Tree Automata and Applications

M1 course, 2023/2024

Organization

Timetable

- Exercises: Thursday 8:30 – 10:30 (Luc Lapointe)
- Course: Thursday 10:45 – 12:45 (Stefan Schwoon)

Organization

Timetable

- Exercises: Thursday 8:30 – 10:30 (Luc Lapointe)
- Course: Thursday 10:45 – 12:45 (Stefan Schwoon)

Exams

- DM or CC (*to be specified by Luc*)
- Final Exam: 2h, 11 January
- First session: DM/CC + Exam (50/50)
- Second session: DM/CC + Repeat Exam (50/50)

Organization

Timetable

- Exercises: Thursday 8:30 – 10:30 (Luc Lapointe)
- Course: Thursday 10:45 – 12:45 (Stefan Schwoon)

Exams

- DM or CC (*to be specified by Luc*)
- Final Exam: 2h, 11 January
- First session: DM/CC + Exam (50/50)
- Second session: DM/CC + Repeat Exam (50/50)

Course materials

- Website: lecturer's homepage + Wiki MPRI, course 1-18
  (exercise sheets, slides, former exams)
- Hubert Comon et al.
  Tree Automata Techniques and Applications.
  `http://tata.gforge.inria.fr/`

Motivations

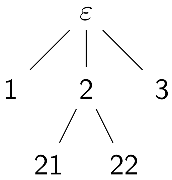
1. Natural extension of formal-language notions (automata, logic, ...)
2. Treatment of tree-like data structures: parse tree, XML documents (XPath, CSS selectors)
3. Applications e.g. in compiler construction, formal verification

Trees

We consider *finite* *ordered* *ranked* trees.

- *ordered* : internal nodes have children $1 \ldots n$
- *ranked* : number of children fixed by node's label

Let N denote the set of positive integers.
Nodes (*positions*) of a tree are associated with elements of N^*:



Definition: Tree

A (finite, ordered) *tree* is a non-empty, finite, prefix-closed set $Pos \subseteq N^*$ such that $w(i+1) \in Pos$ implies $wi \in Pos$ for all $w \in N^*$, $i \in N$.

Ranked Trees

Ranked symbols

Let $\mathcal{F}_0, \mathcal{F}_1, \ldots$ be disjoint sets of symbols of *arity* $0, 1, \ldots$
We note $\mathcal{F} := \bigcup_i \mathcal{F}_i$.

- Notation (example): $\mathcal{F} = \{f(2), g(1), a, b\}$

Let $\mathcal{X}$ denote a set of variables (disjoint from the other symbols).

Ranked Trees

Ranked symbols

Let $\mathcal{F}_0, \mathcal{F}_1, \ldots$ be disjoint sets of symbols of *arity* $0, 1, \ldots$
We note $\mathcal{F} := \bigcup_i \mathcal{F}_i$.

- Notation (example): $\mathcal{F} = \{f(2), g(1), a, b\}$

Let $\mathcal{X}$ denote a set of variables (disjoint from the other symbols).

Definition: Ranked tree

A ranked tree is a mapping $t : Pos \to (\mathcal{F} \cup \mathcal{X})$ satisfying:

- Pos is a tree;
- for all $p \in Pos$, if $t(p) \in \mathcal{F}_n$, $n \geq 1$ then $Pos \cap pN = \{p1, \ldots, pn\}$;
- for all $p \in Pos$, if $t(p) \in \mathcal{X} \cup \mathcal{F}_0$ then $Pos \cap pN = \emptyset$.

Trees and Terms

Definition: Terms

The set of *terms* $T(\mathcal{F}, \mathcal{X})$ is the smallest set satisfying:

- $\mathcal{X} \cup \mathcal{F}_0 \subseteq T(\mathcal{F}, \mathcal{X})$;
- if $t_1, \ldots, t_n \in T(\mathcal{F}, \mathcal{X})$ and $f \in \mathcal{F}_n$, then $f(t_1, \ldots, t_n) \in T(\mathcal{F}, \mathcal{X})$.

We note $T(\mathcal{F}) := T(\mathcal{F}, \emptyset)$. A term in $T(\mathcal{F})$ is called *ground term*.
A term of $T(\mathcal{F}, \mathcal{X})$ is *linear* if every variable occurs at most once.

Trees and Terms

Definition: Terms

The set of *terms* $T(\mathcal{F}, \mathcal{X})$ is the smallest set satisfying:

- $\mathcal{X} \cup \mathcal{F}_0 \subseteq T(\mathcal{F}, \mathcal{X})$;
- if $t_1, \ldots, t_n \in T(\mathcal{F}, \mathcal{X})$ and $f \in \mathcal{F}_n$, then $f(t_1, \ldots, t_n) \in T(\mathcal{F}, \mathcal{X})$.

We note $T(\mathcal{F}) := T(\mathcal{F}, \emptyset)$. A term in $T(\mathcal{F})$ is called *ground term*.
A term of $T(\mathcal{F}, \mathcal{X})$ is *linear* if every variable occurs at most once.

Example: $\mathcal{F} = \{f(2), g(1), a, b\}$, $\mathcal{X} = \{x, y\}$

- $f(g(a), b) \in T(\mathcal{F})$;
- $f(x, f(b, y)) \in T(\mathcal{F}, \mathcal{X})$ is linear;
- $f(x, x) \in T(\mathcal{F}, \mathcal{X})$ is non-linear.

Trees and Terms

Definition: Terms

The set of *terms* $T(\mathcal{F},\mathcal{X})$ is the smallest set satisfying:

- $\mathcal{X} \cup \mathcal{F}_0 \subseteq T(\mathcal{F},\mathcal{X})$;
- if $t_1,\ldots,t_n \in T(\mathcal{F},\mathcal{X})$ and $f \in \mathcal{F}_n$, then $f(t_1,\ldots,t_n) \in T(\mathcal{F},\mathcal{X})$.

We note $T(\mathcal{F}) := T(\mathcal{F},\emptyset)$. A term in $T(\mathcal{F})$ is called *ground term*.
A term of $T(\mathcal{F},\mathcal{X})$ is *linear* if every variable occurs at most once.

Example: $\mathcal{F} = \{f(2), g(1), a, b\}$, $\mathcal{X} = \{x, y\}$

- $f(g(a), b) \in T(\mathcal{F})$;
- $f(x, f(b, y)) \in T(\mathcal{F},\mathcal{X})$ is linear;
- $f(x, x) \in T(\mathcal{F},\mathcal{X})$ is non-linear.

We confuse terms and trees in the obvious manner.

Height and size

Definition

Let $t \in T(\mathcal{F}, \mathcal{X})$. We note $\mathcal{H}(t)$ the *height* of t and $|t|$ the *size* of t.

- if $t \in \mathcal{X}$, then $\mathcal{H}(t) := 0$ and $|t| := 0$; (for notational convenience)
- if $t \in \mathcal{F}_0$, then $\mathcal{H}(t) := 1$ and $|t| := 1$;
- if $t = f(t_1, \ldots, t_n)$, then $\mathcal{H}(t) := 1 + \max\{\mathcal{H}(t_1), \ldots, \mathcal{H}(t_n)\}$ and $|t| := 1 + |t_1| + \cdots + |t_n|$.

Subterms / subtrees

Definition: Subtree

Let $t, u \in T(\mathcal{F}, \mathcal{X})$ and p a position. Then $t|_p : Pos_p \to T(\mathcal{F}, \mathcal{X})$ is the ranked tree defined by

- $Pos_p := \{ q \mid pq \in Pos \}$;
- $t|_p(q) := t(pq)$.

