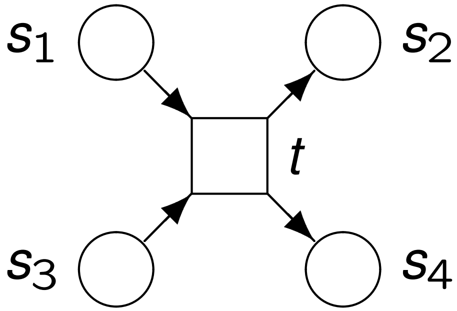
Petri nets

Petri nets

Petri nets are a basic model of parallel and distributed systems.
The basic idea is to describe state changes locally.



Petri nets contain places ○ and transitions □ that may be connected by directed arcs.

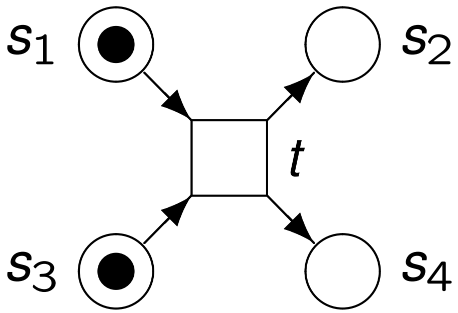
Places symbolise states, conditions, or resources that need to be met/be available before an action can be carried out.

Transitions symbolise actions.

Behaviour of Petri nets

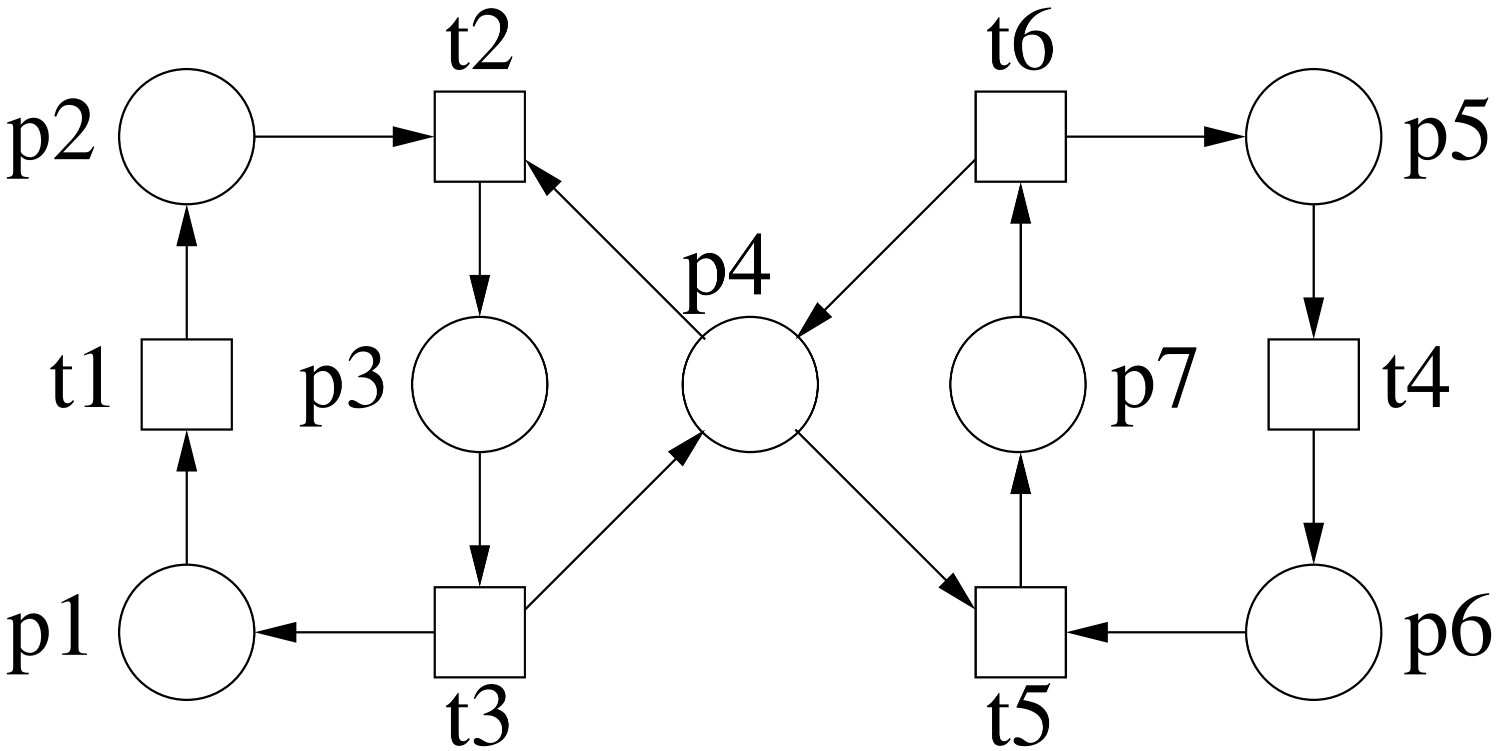
Places may contain tokens that may move to other places by executing ("firing") actions.

A token on a place means that the corresponding condition is fulfilled or that a resource is available:



In the example, transition t may "fire" if there are tokens on places s_1 and s_3. Firing t will remove those tokens and place new tokens on s_2 and s_4.

Example 1



Why Petri Nets?

- low-level model for concurrent systems
- expressly models concurrency, conflict, causality, ...
- finite-state or infinite-state models

Content:

- Semantics of Petri nets
- Modelling with Petri nets
- Analysis methods: finite/infinite-state case, structural analysis

Remark: Many variants of Petri nets exist in the literature; we regard a special simple case also called P/T nets.

Petri Net

A Petri net is a tuple $N = \langle P, T, F, W, m_0 \rangle$, where

- P is a finite set of places,
- T is a finite set of transitions,
- the places P and transitions T are disjoint ($P \cap T = \emptyset$),
- $F \subseteq (P \times T) \cup (T \times P)$ is the flow relation,
- $W \colon ((P \times T) \cup (T \times P)) \to \mathbb{N}$ is the arc weight mapping (where $W(f) = 0$ for all $f \notin F$, and $W(f) > 0$ for all $f \in F$), and
- $m_0 \colon P \to \mathbb{N}$ is the initial marking representing the initial distribution of tokens.

Semantics

Let $N = \langle P, T, F, W, m_0 \rangle$ be a Petri net. We associate with it the transition system $\mathcal{M} = \langle S, \Sigma, \Delta, I, AP, \ell \rangle$, where:

$$S = \{ m \mid m \colon P \to \mathbb{N} \}, \;\; I = \{m_0\}$$

$$\Sigma = T$$

$$\Delta = \{ (m, t, m') \mid \forall p \in P : m(p) \geq W(p, t) \wedge m'(p) = m(p) - W(p, t) + W(t, p) \}$$

$$AP = P, \;\; \ell(m) = \{ p \in P \mid m(p) > 0 \}$$

When $(m, t, m') \in \Delta$, we say that t is enabled in m and that its firing produces the successor marking m'; we also write $m \xrightarrow{t} m'$.

Semantics (remark)

The semantics given on the previous slide is also called interleaving semantics (one transition fires at a time).

Alternatively, one could define a step semantics, which better expresses the concurrent behaviours.

In step semantics, one allows a *multiset* of transitions to fire simultaneously; i.e. a multiset A is enabled in marking m if m contains enough tokens to fire all transitions in A.

However, for our purposes the interleaving semantics is sufficient.

Petri nets: Remarks

If $\langle p, t\rangle \in F$ for a transition t and a place p, then p is an input place of t,

If $\langle t, p\rangle \in F$ for a transition t and a place p, then p is an output place of t,

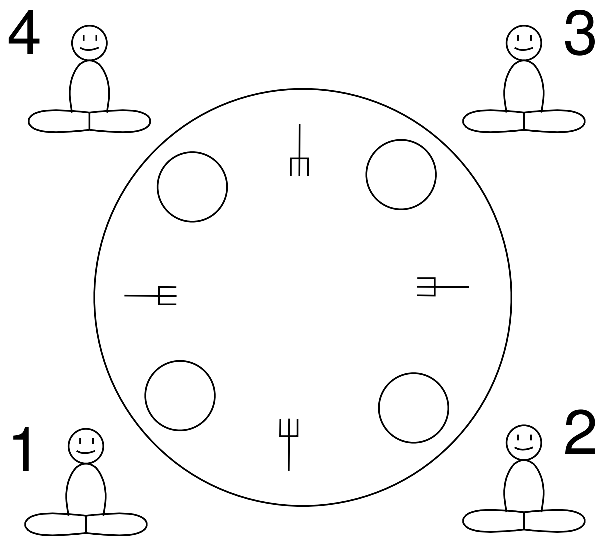
Let $a \in P \cup T$. The set ${}^\bullet a = \{a' \mid \langle a', a\rangle \in F\}$ is called the pre-set of a, and the set $a^\bullet = \{a' \mid \langle a, a'\rangle \in F\}$ is its post-set.

When drawing a Petri net, we usually omit arc weights of 1. Also, we may either denote tokens on a place either by black circles, or by a number.

