$ be the smallest variable (w.r.t. $<$) occurring in $F$. Construct BDDs $B_0$ and $B_1$ for $F[x/1]$ and $F[x/0]$, respectively (these formulae have one variable less than $F$).

Because of the Shannon partitioning, $F$ is representable by a binary decision graph whose root is labelled by $x$ and whose subtrees are $B_0$ and $B_1$. To obtain a BDD, we check whether $B_0$ and $B_1$ are isomorphic; if yes, then $F$ is represented by $B_0$. Otherwise we merge all isomorphic subtrees in $B_0$ and $B_1$. 
BDDs are unique

Given a PL formula $F$ and a variable ordering $<$, there is (up to isomorphism) exactly one BDD that respects $<$ and represents $F$.

Proof: (sketch) by induction on the number of variables, start with 0 (constant functions), then use Shannon partitioning.

Remark: Different orderings still lead to different BDDs. (possibly with vastly different sizes!)
Example: Variable orderings

Recall Example 1 (the Petri net), and let us construct a BDD representing the reachable markings:

![Petri net diagram]

Remark: $P_1$ is marked iff $P_2$ is not, etc.
The corresponding BDD for the ordering $p_1 < p_2 < p_3 < p_4 < p_5 < p_6$:
Remarks:

If we increase the number of components from 3 to $n$ (for some $n \geq 0$), the size of the corresponding BDD will be linear in $n$.

In other words, a BDD of size $n$ can represent $2^n$ (or even more) valuations.

However, the size of a BDD strongly depends on the ordering!
Example: Repeat the previous construction for the ordering

$$p_1 < p_3 < p_5 < p_2 < p_4 < p_6.$$
Equivalence test

To implement CTL model checking, we need a test for equivalence between BDDs (e.g., to check the termination of a fixed-point computation).

**Problem:** Given BDDs $B$ and $C$ (w.r.t. the same ordering) do $B$ and $C$ represent equivalent formulae?

**Solution:** Test whether $B$ and $C$ are isomorphic.

**Special cases:**

Unsatisfiability test: Check if the BDD consists just of the 0 leaf.

Tautology test: Check if the BDD consists just of the 1 leaf.
Implementing BDDs with hash tables

Suppose we want to write an application in which we need to manipulate multiple BDDs.

Efficient BDDs implementations exploit the uniqueness property by storing all BDD nodes in a hash table. (Recall that each node is in fact the root of some BDD.)

Initially, the hash table has only two unique entries, the leaves 0 and 1.

Every other node is uniquely identified by the triple \((x, B_0, B_1)\), where \(x\) is the atomic proposition labelling that node and \(B_0, B_1\) are the subtrees of that node, represented by pointers to their respective roots.
Usually, one implements a function \( mk(x, B_0, B_1) \) that checks whether the hash table already contains such a node; if yes, then the pointer to that node is returned, otherwise a new node is created.

Each BDD is then simply represented by a pointer to its root.

A multitude of BDDs is then stored as a “forest” (a DAG with multiple roots).

Problem: garbage collection (by reference counting)
Equivalence test II

Let us reconsider the equivalence-checking problem. (Given two BDDs \( B \) and \( C \), do \( B \) and \( C \) represent equivalent formulae?)

If \( B \) and \( C \) are stored in hash tables (as described previously), then \( B \) and \( C \) are representable by pointers to their roots.

Due to the uniqueness property, one then simply has to check whether the pointers are the same (a constant-time procedure).
Logical operations I: Complement

Let $F$ be a PL formula and $B$ a BDD representing $F$.

**Problem:** Compute a BDD for $\neg F$.

**Solution:** Exchange the two leaves of $B$.

(Caution: This is not quite so simple with the hash-table implementation.)
Let $F, G$ be PL formulae and $B, C$ the corresponding BDDs (with the same ordering).

**Problem:** Compute a BDD for $F \land G$ from $B$ and $C$.

We have the following equivalence:

\[ F \land G \equiv \text{ite}(x, (F \land G)[x/1], (F \land G)[x/0]) \equiv \text{ite}(x, F[x/1] \land G[x/1], F[x/0] \land G[x/0]) \]

If $x$ is the smallest variable occurring in either $F$ or $G$, then $F[x/1], F[x/0], G[x/1], G[x/0]$ are either the children of the roots of $B$ and $C$ (or the roots themselves).
We construct a BDD for conjunction according to the following, recursive strategy:

If $B$ and $C$ are equal, then return $B$.

If either $B$ or $C$ are the 0 leaf, then return 0.

If either $B$ or $C$ are the 1 leaf, then return the other BDD.

Otherwise, compare the two variables labelling the roots of $B$ and $C$, and let $x$ be the smaller among the two (or the one labelling both).

If the root of $B$ is labelled by $x$, then let $B_1, B_0$ be the subtrees of $B$; otherwise, let $B_1, B_0 := B$. We define $C_1, C_0$ analogously.

Apply the strategy recursively to the pairs $B_1, C_1$ and $B_0, C_0$, yielding BDDs $E$ and $F$. If $E = F$, return $E$, otherwise $mk(x, E, F)$. 


Logical operations III: Union/Disjunction

Let $F, G$ be PL formulae and $B, C$ the corresponding BDDs (with the same ordering).

Problem: Compute a BDD for $F \lor G$ from $B$ and $C$.

Solution: Analogous to conjunction, with the rules for 1 and 0 leaves adapted accordingly.

Complexity: With dynamic programming: $O(|B| \cdot |C|)$ (every pair of nodes at most once).
Computing predecesors

In the following, we derive a strategy for computing the set

\[ \text{pre}(M) = \{ s \mid \exists s': (s, s') \in \rightarrow \land s' \in M \}. \]

Note that the relation \( \rightarrow \) is a subset of \( S \times S \) whereas \( M \subset S \).

We represent \( M \) by a BDD with variables \( y_1, \ldots, y_m \).

\( \rightarrow \) will be represented by a BDD with variables \( x_1, \ldots, x_m \) and \( y_1, \ldots, y_m \) (states “before” and “after”).
Remark: Every BDD for $M$ is at the same time a BDD for $S \times M$!

Thus, we can rewrite $\text{pre}(M)$ as follows:

$$\{ s | \exists s': (s, s') \in \rightarrow \cap (S \times M) \}$$

Then, $\text{pre}$ reduces to the operations intersection and existential abstraction.
Existential abstraction

Existential abstraction w.r.t. an atomic proposition $x$ is defined as follows:

$$
\exists x : F \equiv F[x/0] \lor F[x/1]
$$

I.e., $\exists x : F$ is true for those valuations that can be extended with a value for $x$ in such a way that they become models for $F$.

Example: Let $F \equiv (x_1 \land x_2) \lor x_3$. Then

$$
\exists x_1 : F \equiv F[x_1/0] \lor F[x_1/1] \equiv (x_3) \lor (x_2 \lor x_3) \equiv x_2 \lor x_3
$$

By extension, we can consider existential abstraction over sets of atomic propositions (abstract from each of them in turn).