Moreover, $t[u]_p$ is the tree obtained by replacing $t|_p$ by u in t.

$t \trianglerighteq t'$ (resp. $t \triangleright t'$) denotes that t' is a (proper) subtree of t.

Substitutions and Context

Definition: Substitution

- (Ground) substitution σ: mapping from $\mathcal{X}$ to $T(\mathcal{F}, \mathcal{X})$ resp. $T(\mathcal{F})$
- Notation: $\sigma := \{x_1 \leftarrow t_1, \ldots, x_n \leftarrow t_n\}$, with $\sigma(x) := x$ for all $x \in \mathcal{X} \setminus \{x_1, \ldots, x_n\}$
- Extension to terms: for all $f \in \mathcal{F}_m$ and $t'_1, \ldots, t'_m \in T(\mathcal{F}, \mathcal{X})$
$$\sigma(f(t'_1, \ldots, t'_m)) = f(\sigma(t'_1), \ldots, \sigma(t'_m))$$
- Notation: $t\sigma$ for $\sigma(t)$

Substitutions and Context

Definition: Substitution

- (Ground) substitution σ: mapping from $\mathcal{X}$ to $T(\mathcal{F},\mathcal{X})$ resp. $T(\mathcal{F})$
- Notation: $\sigma := \{x_1 \leftarrow t_1, \dots, x_n \leftarrow t_n\}$, with $\sigma(x) := x$ for all $x \in \mathcal{X} \setminus \{x_1, \dots, x_n\}$
- Extension to terms: for all $f \in \mathcal{F}_m$ and $t'_1, \dots, t'_m \in T(\mathcal{F},\mathcal{X})$

$$\sigma(f(t'_1, \dots, t'_m)) = f(\sigma(t'_1), \dots, \sigma(t'_m))$$

- Notation: $t\sigma$ for $\sigma(t)$

Definition: Context

A *context* is a linear term $C \in T(\mathcal{F},\mathcal{X})$ with variables $x_1, \dots, x_n$.
We note $C[t_1, \dots, t_n] := C\{x_1 \leftarrow t_1, \dots, x_n \leftarrow t_n\}$.

$\mathcal{C}^n(\mathcal{F})$ denotes the contexts with n variables and $\mathcal{C}(\mathcal{F}) := \mathcal{C}^1(\mathcal{F})$.
Let $C \in \mathcal{C}(\mathcal{F})$. We note $C^0 := x_1$ and $C^{n+1} = C^n[C]$ for $n \geq 0$.

Tree automata

Basic idea: Extension of finite automata from words to trees
Direct extension of automata theory when words seen as unary terms:

$$abc \mathrel{\widehat{=}} a(b(c(\$)))$$

Finite automaton: labels every prefix of a word with a state.
Tree automaton: labels every position/subtree of a tree with a state.

Two variants: bottom-up vs top-down labelling

Tree automata

Basic idea: Extension of finite automata from words to trees

Direct extension of automata theory when words seen as unary terms:

$$abc \mathrel{\widehat{=}} a(b(c(\$)))$$

Finite automaton: labels every prefix of a word with a state.

Tree automaton: labels every position/subtree of a tree with a state.

Two variants: bottom-up vs top-down labelling

Basic results (preview)

- Non-deterministic bottom-up and top-down are equally powerful
- Deterministic bottom-up equally powerful
- Deterministic top-down less powerful

Bottom-up automata

Definition: (Bottom-up tree automata)

A *(finite bottom-up) tree automaton* (NFTA) is a tuple $\mathcal{A} = \langle Q, \mathcal{F}, G, \Delta \rangle$, where:

- Q is a finite set of *states*;
- $\mathcal{F}$ a finite ranked alphabet;
- $G \subseteq Q$ are the *final states*;
- Δ is a finite set of rules of the form

$$f(q_1, \ldots, q_n) \to q$$

for $f \in \mathcal{F}_n$ and $q, q_1, \ldots, q_n \in Q$.

Bottom-up automata

Definition: (Bottom-up tree automata)

A *(finite bottom-up) tree automaton* (NFTA) is a tuple $\mathcal{A} = \langle Q, \mathcal{F}, G, \Delta \rangle$, where:

- Q is a finite set of *states*;
- $\mathcal{F}$ a finite ranked alphabet;
- $G \subseteq Q$ are the *final states*;
- Δ is a finite set of rules of the form

$$f(q_1, \ldots, q_n) \to q$$

  for $f \in \mathcal{F}_n$ and $q, q_1, \ldots, q_n \in Q$.

Example: $Q := \{q_0, q_1, q_f\}$, $\mathcal{F} = \{f(2), g(1), a\}$, $G := \{q_f\}$, and rules

$$a \to q_0 \quad g(q_0) \to q_1 \quad g(q_1) \to q_1 \quad f(q_1, q_1) \to q_f$$

Move relation and computation tree

Move relation

Let $t, t' \in T(\mathcal{F}, Q)$. We write $t \to_{\mathcal{A}} t'$ if the following are satisfied:

- $t = C[f(q_1, \ldots, q_n)]$ for some context C;
- $t' = C[q]$ for some rule $f(q_1, \ldots, q_n) \to q$ of $\mathcal{A}$.

Idea: successively reduce t to a single state, starting from the leaves.

As usual, we write $\to^*_{\mathcal{A}}$ for the transitive and reflexive closure of $\to_{\mathcal{A}}$.

Move relation and computation tree

Move relation

Let $t, t' \in T(\mathcal{F}, Q)$. We write $t \to_{\mathcal{A}} t'$ if the following are satisfied:

- $t = C[f(q_1, \ldots, q_n)]$ for some context C;
- $t' = C[q]$ for some rule $f(q_1, \ldots, q_n) \to q$ of $\mathcal{A}$.

Idea: successively reduce t to a single state, starting from the leaves.
As usual, we write $\to_{\mathcal{A}}^{*}$ for the transitive and reflexive closure of $\to_{\mathcal{A}}$.

Computation

Let $t : Pos \to \mathcal{F}$ a ground tree. A *run* or *computation* of $\mathcal{A}$ on t is a labelling $t' : Pos \to Q$ compatible with Δ, i.e.:

- for all $p \in Pos$, if $t(p) = f \in \mathcal{F}_n$, $t'(p) = q$, and $t'(pj) = q_j$ for all $pj \in Pos \cap pN$, then $f(q_1, \ldots, q_n) \to q \in \Delta$

Regular tree languages

A tree t is *accepted* by $\mathcal{A}$ iff $t \rightarrow^*_{\mathcal{A}} q$ for some $q \in G$.

$\mathcal{L}(\mathcal{A})$ denotes the set of trees accepted by $\mathcal{A}$.

L is *regular/recognizable* iff $L := \mathcal{L}(\mathcal{A})$ for some NFTA $\mathcal{A}$.

Two NFTAs $\mathcal{A}_1$ and $\mathcal{A}_2$ are *equivalent* iff $\mathcal{L}(\mathcal{A}_1) = \mathcal{L}(\mathcal{A}_2)$.

NFTA with ε-moves

Definition:

An ε-NFTA is an NFTA $\mathcal{A} = \langle Q, \mathcal{F}, G, \Delta \rangle$, where Δ can additionally contain rules of the form $q \to q'$, with $q, q' \in Q$.

Semantics: Allow to re-label a position from q to q'.

Equivalence of ε-NFTA

For every ε-NFTA $\mathcal{A}$ there exists an equivalent NFTA $\mathcal{A}'$.

Proof (sketch): Construct the rules of $\mathcal{A}'$ by a saturation procedure.