Example: Dining philosophers

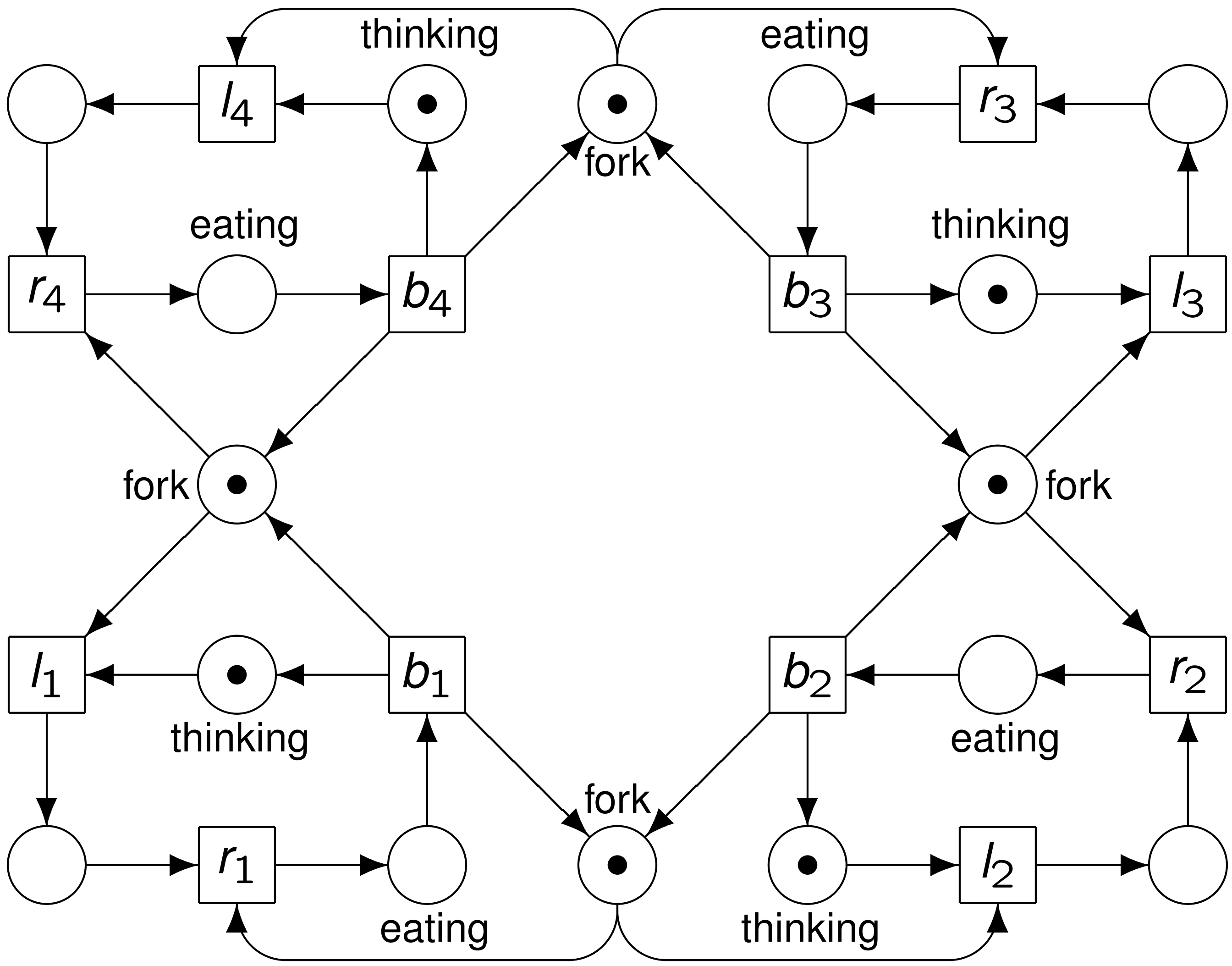
There are philosophers sitting around a round table.

There are forks on the table, one between each pair of philosophers.



The philosophers eat spaghetti from a large bowl in the center of the table.

Dining philosophers: Petri net



Notation for markings

Often we will fix an order on the places (e.g., matching the place numbering), and write, e.g., $m_0 = \langle 2, 5, 0 \rangle$ instead.

When no place contains more than one token, markings are in fact sets, in which case we often use set notation and write instead $m_0 = \{p_5, p_7, p_8\}$.

Alternatively, we could denote a marking as a multiset, e.g.
$m_0 = \{p_1, p_1, p_2, p_2, p_2, p_2, p_2\}$.

$reach(m)$ denotes markings reachable from marking m

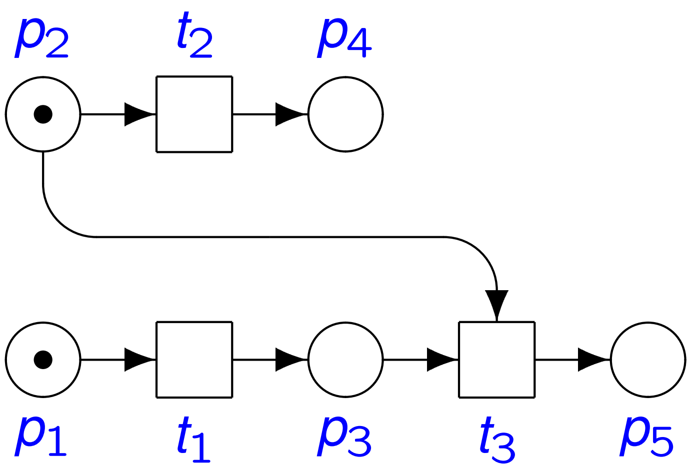
$reach(N) := reach(m_0)$

Reachability graph: Transition system of N restricted to $reach(N)$

k-safeness

Definition: Let N be a net. If no reachable marking of N can contain more than k tokens in any place (where $k \geq 0$ is some constant), then N is said to be k-safe.

Example: The following net is 1-safe (as is the Dining Philosophers example).



Other example: the nets resulting from translating synchronous rendez-vous

k-safeness and Termination

A k-safe net has at most $(k+1)^{|P|}$ reachable markings;
for 1-safe nets, the limit is $2^{|P|}$.

If a net is k-safe for some k, its reachability graph is finite.

On the other hand, if a net is not k-safe for any k, then there are infinitely many reachable markings, and the reachability graph is infinite.

Reachability problem for 1-safe nets

Let N be a Petri net and m be a marking. The *reachability problem* for N, m is to determine whether $m \in reach(N)$.

Theorem: The reachability problem for 1-safe Petri nets is PSPACE-complete.

Proof: (sketch)
upper bound: non-deterministically simulate net for at most $2^{|P|}$ steps;
hardness by reduction from QBF (see following slides).

Reduction QBF $\longrightarrow$ 1-safe PN reachability

Let $\phi = Q_1 x_1 \cdots Q_n x_n \psi$ be a quantified boolean formula.

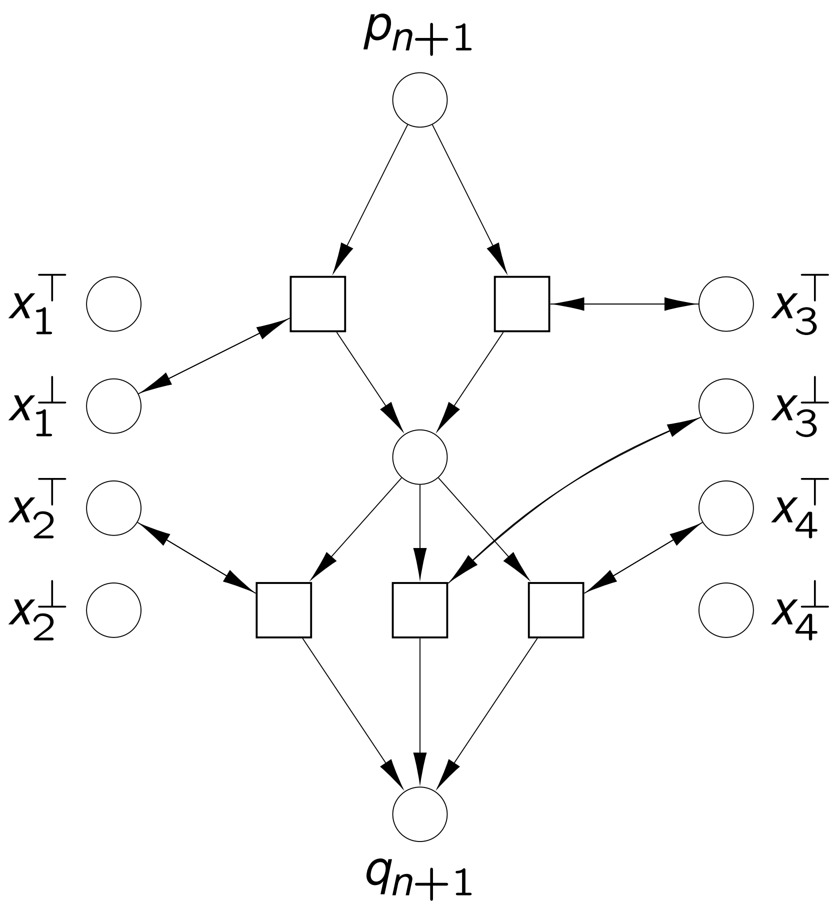
$Q_1, \ldots, Q_n \in \{\exists, \forall\}$ are quantifiers

ψ is in CNF using variables $x_1, \ldots, x_n$.

We construct a net N_1 such that a marking containing only place q_1 is reachable iff ϕ is true.