Deterministic, complete, and reduced NFTA

An NFTA is *deterministic* if no two rules have the same left-hand side.
An NFTA is *complete* if for every $f \in \mathcal{F}_n$ and $q_1, \ldots, q_n \in Q$, there exists at least one rule $f(q_1, \ldots, q_n) \to q \in \Delta$.

As usual, a DFTA has *at most* one run per tree.
A DCFTA as *exactly* one run per tree.

A state q of $\mathcal{A}$ is *accessible* if there exists a tree t s.t. $t \to^*_{\mathcal{A}} q$.
$\mathcal{A}$ is said to be *reduced* if all its states are accessible.

A pumping lemma for tree languages

Lemma

Let L be recognizable. Then there exists a constant k such that for all $t \in L$ with $\mathcal{H}(t) > k$ there exist contexts $C, D \in \mathcal{C}(\mathcal{F})$ and $u \in T(\mathcal{F})$ satisfying:

- D is non-trivial (i.e. not just a variable);
- $t = C[D[u]]$;
- for all $n \geq 0$, we have $C[D^n[u]] \in L$.

A pumping lemma for tree languages

Lemma

Let L be recognizable. Then there exists a constant k such that for all $t \in L$ with $\mathcal{H}(t) > k$ there exist contexts $C, D \in \mathcal{C}(\mathcal{F})$ and $u \in T(\mathcal{F})$ satisfying:

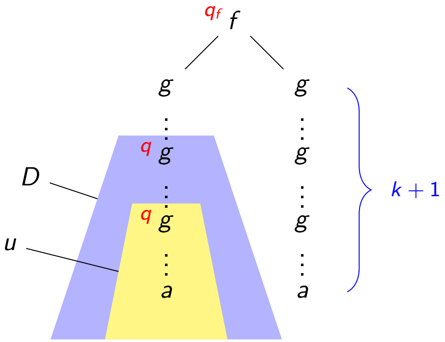
- D is non-trivial (i.e. not just a variable);
- $t = C[D[u]]$;
- for all $n \geq 0$, we have $C[D^n[u]] \in L$.

Proof: Let k be the number of states of an NFTA $\mathcal{A}$ recognizing L.
Then an accepting run for t has positions p, pp' ($p' \neq \varepsilon$) labelled with the same state q. Let $C := t[x]_p$, $D := t|_p[x]_{p'}$, and $u := t|_{pp'}$. We have $t = C[D[u]] \in L$, $D[u] \to_{\mathcal{A}}^* q$, and $u \to_{\mathcal{A}}^* q$, hence the accepting run of t implies $D[q] \to_{\mathcal{A}}^* q$ and $C[q] \to_{\mathcal{A}}^* q_f$, for some final q_f. Therefore, $C[u] \to_{\mathcal{A}}^* q_f$ and for any $n \geq 0$, (by induction)

$$C[D^{n+1}[u]] \to_{\mathcal{A}}^* C[D^n[D[q]]] \to_{\mathcal{A}}^* C[D^n[q]] \to_{\mathcal{A}}^* C[q] \to_{\mathcal{A}}^* q_f$$

Illustration of pumping lemma

Let $L = \{ f(g^i(a), g^i(a)) \mid i \geq 0 \}$ for $\mathcal{F} = \{f(2), g(1), a\}$.
Suppose (by contradiction) that L is recognizable by NFTA $\mathcal{A}$ with k states. Let $t = f(g^k(a), g^k(a))$.



Pumping D creates trees outside L $\Rightarrow$ L not recognizable.

Top-down tree automata

Definition

A *top-down tree automaton* (T-NFTA) is a tuple $\mathcal{A} = \langle Q, \mathcal{F}, I, \Delta \rangle$, where $Q, \mathcal{F}$ are as in NFTA, $I \subseteq Q$ is a set of *initial states*, and Δ contains rules of the form

$$q(f) \to (q_1, \ldots, q_n)$$

for $f \in \mathcal{F}_n$ and $q, q_1, \ldots, q_n \in Q$.

Top-down tree automata

Definition

A *top-down tree automaton* (T-NFTA) is a tuple $\mathcal{A} = \langle Q, \mathcal{F}, I, \Delta \rangle$, where $Q, \mathcal{F}$ are as in NFTA, $I \subseteq Q$ is a set of *initial states*, and Δ contains rules of the form

$$q(f) \to (q_1, \ldots, q_n)$$

for $f \in \mathcal{F}_n$ and $q, q_1, \ldots, q_n \in Q$.

Move relation: $t \to_{\mathcal{A}} t'$ iff

- $t = C[q(f(t_1, \ldots, t_n))]$ for some context C, $f \in \mathcal{F}_n$, and $t_1, \ldots, t_n \in T(\mathcal{F})$;
- $t' = C[f(q_1(t_1), \ldots, q_n(t_n))]$ for some rule $q(f) \to (q_1, \ldots, q_n)$.

t is accepted by $\mathcal{A}$ if $q(t) \to^*_{\mathcal{A}} t$ for some $q \in I$.

From top-down to bottom-up

Theorem (T-NFTA = NFTA)

L is recognizable by an NFTA iff it is recognizable by a T-NFTA.

Claim: L is accepted by NFTA $\mathcal{A} = \langle Q, \mathcal{F}, G, \Delta \rangle$ iff it is accepted by T-NFTA $\mathcal{A}' = \langle Q, \mathcal{F}, G, \Delta' \rangle$, with

$$\Delta' := \{ \, q(f) \to (q_1, \ldots, q_n) \mid f(q_1, \ldots, q_n) \to q \in \Delta \, \}$$

From top-down to bottom-up

Theorem (T-NFTA = NFTA)

L is recognizable by an NFTA iff it is recognizable by a T-NFTA.

Claim: L is accepted by NFTA $\mathcal{A} = \langle Q, \mathcal{F}, G, \Delta \rangle$ iff it is accepted by T-NFTA $\mathcal{A}' = \langle Q, \mathcal{F}, G, \Delta' \rangle$, with

$$\Delta' := \{\, q(f) \to (q_1, \ldots, q_n) \mid f(q_1, \ldots, q_n) \to q \in \Delta \,\}$$

Proof: Let $t \in T(\mathcal{F})$. We show $t \to^*_{\mathcal{A}} q$ iff $q(t) \to^*_{\mathcal{A}'} t$.

- Base: $t = a$ (for some $a \in \mathcal{F}_0$)

$$t = a \to^*_{\mathcal{A}} q \iff a \to_{\Delta} q \iff q(a) \to_{\Delta'} \varepsilon \iff q(a) \to^*_{\mathcal{A}'} a$$

From top-down to bottom-up

Theorem (T-NFTA = NFTA)

L is recognizable by an NFTA iff it is recognizable by a T-NFTA.

Claim: L is accepted by NFTA $\mathcal{A} = \langle Q, \mathcal{F}, G, \Delta \rangle$ iff it is accepted by T-NFTA $\mathcal{A}' = \langle Q, \mathcal{F}, G, \Delta' \rangle$, with

$$\Delta' := \{\, q(f) \to (q_1, \ldots, q_n) \mid f(q_1, \ldots, q_n) \to q \in \Delta \,\}$$

Proof: Let $t \in T(\mathcal{F})$. We show $t \to^*_{\mathcal{A}} q$ iff $q(t) \to^*_{\mathcal{A}'} t$.