Reduction (2)

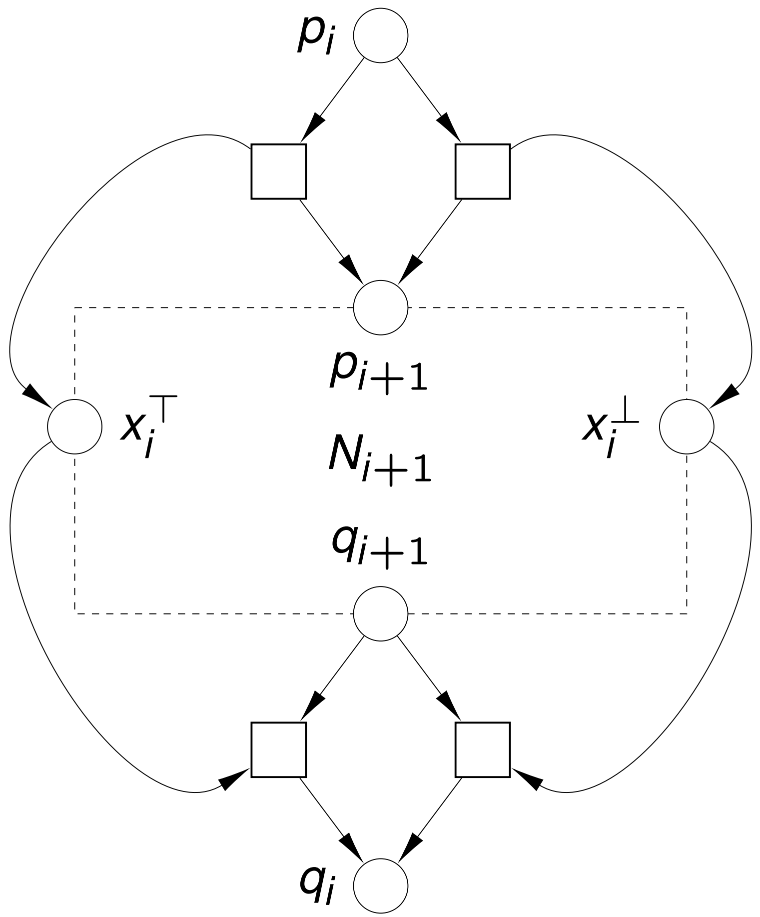
First step: Construct a (partial) net N_{n+1} for ψ, p.ex. for
$\psi = (\neg x_1 \vee x_3) \wedge (x_2 \vee \neg x_3 \vee x_4)$:



Reduction (3)

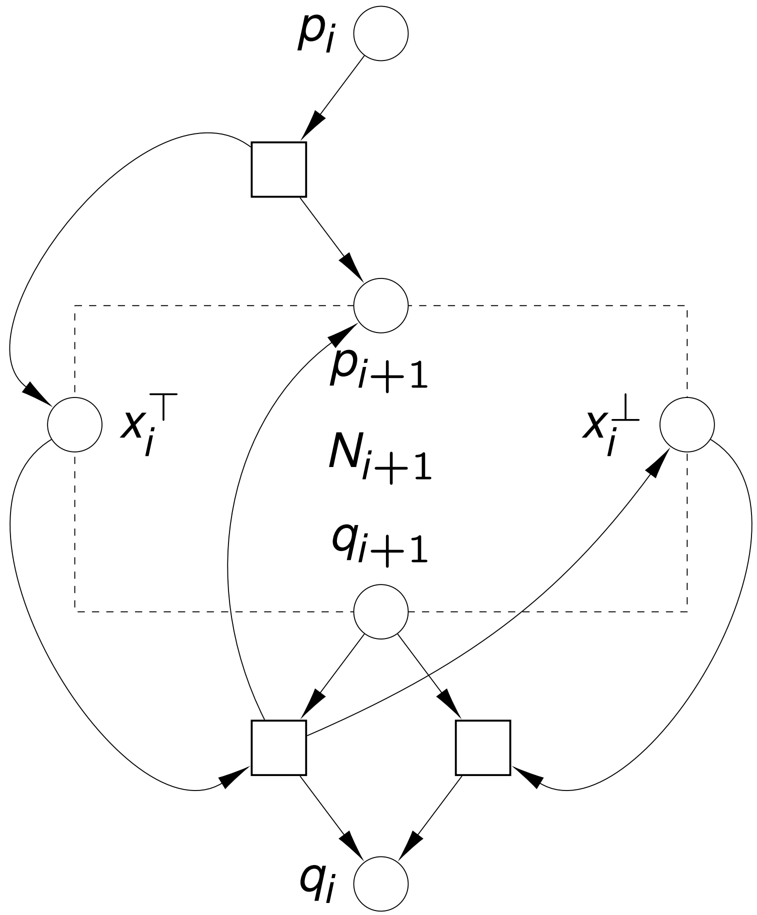
Then for $i = n, \ldots, 1$, construct N_i from N_{i+1} as follows:

if $Q_i = \exists$:



Then for $i = n, \dots, 1$, construct N_i from N_{i+1} as follows:

if $Q_i = \forall$:



Reduction (4)

The initial marking of N_1 consists of a single token on p_1.

For $\exists x_i$, N_i chooses a truth value for x_i and goes to N_{i+1}.

For $\forall x_i$, N_i first chooses $x_i = \top$ then $x_i = \bot$.

N_{n+1} tests if ψ is true under the current assignment.

Finally, one can obtain the marking q_1 iff ϕ is true.

Corollary: Given a 1-safe net N and a place p, it is PSPACE-complete to determine whether $reach(N)$ contains a marking m such that $m(p) = 1$.

Unbounded nets: Coverability graphs

Use of reachability graphs

If the net is not k-safe for any k, then it has infinitely many reachable markings, and one cannot effectively compute the reachability graph.

Nevertheless, the following problem is decidable: Given a (non-safe) net $\mathcal{P}$ and a marking m, is m reachable in $\mathcal{P}$?

This result is due to Mayr and Kosaraju (1981/82).
Precise complexity: non-primitive recursive (Leroux/Czerwiński+Orlikowski 2021)

Coverability problem

Sometimes, one is interested in a checking whether m is *part of* a reachable marking (one says that m is *coverable* in this case).

We discuss a construction that constructs, instead of the (possibly infinite) reachability graph, a variant of the graph containing all coverable markings.

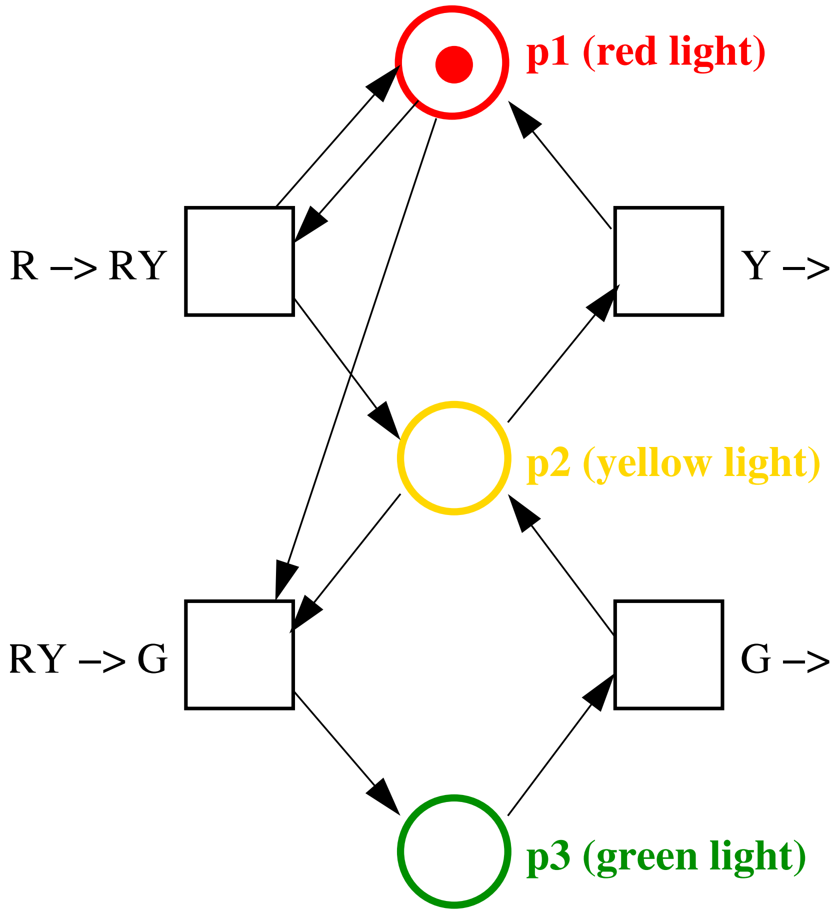
While that algorithm can result in a graph of non-primitive recursive size, it is relatively easy to understand.

The coverability graph will have the following properties:

- It can be used to find out whether the reachability graph is infinite.
- It is always finite, and its construction always terminates.

Example

Consider the following (slightly inept) attempt at modelling a traffic light:



ω-markings

We use ω to represent “arbitrarily many” (not infinitely many!) tokens on a place.

For the firing rule, we extend the arithmetic as usual (for all $n \in \mathbb{N}$):

$$n + \omega = \omega + n = \omega \qquad \omega + \omega = \omega$$

$$\omega - n = \omega \qquad 0 \cdot \omega = 0$$

$$\omega \cdot \omega = \omega \qquad n \geq 1 \Rightarrow n \cdot \omega = \omega \cdot n = \omega$$

$$n \leq \omega \qquad \omega \leq \omega$$

An ω-marking M' covers an ω-marking M, denoted $M \leq M'$, iff

$$\forall p \in P\colon\ M(p) \leq M'(p).$$

An ω-marking M' strictly covers an ω-marking M, denoted $M < M'$, iff

$$M \leq M' \quad \text{and} \quad M' \neq M.$$

Coverability and Transition Sequences

Observation: Let M and M' be two markings such that $M \leq M'$.
Then for all transitions t, if $M \xrightarrow{t}$ then $M' \xrightarrow{t}$. as M does.

This observation can be extended to *sequences* of transitions.
Define $M \xrightarrow{t_1 t_2 \ldots t_n} M'$ to denote:

$$\exists M_1, M_2, \ldots, M_n : M \xrightarrow{t_1} M_1 \xrightarrow{t_2} M_2 \cdots \xrightarrow{t_n} M_n = M'.$$

Now, if $M \xrightarrow{t_1 t_2 \ldots t_n}$ and $M \leq M'$, then by firing the transition sequence $t_1 t_2 \ldots t_n$ repeatedly we can "pump" an arbitrary number of tokens to all the places having a non-zero marking in $\triangle M := M' - M$.

The basic idea for constructing the coverability graph is now to replace the marking M' with a marking where all the places with non-zero tokens in $\triangle M$ are replaced by ω.