- Base: $t = a$ (for some $a \in \mathcal{F}_0$)

$$t = a \to^*_{\mathcal{A}} q \iff a \to_{\Delta} q \iff q(a) \to_{\Delta'} \varepsilon \iff q(a) \to^*_{\mathcal{A}'} a$$

- Induction: $t = f(t_1, \ldots, t_n)$, hypothesis holds for $t_1, \ldots, t_n$

$$f(t_1, \ldots, t_n) \to^*_{\mathcal{A}} q \iff \exists q_1, \ldots q_n : f(q_1, \ldots, q_n) \to_{\Delta} q \wedge \forall i : t_i \to^*_{\mathcal{A}} q_i$$

$$\iff \exists q_1, \ldots, q_n : q(f) \to_{\Delta'} (q_1, \ldots, q_n) \wedge \forall i : q_i(t_i) \to^*_{\mathcal{A}'} t_i$$

$$\iff q(f(t_1, \ldots, t_n)) \to_{\mathcal{A}'} f(q_1(t_1), \ldots, q_n(t_n)) \to^*_{\mathcal{A}'} f(t_1, \ldots, t_n)$$

From NFTA to DFTA

Theorem (NFTA=DFTA)

If L is recognizable by an NFTA, then it is recognizable by a DFTA.

From NFTA to DFTA

Theorem (NFTA=DFTA)

If L is recognizable by an NFTA, then it is recognizable by a DFTA.

Claim (subset construct.): Let $\mathcal{A} = \langle Q, \mathcal{F}, G, \Delta \rangle$ an NFTA recognizing L. The following DCFTA $\mathcal{A}' = \langle 2^Q, \mathcal{F}, G', \Delta' \rangle$ also recognizes L:

- $G' = \{ S \subseteq Q \mid S \cap G \neq \emptyset \}$
- for every $f \in \mathcal{F}_n$ and $S_1, \ldots, S_n \subseteq Q$, let $f(S_1, \ldots, S_n) \to S \in \Delta'$, where $S = \{ q \in Q \mid \exists q_1 \in S_1, \ldots, q_n \in S_n : f(q_1, \ldots, q_n) \to q \in \Delta \}$

Proof: For $t \in T(\mathcal{F})$, show $t \to^*_{\mathcal{A}'} \{ q \mid t \to^*_{\mathcal{A}} q \}$, by structural induction.

From NFTA to DFTA

Theorem (NFTA=DFTA)

If L is recognizable by an NFTA, then it is recognizable by a DFTA.

Claim (subset construct.): Let $\mathcal{A} = \langle Q, \mathcal{F}, G, \Delta \rangle$ an NFTA recognizing L.
The following DCFTA $\mathcal{A}' = \langle 2^Q, \mathcal{F}, G', \Delta' \rangle$ also recognizes L:

- $G' = \{ S \subseteq Q \mid S \cap G \neq \emptyset \}$
- for every $f \in \mathcal{F}_n$ and $S_1, \ldots, S_n \subseteq Q$, let $f(S_1, \ldots, S_n) \to S \in \Delta'$, where $S = \{ q \in Q \mid \exists q_1 \in S_1, \ldots, q_n \in S_n : f(q_1, \ldots, q_n) \to q \in \Delta \}$

Proof: For $t \in T(\mathcal{F})$, show $t \to^*_{\mathcal{A}'} \{ q \mid t \to^*_{\mathcal{A}} q \}$, by structural induction.

DFTA with accessible states

In practice, the construction of $\mathcal{A}'$ can be restricted to accessible states:
Start with transitions $a \to S$, then saturate.

From NFTA to DFTA

Theorem (NFTA=DFTA)

If L is recognizable by an NFTA, then it is recognizable by a DFTA.

Claim (subset construct.): Let $\mathcal{A} = \langle Q, \mathcal{F}, G, \Delta \rangle$ an NFTA recognizing L.
The following DCFTA $\mathcal{A}' = \langle 2^Q, \mathcal{F}, G', \Delta' \rangle$ also recognizes L:

- $G' = \{ S \subseteq Q \mid S \cap G \neq \emptyset \}$
- for every $f \in \mathcal{F}_n$ and $S_1, \ldots, S_n \subseteq Q$, let $f(S_1, \ldots, S_n) \to S \in \Delta'$, where $S = \{ q \in Q \mid \exists q_1 \in S_1, \ldots, q_n \in S_n : f(q_1, \ldots, q_n) \to q \in \Delta \}$

Proof: For $t \in T(\mathcal{F})$, show $t \to^*_{\mathcal{A}'} \{ q \mid t \to^*_{\mathcal{A}} q \}$, by structural induction.

DFTA with accessible states

In practice, the construction of $\mathcal{A}'$ can be restricted to accessible states:
Start with transitions $a \to S$, then saturate.

Deterministic top-down are less powerful

E.g., $L = \{f(a, b), f(b, a)\}$ can be recognized by DFTA but not by T-DFTA.

Closure properties

Theorem (Boolean closure)

Recognizable tree languages are closed under Boolean operations.

Closure properties

Theorem (Boolean closure)

Recognizable tree languages are closed under Boolean operations.

Negation (invert accepting states)

Let $\langle Q, \mathcal{F}, G, \Delta \rangle$ be a DCFTA recognizing L.
Then $\langle Q, \mathcal{F}, Q \setminus G, \Delta \rangle$ recognizes $T(\mathcal{F}) \setminus L$.

Closure properties

Theorem (Boolean closure)

Recognizable tree languages are closed under Boolean operations.

Negation (invert accepting states)

Let $\langle Q, \mathcal{F}, G, \Delta \rangle$ be a DCFTA recognizing L.
Then $\langle Q, \mathcal{F}, Q \setminus G, \Delta \rangle$ recognizes $T(\mathcal{F}) \setminus L$.

Union (juxtapose)

Let $\langle Q_i, \mathcal{F}, G_i, \Delta_i \rangle$ be NFTA recognizing L_i, for $i = 1, 2$.
Then $\langle Q_1 \uplus Q_2, \mathcal{F}, G_1 \cup G_2, \Delta_1 \cup \Delta_2 \rangle$ recognizes $L_1 \cup L_2$.

Cross-product construction

Direct intersection

Let $\mathcal{A}_i = \langle Q_i, \mathcal{F}, G_i, \Delta_i \rangle$ be NFTA recognizing L_i, for $i = 1, 2$.
Then $\mathcal{A} = \langle Q_1 \times Q_2, \mathcal{F}, G_1 \times G_2, \Delta \rangle$ recognizes $L_1 \cap L_2$, where

$$\frac{f(q_1, \ldots, q_n) \to q \in \Delta_1 \quad f(q_1', \ldots, q_n') \to q' \in \Delta_2}{f(\langle q_1, q_1' \rangle, \ldots, \langle q_n, q_n' \rangle) \to \langle q, q' \rangle \in \Delta}$$

Cross-product construction

Direct intersection

Let $\mathcal{A}_i = \langle Q_i, \mathcal{F}, G_i, \Delta_i \rangle$ be NFTA recognizing L_i, for $i = 1, 2$.
Then $\mathcal{A} = \langle Q_1 \times Q_2, \mathcal{F}, G_1 \times G_2, \Delta \rangle$ recognizes $L_1 \cap L_2$, where

$$\frac{f(q_1, \ldots, q_n) \to q \in \Delta_1 \quad f(q'_1, \ldots, q'_n) \to q' \in \Delta_2}{f(\langle q_1, q'_1 \rangle, \ldots, \langle q_n, q'_n \rangle) \to \langle q, q' \rangle \in \Delta}$$

Remarks:

- If $\mathcal{A}_1, \mathcal{A}_2$ are D(C)FTA, then so is $\mathcal{A}$.
- If $\mathcal{A}_1, \mathcal{A}_2$ are complete, replace $G_1 \times G_2$ with $(G_1 \times Q_2) \cup (Q_1 \times G_2)$ to recognize $L_1 \cup L_2$.