Coverability Graph Algorithm (1/2)

$\textsc{Coverability-Graph}(\langle P, T, F, W, M_0 \rangle)$

$\langle V, E, v_0 \rangle := \langle \{M_0\}, \emptyset, M_0 \rangle$;
Work : set $:= \{M_0\}$;
while $\mathit{Work} \neq \emptyset$
do select M from Work;
    $\mathit{Work} := \mathit{Work} \setminus \{M\}$;
    for $t \in \mathrm{enabled}(M)$
    do $M' := \mathrm{fire}(M, t)$;
        $M' := \mathrm{AddOmegas}(M, M', V)$;
        if $M' \notin V$
          then $V := V \cup \{M'\}$
                $\mathit{Work} := \mathit{Work} \cup \{M'\}$;
        $E := E \cup \{\langle M, t, M' \rangle\}$;
return $\langle V, E, v_0 \rangle$;

The subroutine $\mathrm{AddOmegas}(M, M', V)$ will check if the sequences leading to M' can be repeated, strictly increasing the number of tokens on some places, and replace their values with ω.

Coverability Graph Algorithm (2/2)

The following notation us used in the AddOmegas subroutine:

- $M'' \rightarrow^* M$ iff the coverability graph currently contains a path (including the empty path!) leading from M'' to M.

AddOmegas(M, M', V)

```
1  repeat saved := M';
2          for all M'' ∈ V s.t. M'' →* M
3          do if M'' < M'
4                then M' := M' + ((M' − M'') · ω);
5  until saved = M';
6  return M';
```

In other words, repeated check all the predecessor markings of the new marking M' to see if they are strictly covered by M'. Line 5 causes all places whose number of tokens in M' is strictly larger than in the "parent" M'' to contain ω.

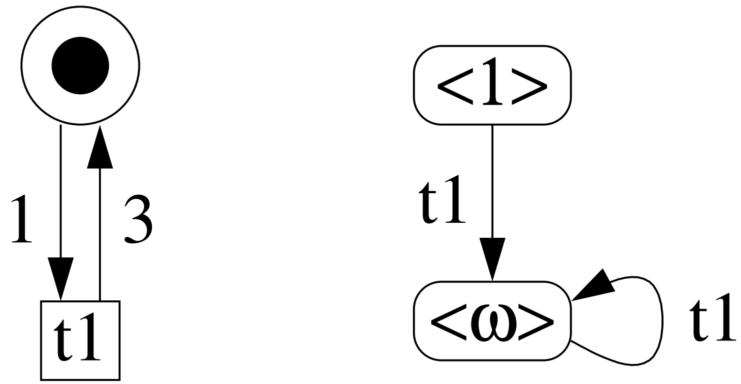
Properties of the coverability graph (1)

Let $N = \langle P, T, F, W, M_0 \rangle$ be a net.

The coverability graph has the following fundamental property:

> If a marking M of N is reachable, then M is covered by some vertex of the coverability graph of N.

Note that the reverse implication *does not* hold: A marking that is covered by some vertex of the coverability graph is not necessarily reachable, as shown by the following example:



Properties of the coverability graph (2)

The coverability graph could thus be said to compute an overapproximation of the reachable markings.

The construction of the coverability graph always terminates (consequence of Dickson's Lemma). If N is bounded, then the coverabilibility graph is identical to the reachability graph.

Coverability graphs are not unique,
i.e. for a given net there may be more than one coverability graph, depending on the order of the worklist and the order in which firing transitions are considered.

Petri nets: Structural analysis

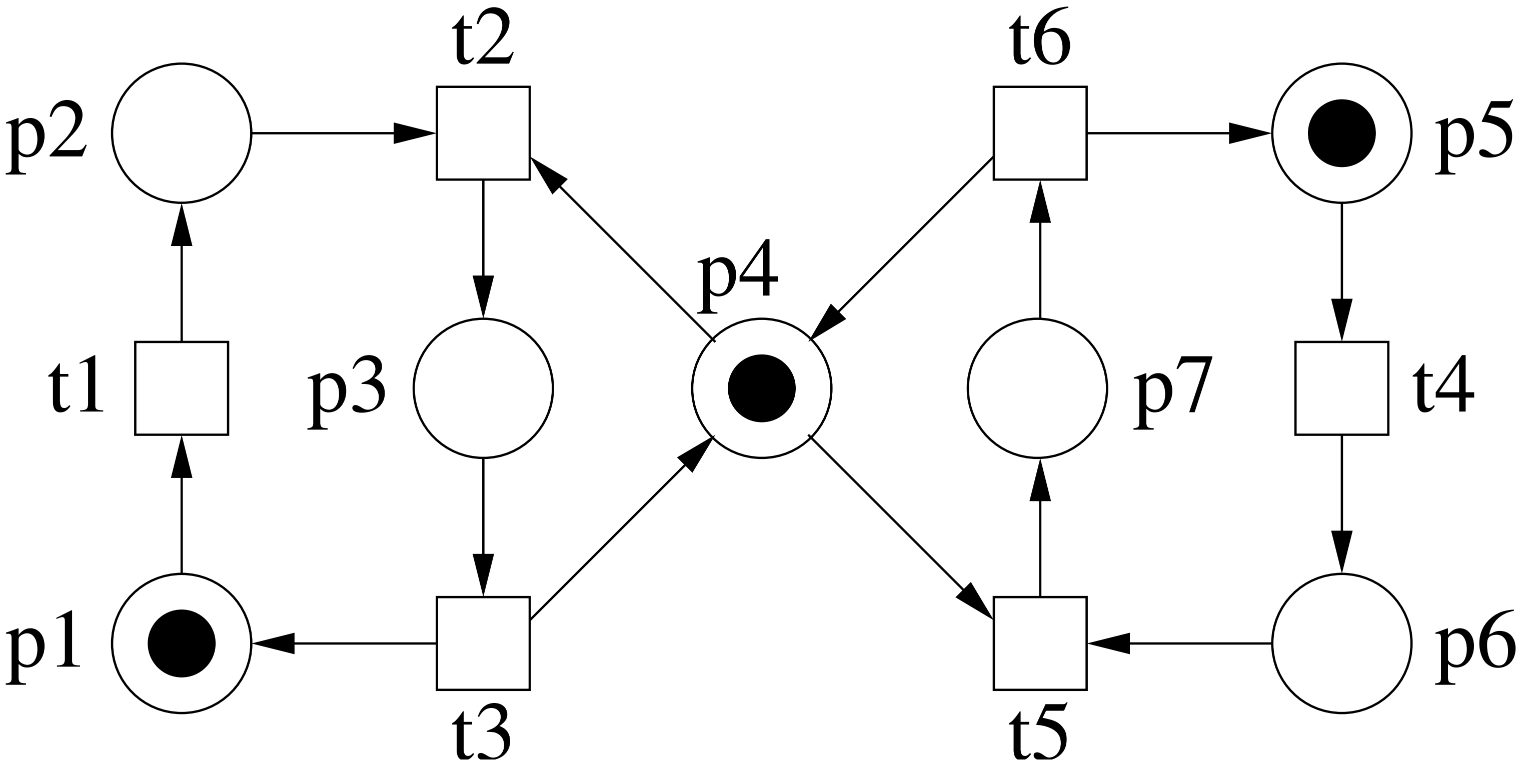
Structural Analysis: Motivation

We shall consider another class of techniques that can extract information about the behaviour of the system by analyzing it locally (i.e., without first constructing an object that represents the entire behaviour of the net).

This class of techniques is called structural analysis. Some its components are:

- Place invariants
- Traps

Example 1



Incidence Matrix

Let $N = \langle P, T, F, W, M_0 \rangle$ be a P/T net. The corresponding incidence matrix $C\colon P \times T \to \mathbb{Z}$ is the matrix whose rows correspond to places and whose columns correspond to transitions. Column $t \in T$ denotes how the firing of t affects the marking of the net: $C(t, p) = W(t, p) - W(p, t)$.

The incidence matrix of Example 1:

$$
\begin{array}{cc}
\begin{array}{cccccc} t_1 & t_2 & t_3 & t_4 & t_5 & t_6 \end{array} & \\
\begin{pmatrix}
-1 & 0 & 1 & 0 & 0 & 0 \\
1 & -1 & 0 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 & 0 \\
0 & -1 & 1 & 0 & -1 & 1 \\
0 & 0 & 0 & -1 & 0 & 1 \\
0 & 0 & 0 & 1 & -1 & 0 \\
0 & 0 & 0 & 0 & 1 & -1
\end{pmatrix} &
\begin{array}{c} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \\ p_7 \end{array}
\end{array}
$$

Markings as vectors

Let us now write markings as column vectors. E.g., the initial marking in Example 1 is $M_0 = (1\ 0\ 0\ 1\ 1\ 0\ 0)^T$.

Likewise, we can write firing counts as column vectors with one entry for each transition. E.g., if each of the transitions t_1, t_2, and t_4 fires once, we can express this with $u = (1\ 1\ 0\ 1\ 0\ 0)^T$.

Then, the result of firing these transitions can be computed as $M_0 + C \cdot u$.

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

Let N be a P/T net with indicence matrix C, and let M, M' be two markings of N. The following implication holds:

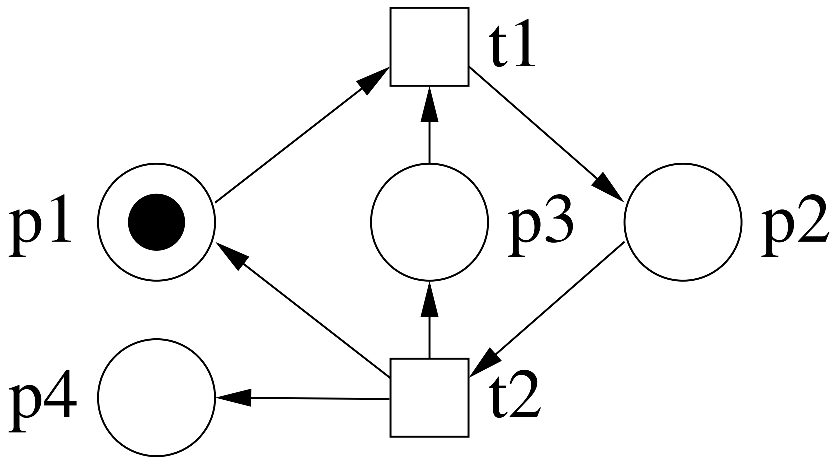
> If $M' \in reach(M)$, then there exists a vector u such that $M' = M + C \cdot u$ such that all entries in u are natural numbers.

Notice that the reverse implication does not hold in general!

E.g., bi-directional arcs (an arc from a place to a transition and back) cancel each other out in the matrix. For instance, if Example 1 contained a bi-directional arc between t_1 and p_3, the matrix would remain the same, but the marking $\{p_3, p_6\}$ (obtained on the previous slide) would be unreachable!

Example 2

An example without “back-and-forth” arcs:



Even though we have

$$\begin{pmatrix}1\\0\\0\\0\end{pmatrix} + \begin{pmatrix}-1 & 1\\1 & -1\\-1 & 1\\0 & 1\end{pmatrix} \cdot \begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1\\0\\0\\1\end{pmatrix},$$

none of the sequences corresponding to $(1\ 1)^T$, i.e. t_1t_2 or t_2t_1, can happen.

Proving unreachability using the incidence matrix

To summarize: The markings obtained by computing with the incidence matrix are an over-approximation of the actual reachable markings

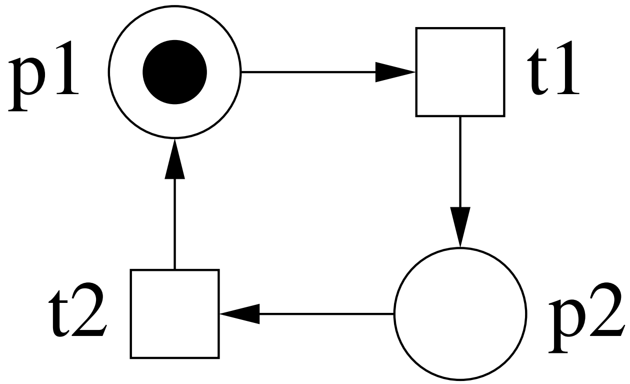
However, we *can* sometimes use the matrix equations to show that a marking M is *unreachable*. (Compare coverability graphs...)

I.e., a corollary of the previous implication is that if $M' = M + Cu$ has no natural solution for u, then $M' \notin reach(M)$.

Note: When we are talking about natural (integral) solutions of equations, we mean those whose components are natural (integral) numbers.

Example 3

Consider the following net and the marking $M = (1\ 1)^T$.



$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

has no solution, and therefore M is not reachable.

Place invariants

Let N be a net and C its incidence matrix. A natural solution of the equation $C^T x = \mathbf{0}$ such that $x \neq \mathbf{0}$ is called a place invariant (or: P-invariant) of N.

Notice that a P-invariant is a vector with one entry for each place.

For instance, in Example 1, $x_1 = (1\ 1\ 1\ 0\ 0\ 0\ 0)^T$, $x_2 = (0\ 0\ 1\ 1\ 0\ 0\ 1)^T$, and $x_3 = (0\ 0\ 0\ 0\ 1\ 1\ 1)^T$ are all P-invariants.

A P-invariant indicates that the number of tokens in all reachable markings satisfies some linear invariant (see next slide).

Properties of P-invariants

Let M be marking reachable with a transition sequence whose firing count is expressed by u, i.e. $M = M_0 + Cu$. Let x be a P-invariant. Then, the following holds:

$$M^T x = (M_0 + Cu)^T x = M_0^T x + (Cu)^T x = M_0^T x + u^T C^T x = M_0^T x$$

For instance, invariant x_2 means that all reachable markings M satisfy

$$M(p_3) + M(p_4) + M(p_7) = M_0(p_3) + M_0(p_4) + M_0(p_7) = 1 \qquad (1)$$

In particular, p_3 and p_7 cannot be marked concurrently!

As a special case, a P-invariant in which all entries are either 0 or 1 indicates a set of places in which the number of tokens remains unchanged in all reachable markings.

Note that linear combinations of P-invariants are also P-invariants.

Traps

Let $\langle P, T, F, W, M_0 \rangle$ be a P/T net. A trap is a set of places $S \subseteq P$ such that $S^\bullet \subseteq {}^\bullet S$.

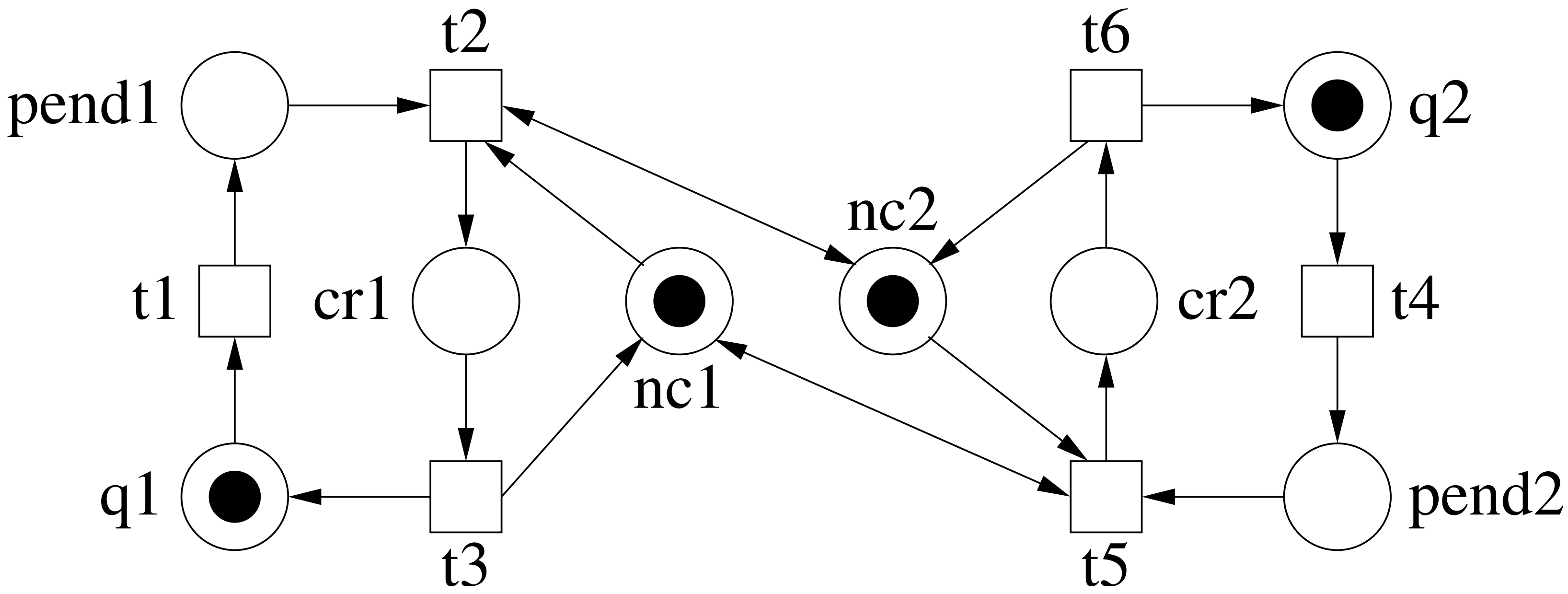
In other words, each transition which removes tokens from a trap must also put at least one token back to the trap.

A trap S is called marked in marking M iff for at least one place $s \in S$ it holds that $M(s) \geq 1$.

Note: If a trap S is marked initially (i.e. in M_0), then it is also marked in all reachable markings.

Example 4

Consider the following attempt at a mutual exlusion algorithm for cr_1 and cr_2:



The idea is to achieve mutual exclusion by entering the critical section only if the other process is not already there.

In Example 4, $S_1 = \{nc_1, nc_2\}$ is a trap.

The only transitions that remove tokens from this set are t_2 and t_5. However, both also add new tokens to S_1.

S_1 is marked initially, and therefore in all reachable markings M the following inequality holds: $M(nc_1) + M(nc_2) \geq 1$

Traps can be useful in combination with place invariants to recapture information lost in the incidence matrix due to the cancellation of self-loop arcs.

Proving mutual exclusion properties using traps

In Example 4, we want to prove that in all reachable markings M, cr_1 and cr_2 cannot be marked at the same time. This can be expressed by the following inequality:

$$M(cr_1) + M(cr_2) \leq 1$$

The P-invariants we can derive in the net yield these equalities:

$$\begin{aligned}
M(q_1) + M(pend_1) + M(cr_1) &= 1 && (2)\\
M(q_2) + M(pend_2) + M(cr_2) &= 1 && (3)\\
M(cr_1) + M(nc_1) &= 1 && (4)\\
M(cr_2) + M(nc_2) &= 1 && (5)
\end{aligned}$$

However, these equalities are insufficient to prove the desired property!

Recall that $S_1 = \{nc_1, nc_2\}$ is a trap.

S_1 is marked initially and therefore in all reachable markings M. Thus:

$$M(nc_1) + M(nc_2) \geq 1 \tag{6}$$

Now, adding (4) and (5) and subtracting (6) yields $M(cr_1) + M(cr_2) \leq 1$, which proves the mutual exclusion property.