Tree languages and context-free languages

Front

Let t be a ground tree. Then $fr(t) \in \mathcal{F}_0^*$ denotes the word obtained from reading the leaves from left to right (in increasing lexicographical order of their positions).

Example: $t = f(a, g(b, a), c)$, $fr(t) = abac$

Tree languages and context-free languages

Front

Let t be a ground tree. Then $fr(t) \in \mathcal{F}_0^*$ denotes the word obtained from reading the leaves from left to right (in increasing lexicographical order of their positions).

Example: $t = f(a, g(b, a), c)$, $fr(t) = abac$

Leaf languages

- Let L be a recognizable tree language. Then $fr(L)$ is context-free.
- Let L be a context-free language that does not contain the empty word. Then there exists an NFTA $\mathcal{A}$ with $L = fr(\mathcal{L}(\mathcal{A}))$.

Tree languages and context-free languages

Front

Let t be a ground tree. Then $fr(t) \in \mathcal{F}_0^*$ denotes the word obtained from reading the leaves from left to right (in increasing lexicographical order of their positions).

Example: $t = f(a, g(b, a), c)$, $fr(t) = abac$

Leaf languages

- Let L be a recognizable tree language. Then $fr(L)$ is context-free.
- Let L be a context-free language that does not contain the empty word. Then there exists an NFTA $\mathcal{A}$ with $L = fr(\mathcal{L}(\mathcal{A}))$.

Proof (idea):

- Given a T-NFTA recognizing L, construct a CFG from it.
- L is generated by a CFG using productions of the form $A \to BC \mid a$ only. Replace $A \to BC$ by $A \to A_2$ and $A_2 \to BC$, construct a T-NFTA from the result.

Visibly pushdown automata

Visibly pushdown automaton

Let $\mathcal{A} = \langle Q, \Sigma, \Gamma, T, q_0 z_0, F \rangle$ be a pushdown automaton.
$\mathcal{A}$ is called *visibly pushdown* (VPA) if there exist $\Sigma_0, \Sigma_1, \Sigma_2$ such that

- $\Sigma = \Sigma_0 \uplus \Sigma_1 \uplus \Sigma_2$
- $T \subseteq \bigcup_{i=0}^{2} (Q \times \Gamma) \times \Sigma_i \times (Q \times \Gamma^i)$

Visibly pushdown automata

Visibly pushdown automaton

Let $\mathcal{A} = \langle Q, \Sigma, \Gamma, T, q_0 z_0, F \rangle$ be a pushdown automaton.
$\mathcal{A}$ is called *visibly pushdown* (VPA) if there exist $\Sigma_0, \Sigma_1, \Sigma_2$ such that

- $\Sigma = \Sigma_0 \uplus \Sigma_1 \uplus \Sigma_2$
- $T \subseteq \bigcup_{i=0}^{2} (Q \times \Gamma) \times \Sigma_i \times (Q \times \Gamma^i)$

Closure properties

Languages accepted by VPA are closed under boolean operations.

Visibly pushdown automata

Visibly pushdown automaton

Let $\mathcal{A} = \langle Q, \Sigma, \Gamma, T, q_0 z_0, F \rangle$ be a pushdown automaton.
$\mathcal{A}$ is called *visibly pushdown* (VPA) if there exist $\Sigma_0, \Sigma_1, \Sigma_2$ such that

- $\Sigma = \Sigma_0 \uplus \Sigma_1 \uplus \Sigma_2$
- $T \subseteq \bigcup_{i=0}^{2} (Q \times \Gamma) \times \Sigma_i \times (Q \times \Gamma^i)$

Closure properties

Languages accepted by VPA are closed under boolean operations.

VPA and tree languages

Let $L \subseteq T(\mathcal{F})$ be a recognizable tree language.
Then L, seen as a word language of terms, is accepted by a VPA.

From TA to VPA

Let $\mathcal{A} = \langle Q, \mathcal{F}, I, \Delta \rangle$ be a T-NFTA accepting L.
For convenience, assume $I = \{q_0\}$ is a singleton (closure under union). We construct a single-state VPA $\mathcal{B} = \langle \Sigma, \Gamma, T, q_0 \rangle$ accepting by empty stack and recognizing the terms of L (can be converted into a normal VPA).

- $\Sigma_0 = \mathcal{F}_0 \cup \{\ )\ \}$, $\Sigma_1 = \mathcal{F} \setminus \mathcal{F}_0$, $\Sigma_2 = \{\ ,\ ,\ (\ \}$
- $\Gamma = Q \cup \{\ r_i \mid r \in \Delta,\ r = q(f) \to (q_1, \ldots, q_n),\ n \geq 1,\ 0 \leq i \leq n \}$
- $T = \bigcup_{r \in \Delta} T_r$
  - for $r = q(a) \to \varepsilon$, we have $T_r = \{\ \langle q, a, \varepsilon \rangle\ \}$;
  - for $r = q(f) \to (q_1, \ldots, q_n)$, $n \geq 1$, we have
    $T_r = \{\langle q, f, r_0 \rangle, \langle r_0,\ (\ , q_1 r_1 \rangle, \langle r_n,\ )\ , \varepsilon \rangle\}$
    $\cup \{\ \langle r_i,\ ,\ , q_{i+1} r_{i+1} \rangle \mid 1 \leq i < n \}$

Idea: $q \xrightarrow{t}^*_{\mathcal{B}} \varepsilon$ iff $q(t) \to^*_{\mathcal{A}} t$

From TA to VPA: Example

Consider a T-NFTA $\langle Q, \mathcal{F}, I, \Delta \rangle$ accepting $L = \{\, f(g^i(a)) \mid i \geq 0 \,\}$:

- $Q = \{q_0, q_1, q_f\}$, $\mathcal{F} = \{f(2), g(1), a\}$, $I = \{q_f\}$;
- $\Delta := \{\alpha : q_0(a) \to \varepsilon, \quad \beta : q_1(g) \to q_0, \quad \gamma : q_1(g) \to q_1, \quad \delta : q_f(f) \to (q_1, q_1)\}$.

We construct the single-state VPA $\langle \Sigma, \Gamma, T, q_f \rangle$, where:

- $\Sigma_0 = \{a, \ )\ \}$, $\Sigma_1 = \{f, g\}$, $\Sigma_2 = \{\ ,\ ,\ (\ \}$;
- $\Gamma = Q \cup \{\beta_0, \beta_1, \gamma_0, \gamma_1, \delta_0, \delta_1, \delta_2\}$;
- $T_\alpha = \{\langle q_0, a, \varepsilon \rangle\}$;
- $T_\beta = \{\langle q_1, g, \beta_0 \rangle, \langle \beta_0, (\ , q_0\beta_1 \rangle, \langle \beta_1,)\ \varepsilon \rangle\}$;
- $T_\gamma = \{\langle q_1, g, \gamma_0 \rangle, \langle \gamma_0, (\ , q_1\gamma_1 \rangle, \langle \gamma_1,)\ \varepsilon \rangle\}$;
- $T_\delta = \{\langle q_f, f, \delta_0 \rangle, \langle \delta_0, (\ , q_1\delta_1 \rangle, \langle \delta_1, \ ,\ , q_1\delta_2 \rangle, \langle \delta_2,)\ \varepsilon \rangle\}$.