Unfoldings

Unfoldings

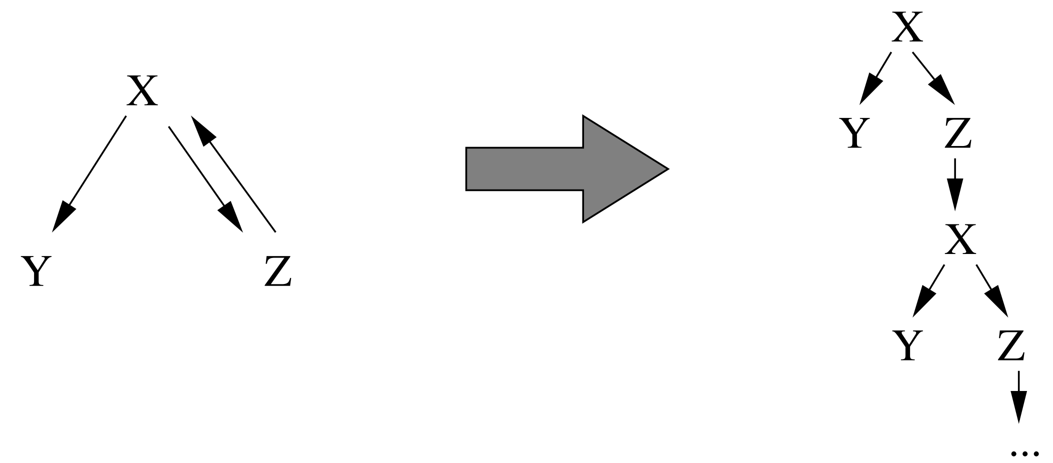
Unfoldings are a data structure that represents the behaviour of a Petri net.

We will study it for 1-safe nets.

Unfoldings represent a trade-off in terms of time/space requirements; their size is in between that of a net and its reachability graph, and checking whether a marking is reachable becomes easier than for the net, but more difficult than from the reachability graph.

Unfoldings for finite transition systems

Let $\mathcal{T}$ be a finite transition system with initial state X. One can define the acyclic unfolding $\mathcal{U}_{\mathcal{T}}$ (which is used for CTL model checking):

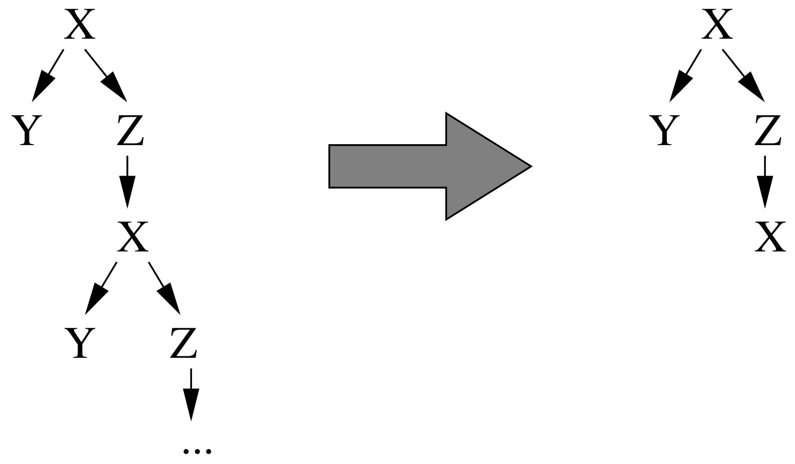


Remark: $\mathcal{U}_{\mathcal{T}}$ can be viewed as a structure in which every state is labelled by a state from $\mathcal{T}$. We denote this labelling by the function B.

$\mathcal{U}_{\mathcal{T}}$ contains the same behaviours as $\mathcal{T}$ (and the same reachable states). Additionally, $\mathcal{U}_{\mathcal{T}}$ has a simpler structure (acyclic, in fact, a tree). However, in general, $\mathcal{U}_{\mathcal{T}}$ is *infinite*.

Prefixes

$\mathcal{P}$ is called a prefix of $\mathcal{U}_\mathcal{T}$ if $\mathcal{P}$ is obtained by "pruning" arbitrary branches of $\mathcal{U}_\mathcal{T}$.
Example:



Observation: One can always find a *finite* prefix containing the same reachable states as the infinite unfolding (by unrolling loops exactly once). We shall call such a prefix complete.

Unfoldings of Petri nets

The unfolding of a Petri net $\mathcal{P}$ (or, a prefix of the same) is an infinite *acyclic* Petri net $\mathcal{U}$. We shall be interested in computing a finite prefix $\mathcal{Q}$ of $\mathcal{U}$.

Remark: In the following, we call the places of $\mathcal{Q}$ conditions, the transitions of $\mathcal{Q}$ events. This merely serves to better distinguish the elements of $\mathcal{P}$ and $\mathcal{Q}$, functionally they are the same!

Every condition of $\mathcal{Q}$ is labelled by a place of $\mathcal{P}$,
every event of $\mathcal{Q}$ by a transition of $\mathcal{P}$.

Every event e is of the form (S, t), where S is the preset of e and t the label of e.

Let S be a set of conditions. $B(S)$ denotes the set of places labelling the elements of S.

Every condition has exactly one incoming arc.

Unfolding construction for Petri nets

We first discuss the construction of $\mathcal{U}$ (possibly infinite).

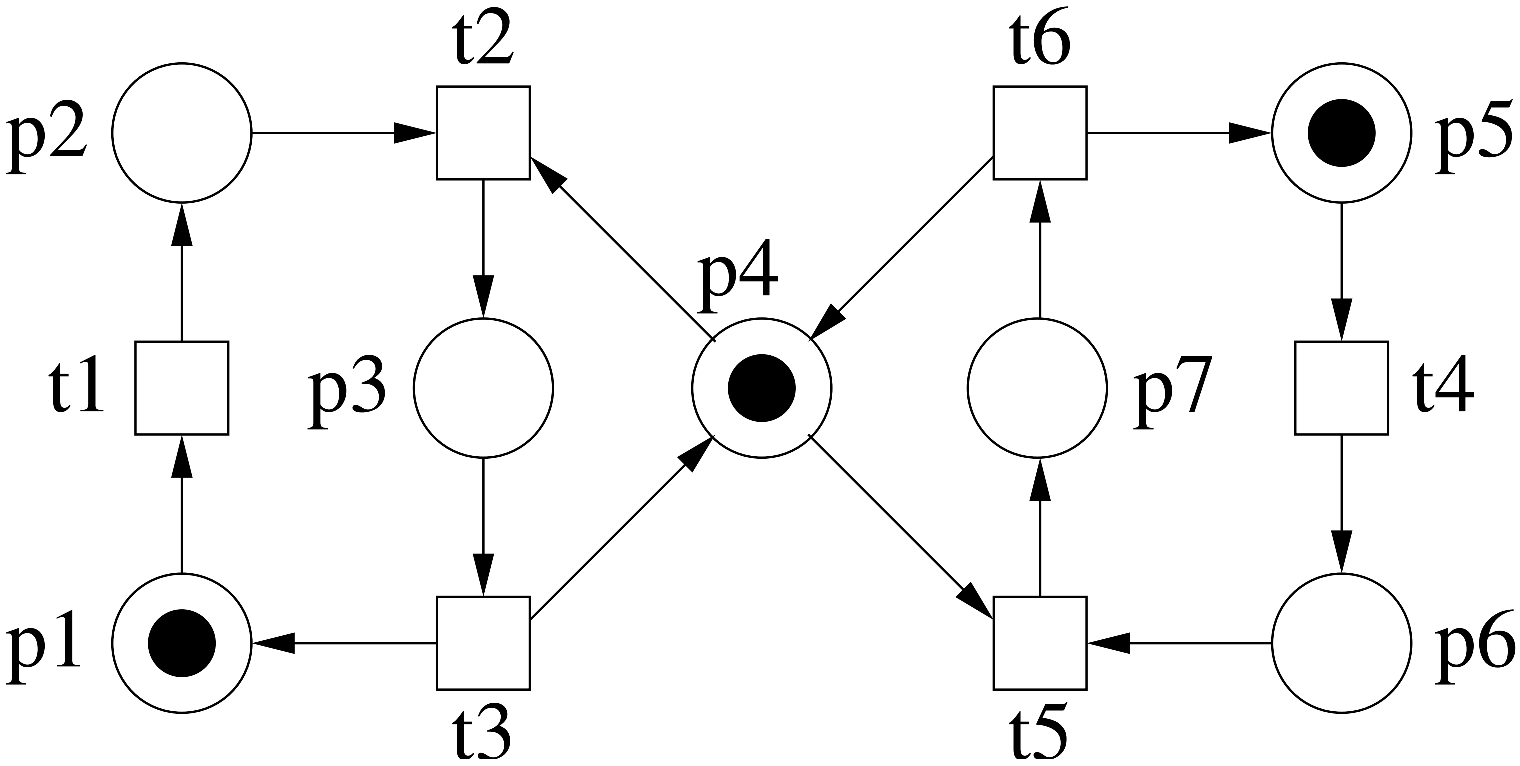
1. Let m_0 be the initial marking of $\mathcal{P}$. Then the initial marking of $\mathcal{U}$ contains exactly one condition for each place in m_0.

2. Let S the subset of a reachable marking in $\mathcal{U}$. Let $B(S) = {}^\bullet t$ for some transition t of $\mathcal{P}$ such that (S, t) is not yet contained in $\mathcal{U}$.

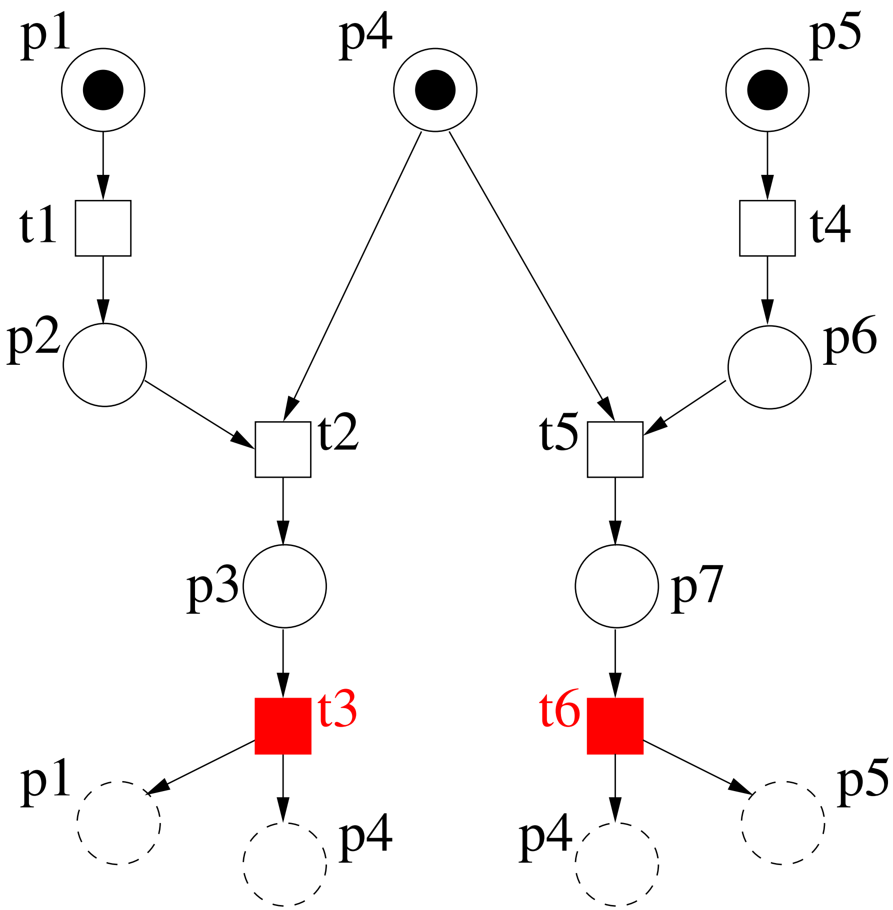
2a. If no such pair (S, t) exists, we are done.

2b. Add the event $e := (S, t)$ to the prefix (with S as preset and label t). Moreover, extend the prefix by one condition for every output place of t and make it an output place of e.

Example 1: Petri net. . .



. . . and a possible prefix of the unfolding



We shall now amend the construction so that it prunes certain branches of the unfolding, creating a *finite* prefix.

More precisely, certain events will be called cutoffs. These events lead to markings that we have already seen.

Prefix construction for Petri nets

1. Let m_0 be the initial marking of $\mathcal{P}$. Then the initial marking of $\mathcal{Q}$ contains exactly one condition for each place in m_0. We set $\mathcal{E} := \{m_0\}$.

2. Let S the subset of a reachable marking in $\mathcal{Q}$. where no element of S is the output place of a cutoff event. Let $B(S) = {}^\bullet t$ for some transition t of $\mathcal{P}$ such that (S, t) is not yet contained in $\mathcal{Q}$.

2a. If no such pair (S, t) exists, we are done.

2b. Add the event $e := (S, t)$ to the prefix (with S as preset and label t). Moreover, extend the prefix by one condition for every output place of t and make it an output place of e.

2c. We associate with e a marking m_e (which is reachable in $\mathcal{P}$) (see below). If $m_e \in \mathcal{E}$, then e is a cutoff. Otherwise $\mathcal{E} := \mathcal{E} \cup \{m_e\}$.

Determining m_e

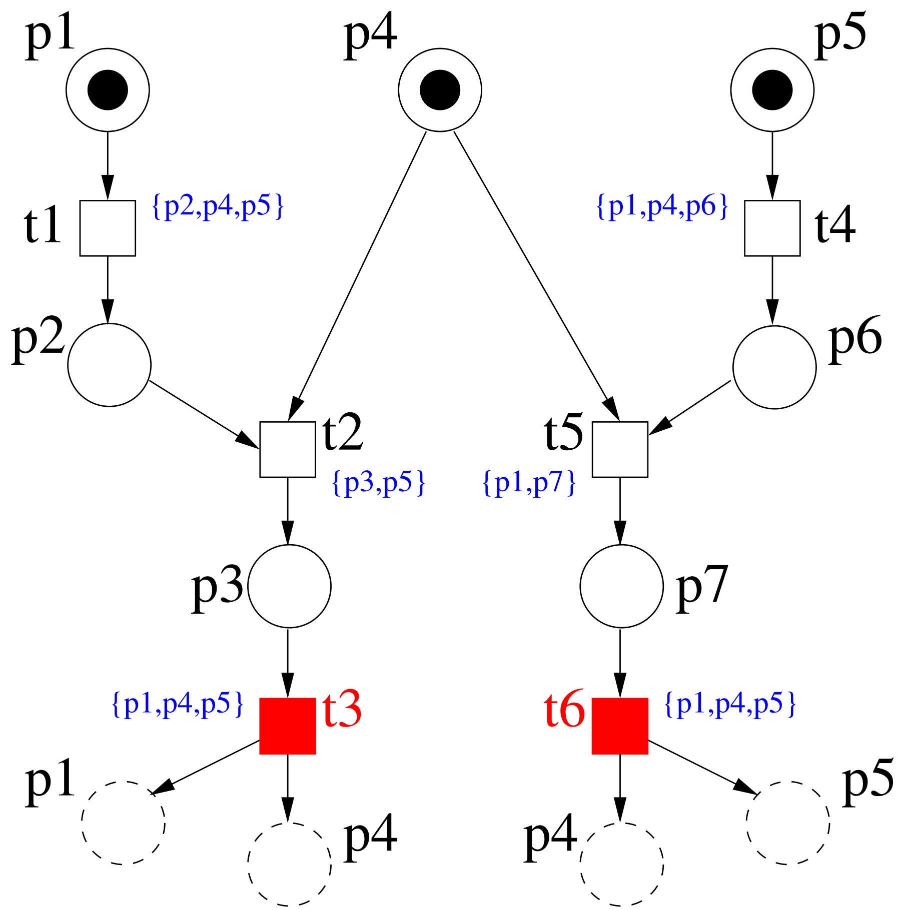
For the event $e = (S, t)$, we determine m_e, a marking of $\mathcal{P}$, as follows:

Idea: m_e is the label of the marking obtained by making the “minimal” effort to fire e.

Let x, y be two nodes (conditions *or* events) in $\mathcal{Q}$. Let $<$ be the smallest partial order where $x < y$ if there is an edge from x to y.

Let x be a node of $\mathcal{Q}$. We define $\lfloor x \rfloor := \{ y \mid y \leq x \}$.

Let m_e be the labels of the marking obtained by firing the events of $\lfloor e \rfloor$ (in any order). Note: Such a firing sequence exists due to the properties of S.


p1
p4
p5
t1
{p2,p4,p5}
{p1,p4,p6}
t4
p2
p6
t2
t5
{p3,p5}
{p1,p7}
p3
p7
{p1,p4,p5}
t3
t6
{p1,p4,p5}
p1
p5
p4
p4

Complete prefixes

Let $\mathcal{P}$ be a Petri net and $\mathcal{Q}$ a prefix of its unfolding $\mathcal{U}$ with labelling B. We call $\mathcal{Q}$ complete if it satisfies the following property:

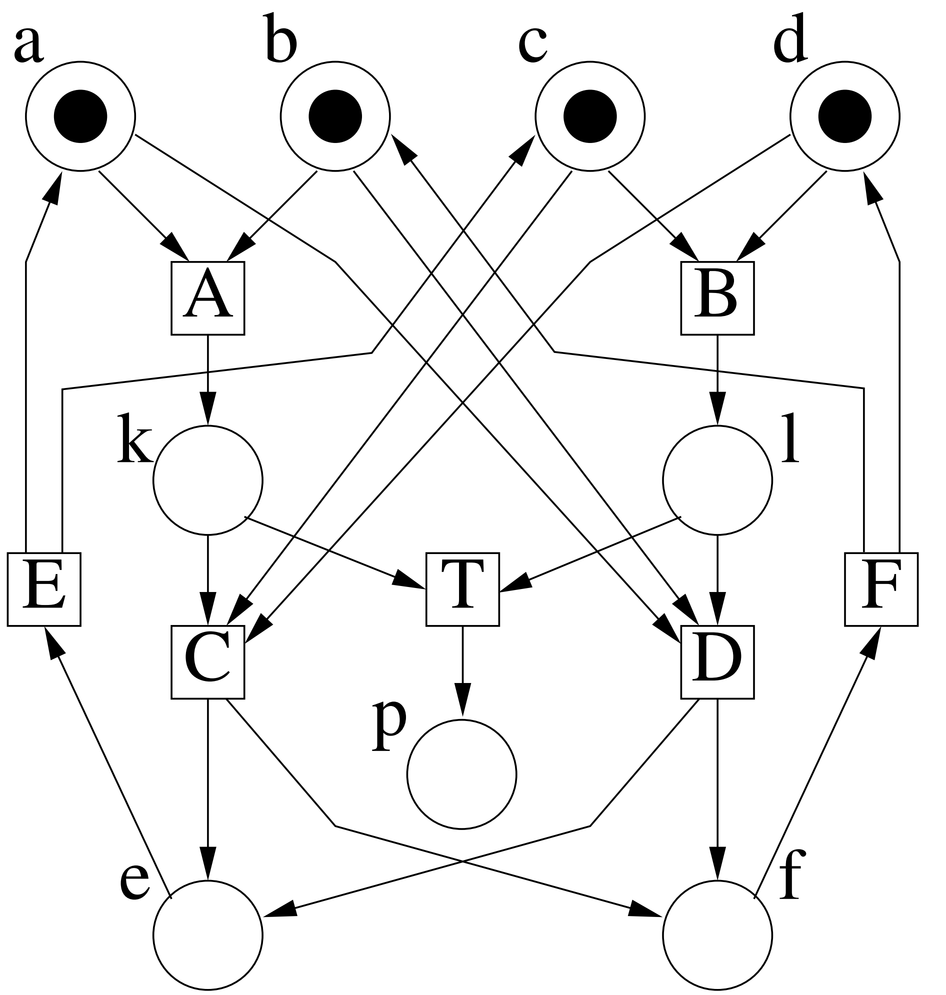
> A marking m is reachable in $\mathcal{P}$ *iff* a marking m' with $B(m') = m$ is reachable in $\mathcal{Q}$.

Thus, if $\mathcal{Q}$ is complete, we can decide reachability in $\mathcal{P}$ by examining $\mathcal{Q}$.