Run on $f(g(a), g(g(a)))$:

$$q_f \xrightarrow{f} \delta_0 \xrightarrow{(} q_1\delta_1 \xrightarrow{g} \beta_0\delta_1 \xrightarrow{(} q_0\beta_1\delta_1 \xrightarrow{a} \beta_1\delta_1 \xrightarrow{)} \delta_1 \xrightarrow{,} q_1\delta_2 \xrightarrow{g} \gamma_0\delta_2 \xrightarrow{(} q_1\gamma_1\delta_2$$

$$\xrightarrow{g} \beta_0\gamma_1\delta_2 \xrightarrow{(} q_0\beta_1\gamma_1\delta_2 \xrightarrow{a} \beta_1\gamma_1\delta_2 \xrightarrow{)} \gamma_1\delta_2 \xrightarrow{)} \delta_2 \xrightarrow{)} \varepsilon$$

Tree homomorphism

Definition

Let $\mathcal{X}_n := \{x_1, \ldots, x_n\}$ and $\mathcal{F}, \mathcal{F}'$ ranked alphabets.
A *tree homomorphism* is a mapping $h : \mathcal{F} \to T(\mathcal{F}', \mathcal{X})$,
with $h(f) \in T(\mathcal{F}, \mathcal{X}_n)$ if $f \in \mathcal{F}_n$.

Extension of h to trees $(T(\mathcal{F}) \to T(\mathcal{F}'))$:

- $h(f(t_1, \ldots, t_n)) = h(f)\{x_1 \leftarrow h(t_1), \ldots, x_n \leftarrow h(t_n)\}$

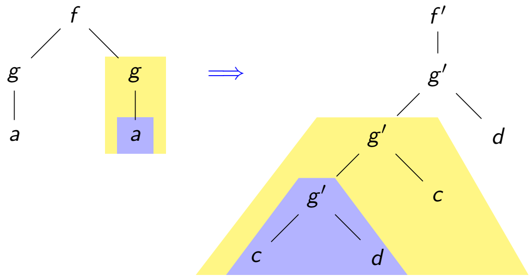
Intuition:

- $h(f)$ "explodes" f-positions into trees
- reorders/copies/deletes subtrees.

Examples

Example

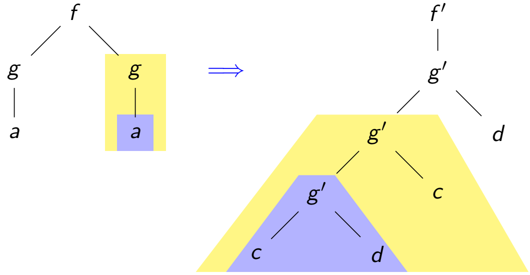
- $\mathcal{F} = \{f(2), g(1), a\}$, $\mathcal{F}' = \{f'(1), g'(2), c, d\}$
- $h(f) = f'(g'(x_2, d))$, $h(g) = g'(x_1, c)$, $h(a) = g'(c, d)$



Examples

Example

- $\mathcal{F} = \{f(2), g(1), a\}$, $\mathcal{F}' = \{f'(1), g'(2), c, d\}$
- $h(f) = f'(g'(x_2, d))$, $h(g) = g'(x_1, c)$, $h(a) = g'(c, d)$



Example (ternary to binary tree)

- $\mathcal{F} = \{f(3), a, b\}$, $\mathcal{F}' = \{g(2), a, b\}$
- $h_{32}(f) = g(x_1, g(x_2, x_3))$, $h_{32}(a) = a$, $h_{32}(b) = b$

Properties of homomorphisms

A homomorphism h is

- *linear* if $h(f)$ linear for all f;
- *non-erasing* if $\mathcal{H}(h(f)) > 0$ for all f;
- *flat* if $\mathcal{H}(h(f)) = 1$ for all f;
- *complete* if $f \in \mathcal{F}_n$ implies that $h(f)$ contains all of $\mathcal{X}_n$;
- *permuting* if h is complete, linear, and flat;
- *alphabetic* if $h(f)$ has the form $g(x_1, \ldots, x_n)$ for all f.

Example: h_{32} is linear, non-erasing, and complete.

Properties of homomorphisms

A homomorphism h is

- *linear* if $h(f)$ linear for all f;
- *non-erasing* if $\mathcal{H}(h(f)) > 0$ for all f;
- *flat* if $\mathcal{H}(h(f)) = 1$ for all f;
- *complete* if $f \in \mathcal{F}_n$ implies that $h(f)$ contains all of $\mathcal{X}_n$;
- *permuting* if h is complete, linear, and flat;
- *alphabetic* if $h(f)$ has the form $g(x_1, \ldots, x_n)$ for all f.

Example: h_{32} is linear, non-erasing, and complete.

Non-linear homomorphisms do not preserve recognizability

- Example: $h(f) = f'(x_1, x_1)$, $h(g) = g(x_1)$, $h(a) = a$
- $L = \{ f(g^i(a)) \mid i \geq 0 \}$ (recognizable)
- $h(L) = \{ f'(g^i(a), g^i(a)) \mid i \geq 0 \}$ (not recognizable)

Linear homomorphisms

Theorem: Linear homomorphisms preserve recognizability

Let $L \subseteq T(\mathcal{F})$ be recognizable and $h : \mathcal{F} \to \mathcal{F}'$ a linear tree homomorphism. Then $h(L)$ is recognizable.

Linear homomorphisms

Theorem: Linear homomorphisms preserve recognizability

Let $L \subseteq T(\mathcal{F})$ be recognizable and $h : \mathcal{F} \to \mathcal{F}'$ a linear tree homomorphism. Then $h(L)$ is recognizable.

Illustrating example:

- $\mathcal{F} = \{f(2), g(1), a\}$, $\mathcal{F}' = \{f'(1), g'(2), c, d\}$
- $h(f) = f'(g'(x_2, d))$, $h(g) = g'(x_1, c)$, $h(a) = g'(c, d)$
- $L = \{\, f(g^i(a), g^k(a)) \mid i, k \geq 0 \,\}$
- $\mathcal{A} = \langle \{q_0, q_1, q_f\}, \mathcal{F}, \{q_f\}, \Delta \rangle$ recognizes L with
  $\Delta := \{\alpha : a \to q_0, \quad \beta : g(q_0) \to q_1, \quad \gamma : g(q_1) \to q_1, \quad \delta : f(q_1, q_1) \to q_f\}$

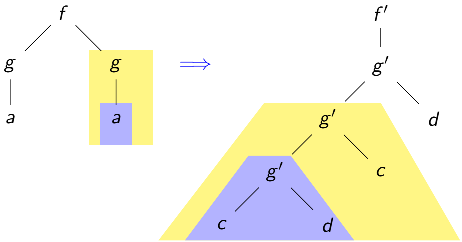
Linear homomorphisms

Theorem: Linear homomorphisms preserve recognizability

Let $L \subseteq T(\mathcal{F})$ be recognizable and $h : \mathcal{F} \to \mathcal{F}'$ a linear tree homomorphism. Then $h(L)$ is recognizable.