Unfortunately, the algorithm given previously does not always produce a complete prefix. Indeed, its shape depends on the order in which events are added. We shall discuss an example that demonstrates this effect.

Example 2

Consider the following Petri net:

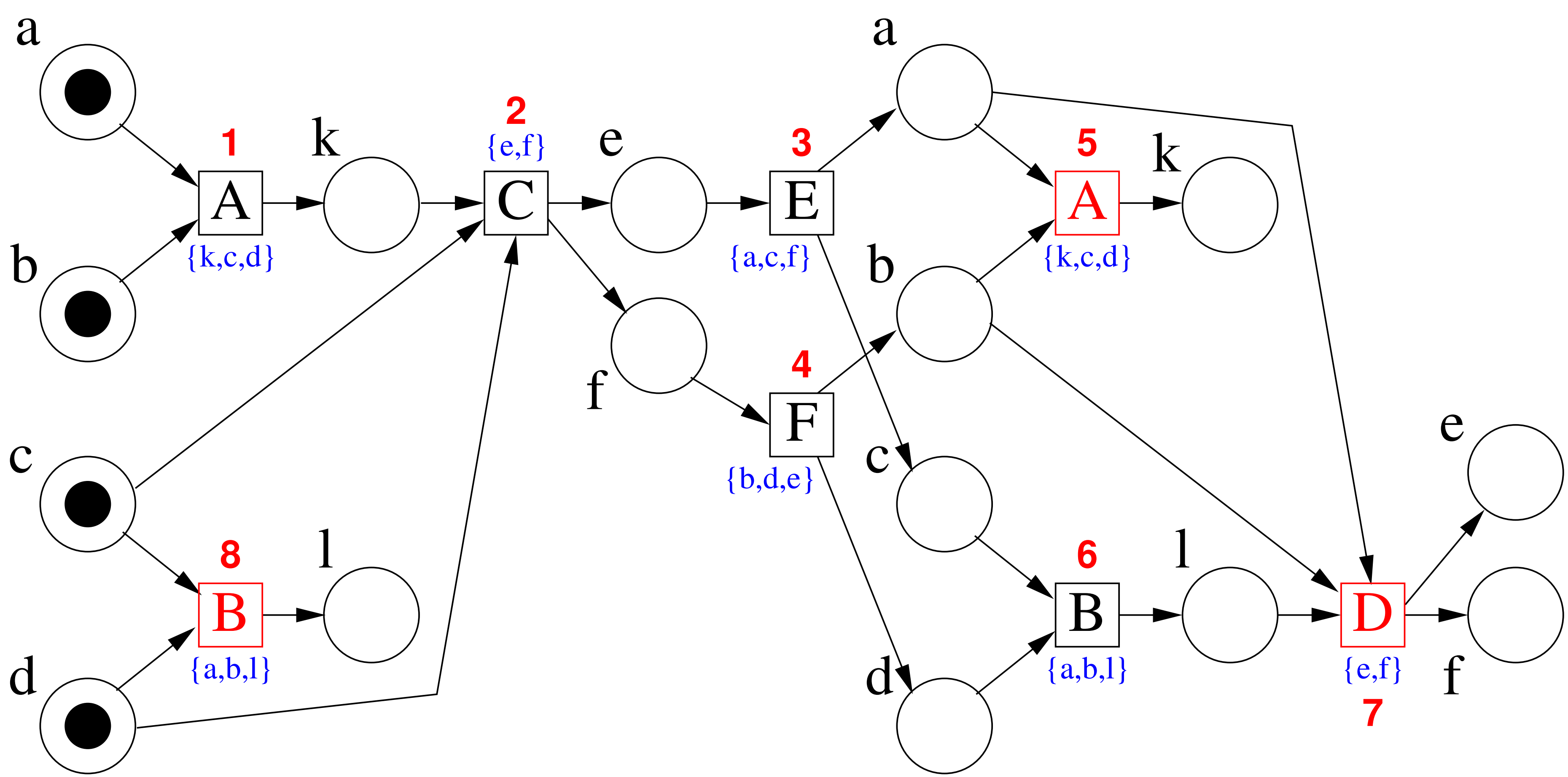


In Example 2 the marking $\{p\}$ is reachable, e.g. by firing $A\,B\,T$.

The net can also reach the marking $\{e, f\}$ by firing either $A\,C$ or $B\,D$, and then return by firing $E\,F$ to the initial marking.

We shall see that a prefix generated according to depth-first order will “overlook” the transition T.

Depth-first order generated the prefix shown below (order and cutoffs indicated in red):



Configurations

Let m be a reachable marking in $\mathcal{U}$ and let C be the set of events in $\bigcup_{c \in m} \lfloor c \rfloor$. Then we call C a configuration.

A configuration C represents a set of events that can all fire in one execution. Given C, we denote the marking (of $\mathcal{U}$) reached by such an execution by m_C.

Remark: For every event e, the set $\lfloor e \rfloor =: C_e$ is a configuration. We have $B(m_{C_e}) = m_e$.

We call E an extension of C iff $C \cap E = \emptyset$ and $C \cup E$ is a configuration. In this case, we write $C \oplus E$ to denote the configuration $C \cup E$.

Let C, C' be two configurations such that $B(m_C) = B(m_{C'})$. If E is an extension of C, then there is an extension E' of C' that is isomorphic to E.

Adequate orders

Let $\prec$ be a well-founded total order on configurations that refines $\subset$ (i.e. $C \subset C'$ implies $C \prec C'$).

Intuition: $\prec$ is a possible order in which the events of $\mathcal{U}$ can be generated; i.e., e is added before e' if $C_e \prec C_{e'}$.

Let $\mathcal{Q}_{\prec}$ be the prefix of $\mathcal{U}$ generated by adding events in the order given by $\prec$ as above.

We call $\prec$ adequate iff $\mathcal{Q}_{\prec}$ is complete.

A sufficient condition for adequate orders

The following condition guarantees that $\prec$ is adequate:

> Let C, C' be two configurations with $C \prec C'$ and $B(m_C) = B(m_{C'})$, and let E an extension of C and E' the extension of C' isomorphic to E. Then $D \prec D'$ must hold, where $D = C \oplus E$ and $D' = C' \oplus E'$.

Proof: Let $\prec$ be an order satisfying the above constraint. We show that $\mathcal{Q}_\prec$ is complete. So let m be a marking reachable in $\mathcal{P}$. Then there is a marking m' in $\mathcal{U}$ with $B(m') = m$. Let C' be the configuration containing the events in $\bigcup_{c \in m'} \lfloor c \rfloor$. Either C' is contained in $\mathcal{Q}_\prec$, or $C' = C_{e'} \oplus E'$ for some cutoff event e'. But then there is another event e with $m_e = m_{e'}$ and $C_e \prec C_{e'}$ and therefore a configuration $C := C_e \oplus E$, where E is isomorphic to E', and we have $B(m_C) = B(m_{C'}) = m$. Now, since $C_e \prec C_{e'}$ we have $C \prec C'$. Either C is contained in $\mathcal{Q}_\prec$, or one repeats the argument, but only finitely often since $\prec$ is well-founded.

Conflict, causality, concurrency

From the structure of the unfolding we can derive statements about the mutual relationships of conditions:

Let c, d be two (different) conditions of $\mathcal{Q}$.

c, d are called causally dependent if $c < d$ or $d < c$. (I.e., in every firing sequence containing both conditions, one condition must be consumed to generate the other.)

c, d are in conflict if there are events e, f (where $e \neq f$), $e \in \lfloor c \rfloor$, $f \in \lfloor d \rfloor$, and ${}^\bullet e \cap {}^\bullet f \neq \emptyset$. (I.e., c, d can *never* occur in a reachable marking of $\mathcal{Q}$!)

c, d are called concurrent if they are neither causally dependent nor in conflict with one another

Concurrent conditions are jointly reachable

Let $\mathcal{C}$ be a set of conditions. Then $\mathcal{C}$ is a subset of a reachable marking in $\mathcal{U}$ iff all conditions in $\mathcal{C}$ are mutually concurrent.

Proof ("$\Rightarrow$"): Obvious.

Proof ("$\Leftarrow$"): (sketch) Let $\mathcal{E}$ be the set of events in $\bigcup_{c \in \mathcal{C}} \lfloor c \rfloor$. Induction on the size of $\mathcal{E}$: obvious for $E = \emptyset$, otherwise remove a maximal event e from $\mathcal{E}$ and prove that $(\mathcal{C} \setminus e^\bullet) \cup {}^\bullet e$ is mutually concurrent.

Reachability checking using complete prefixes

Theorem: Let $\mathcal{P}$ be a Petri net and $\mathcal{Q}$ a complete unfolding prefix. Given $\mathcal{Q}$ and a marking m of $\mathcal{P}$, it is NP-complete to determine whether m is reachable in $\mathcal{P}$.

Proof: Membership in NP: guess a marking m' of $\mathcal{Q}$ such that $B(m') = m$, check if it does not contain causally dependent or conflicting conditions.

Hardness: polynomial reduction from SAT

Reducing reachability to SAT

In the other direction, we can, given m and $\mathcal{Q}$, produce a propositional logic formula, of polynomial size in $|m| + |\mathcal{Q}|$, that is satisfiable iff m is reachable in $\mathcal{P}$.

The formula uses one boolean variable for each event and each condition. Its satisfying assignments are those that correspond to a reachable marking m' (i.e. concurrent sets of conditions) in $\mathcal{Q}$.

The formula assigns “true” to the conditions and events in $\bigcup_{c \in m'} \lfloor c \rfloor$ and false to all others; then it checks that no condition in m' is consumed by one of the events in that set and that no condition is consumed twice.

Finally, one demands that the image of m' in $\mathcal{P}$ is m.

Remarks

Remark (1): Notice that the unfolding (and most of the formula) is independent of m and needs to be generated from $\mathcal{P}$ only once for any number of reachability queries.

Remark (2): In a very similar way, one can check whether $\mathcal{P}$ contains a deadlock, i.e. a reachable marking that does not enable any transition.