Illustrating example:

- $\mathcal{F} = \{f(2), g(1), a\}$, $\mathcal{F}' = \{f'(1), g'(2), c, d\}$
- $h(f) = f'(g'(x_2, d))$, $h(g) = g'(x_1, c)$, $h(a) = g'(c, d)$
- $L = \{\, f(g^i(a), g^k(a)) \mid i, k \geq 0 \,\}$
- $\mathcal{A} = \langle \{q_0, q_1, q_f\}, \mathcal{F}, \{q_f\}, \Delta \rangle$ recognizes L with
  $\Delta := \{\alpha : a \to q_0, \quad \beta : g(q_0) \to q_1, \quad \gamma : g(q_1) \to q_1, \quad \delta : f(q_1, q_1) \to q_f\}$



Linear homomorphisms

Theorem: Linear homomorphisms preserve recognizability

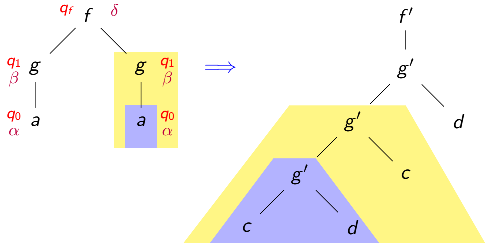
Let $L \subseteq T(\mathcal{F})$ be recognizable and $h : \mathcal{F} \to \mathcal{F}'$ a linear tree homomorphism. Then $h(L)$ is recognizable.

Illustrating example:

- $\mathcal{F} = \{f(2), g(1), a\}$, $\mathcal{F}' = \{f'(1), g'(2), c, d\}$
- $h(f) = f'(g'(x_2, d))$, $h(g) = g'(x_1, c)$, $h(a) = g'(c, d)$
- $L = \{\, f(g^i(a), g^k(a)) \mid i, k \geq 0\,\}$
- $\mathcal{A} = \langle \{q_0, q_1, q_f\}, \mathcal{F}, \{q_f\}, \Delta \rangle$ recognizes L with
  $\Delta := \{\alpha : a \to q_0, \quad \beta : g(q_0) \to q_1, \quad \gamma : g(q_1) \to q_1, \quad \delta : f(q_1, q_1) \to q_f\}$

Run on $\mathcal{A}$

Rules used to produce states



Linear homomorphisms

Theorem: Linear homomorphisms preserve recognizability

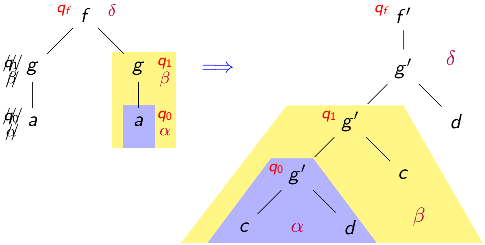
Let $L \subseteq T(\mathcal{F})$ be recognizable and $h : \mathcal{F} \to \mathcal{F}'$ a linear tree homomorphism.
Then $h(L)$ is recognizable.

Illustrating example:

- $\mathcal{F} = \{f(2), g(1), a\}$, $\mathcal{F}' = \{f'(1), g'(2), c, d\}$
- $h(f) = f'(g'(x_2, d))$, $h(g) = g'(x_1, c)$, $h(a) = g'(c, d)$
- $L = \{\, f(g^i(a), g^k(a)) \mid i, k \geq 0 \,\}$
- $\mathcal{A} = \langle \{q_0, q_1, q_f\}, \mathcal{F}, \{q_f\}, \Delta \rangle$ recognizes L with
  $\Delta := \{\alpha : a \to q_0, \quad \beta : g(q_0) \to q_1, \quad \gamma : g(q_1) \to q_1, \quad \delta : f(q_1, q_1) \to q_f\}$

Run on $\mathcal{A}$

Rules used
to produce states

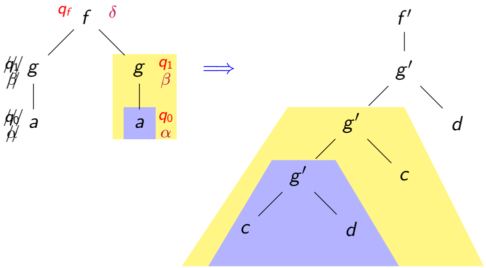


Construct automaton
for $h(L)$ preserving
state labels from $\mathcal{A}$
+
Guess the rules.

Automaton construction for $h(L)$

Given a reduced NFTA $\mathcal{A} = \langle Q, \mathcal{F}, G, \Delta \rangle$ for L, construct NFTA $\mathcal{A}' = \langle Q', \mathcal{F}', G, \Delta' \rangle$ for $h(L)$.

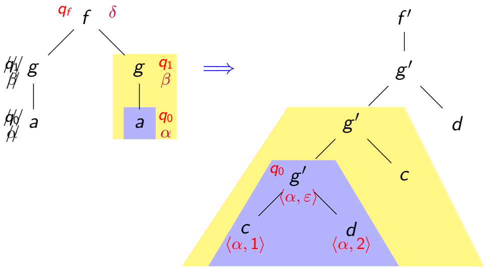
- $Q' := Q \cup \{ \langle r, p \rangle \mid r \in \Delta, \exists f \in \mathcal{F} : r = f(\ldots) \to \ldots, p \in Pos_{h(f)} \}$;
- Δ' contains, for each transition $r : f(s_1, \ldots, s_n) \to s$ in Δ and $p \in Pos_{h(f)}$:
  - $f'(\langle r, p1 \rangle, \ldots, \langle r, pk \rangle) \to \langle r, p \rangle$ if $h(f)(p) = f' \in \mathcal{F}'_k$
  - $s_i \to \langle r, p \rangle$ if $h(f)(p) = x_i$
  - $\langle r, \varepsilon \rangle \to s$



Automaton construction for $h(L)$

Given a reduced NFTA $\mathcal{A} = \langle Q, \mathcal{F}, G, \Delta \rangle$ for L,
construct NFTA $\mathcal{A}' = \langle Q', \mathcal{F}', G, \Delta' \rangle$ for $h(L)$.

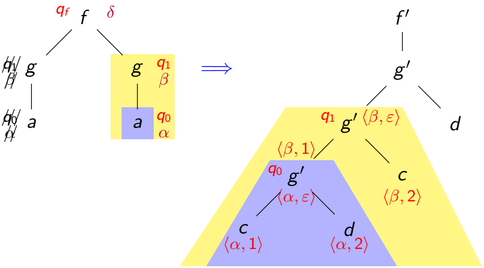
- $Q' := Q \cup \{ \langle r, p \rangle \mid r \in \Delta, \exists f \in \mathcal{F} : r = f(\dots) \to \dots, p \in Pos_{h(f)} \}$;
- Δ' contains, for each transition $r : f(s_1, \dots, s_n) \to s$ in Δ and $p \in Pos_{h(f)}$:
  - $f'(\langle r, p1 \rangle, \dots, \langle r, pk \rangle) \to \langle r, p \rangle$ if $h(f)(p) = f' \in \mathcal{F}'_k$
  - $s_i \to \langle r, p \rangle$ if $h(f)(p) = x_i$
  - $\langle r, \varepsilon \rangle \to s$



Automaton construction for $h(L)$

Given a reduced NFTA $\mathcal{A} = \langle Q, \mathcal{F}, G, \Delta \rangle$ for L,
construct NFTA $\mathcal{A}' = \langle Q', \mathcal{F}', G, \Delta' \rangle$ for $h(L)$.

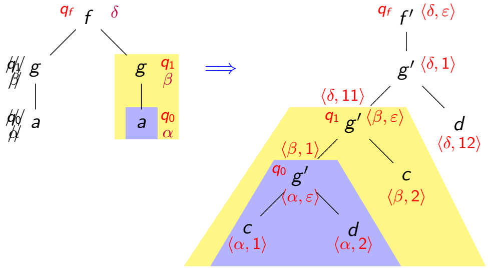
- $Q' := Q \cup \{ \langle r, p \rangle \mid r \in \Delta, \exists f \in \mathcal{F} : r = f(\ldots) \to \ldots, p \in Pos_{h(f)} \}$;
- Δ' contains, for each transition $r : f(s_1, \ldots, s_n) \to s$ in Δ and $p \in Pos_{h(f)}$:
  - $f'(\langle r, p1 \rangle, \ldots, \langle r, pk \rangle) \to \langle r, p \rangle$ if $h(f)(p) = f' \in \mathcal{F}'_k$
  - $s_i \to \langle r, p \rangle$ if $h(f)(p) = x_i$
  - $\langle r, \varepsilon \rangle \to s$



Automaton construction for $h(L)$

Given a reduced NFTA $\mathcal{A} = \langle Q, \mathcal{F}, G, \Delta \rangle$ for L,
construct NFTA $\mathcal{A}' = \langle Q', \mathcal{F}', G, \Delta' \rangle$ for $h(L)$.

- $Q' := Q \cup \{ \langle r, p \rangle \mid r \in \Delta, \exists f \in \mathcal{F} : r = f(\ldots) \to \ldots, p \in Pos_{h(f)} \}$;
- Δ' contains, for each transition $r : f(s_1, \ldots, s_n) \to s$ in Δ and $p \in Pos_{h(f)}$:
  - $f'(\langle r, p1 \rangle, \ldots, \langle r, pk \rangle) \to \langle r, p \rangle$ if $h(f)(p) = f' \in \mathcal{F}'_k$
  - $s_i \to \langle r, p \rangle$ if $h(f)(p) = x_i$
  - $\langle r, \varepsilon \rangle \to s$



Correctness

To prove: $\mathcal{A}'$ accepts $h(L)$.

Correctness

To prove: $\mathcal{A}'$ accepts $h(L)$.

- $h(L) \subseteq \mathcal{L}(\mathcal{A}')$:
  For $t \in T(\mathcal{F})$, prove that $t \rightarrow^*_{\mathcal{A}} q$ implies $h(t) \rightarrow^*_{\mathcal{A}'} q$,
  by structural induction over t.

Correctness

To prove: $\mathcal{A}'$ accepts $h(L)$.

- $h(L) \subseteq \mathcal{L}(\mathcal{A}')$:
  For $t \in T(\mathcal{F})$, prove that $t \to_{\mathcal{A}}^* q$ implies $h(t) \to_{\mathcal{A}'}^* q$,
  by structural induction over t.

- $h(L) \supseteq \mathcal{L}(\mathcal{A}')$:
  For $t' \in T(\mathcal{F}')$, prove that if $t' \to_{\mathcal{A}'}^* q \in Q$,
  then there exists $t \in T(\mathcal{F}) \cap h^{-1}(t')$ with $t \to_{\mathcal{A}}^* q$,
  by induction on number of states (of Q) in the computation $t' \to_{\mathcal{A}'}^* q$.

Inverse tree homomorphisms

Theorem: Inverse homomorphisms preserve recognizability

Let $L \subseteq T(\mathcal{F}')$ be recognizable and $h : \mathcal{F} \to \mathcal{F}'$ a tree homomorphism (not necessarily linear). Then $h^{-1}(L)$ is recognizable.

Inverse tree homomorphisms

Theorem: Inverse homomorphisms preserve recognizability

Let $L \subseteq T(\mathcal{F}')$ be recognizable and $h : \mathcal{F} \to \mathcal{F}'$ a tree homomorphism (not necessarily linear). Then $h^{-1}(L)$ is recognizable.

Given an NFTA $\mathcal{A}' = \langle Q, \mathcal{F}', G, \Delta' \rangle$ for L,
construct NFTA $\mathcal{A} = \langle Q \uplus \{!\}, \mathcal{F}, G, \Delta \rangle$ for $h^{-1}(L)$.

For all $n \geq 0$ and $f \in \mathcal{F}_n$, and $p_1, \ldots, p_n \in Q$,

- add $f(!, \ldots, !) \to\ !$ to Δ;
- if $h(f)\{x_1 \leftarrow p_1, \ldots, x_n \leftarrow p_n\} \to^*_{\mathcal{A}'} q$, add $f(q_1, \ldots, q_n) \to q$ to Δ, with:

$$q_i = \begin{cases} p_i & \text{if } x_i \text{ appears in } h(f) \\ ! & \text{otherwise} \end{cases}$$

Proof: Show $t \to^*_{\mathcal{A}} q$ iff $h(t) \to^*_{\mathcal{A}'} q$, for all $t \in T(\mathcal{F})$.

Intersection problem

Theorem

The following problem is EXPTIME-complete:
Given tree automata $\mathcal{A}_1, \ldots, \mathcal{A}_n$, is $\mathcal{L}(\mathcal{A}_1) \cap \cdots \cap \mathcal{L}(\mathcal{A}_n) \neq \emptyset$?

Intersection problem

Theorem

The following problem is EXPTIME-complete:
Given tree automata $\mathcal{A}_1, \ldots, \mathcal{A}_n$, is $\mathcal{L}(\mathcal{A}_1) \cap \cdots \cap \mathcal{L}(\mathcal{A}_n) \neq \emptyset$?

Proof (sketch):

- Membership: Compute the accessible tuples of states in $\mathcal{A}_1 \times \cdots \times \mathcal{A}_n$.

Intersection problem

Theorem

The following problem is EXPTIME-complete:
Given tree automata $\mathcal{A}_1, \ldots, \mathcal{A}_n$, is $\mathcal{L}(\mathcal{A}_1) \cap \cdots \cap \mathcal{L}(\mathcal{A}_n) \neq \emptyset$?

Proof (sketch):

- Membership: Compute the accessible tuples of states in $\mathcal{A}_1 \times \cdots \times \mathcal{A}_n$.
- Hardness: Simulate an polynomial-space ATM $\mathcal{M}$ with input of length n and space $p(n)$ (using EXPTIME=APSPACE).

Intersection problem

Theorem

The following problem is EXPTIME-complete:
Given tree automata $\mathcal{A}_1, \ldots, \mathcal{A}_n$, is $\mathcal{L}(\mathcal{A}_1) \cap \cdots \cap \mathcal{L}(\mathcal{A}_n) \neq \emptyset$?

Proof (sketch):

- Membership: Compute the accessible tuples of states in $\mathcal{A}_1 \times \cdots \times \mathcal{A}_n$.
- Hardness: Simulate an polynomial-space ATM $\mathcal{M}$ with input of length n and space $p(n)$ (using EXPTIME=APSPACE).
  If $\mathcal{M}$ accepts the input, there is an accepting run.
  Encode runs of $\mathcal{M}$ as configuration trees.

Intersection problem

Theorem

The following problem is EXPTIME-complete:
Given tree automata $\mathcal{A}_1, \ldots, \mathcal{A}_n$, is $\mathcal{L}(\mathcal{A}_1) \cap \cdots \cap \mathcal{L}(\mathcal{A}_n) \neq \emptyset$?

Proof (sketch):

- Membership: Compute the accessible tuples of states in $\mathcal{A}_1 \times \cdots \times \mathcal{A}_n$.
- Hardness: Simulate an polynomial-space ATM $\mathcal{M}$ with input of length n and space $p(n)$ (using EXPTIME=APSPACE).
  If $\mathcal{M}$ accepts the input, there is an accepting run.
  Encode runs of $\mathcal{M}$ as configuration trees.
  Construct a collection of T-NFTA $\mathcal{A}_i$, for $i = 1, \ldots, p(n)$, such that the intersection of their languages is non-empty iff $\mathcal{M}$ has an accepting run. $\mathcal{A}_i$ checks the following:
  1. if $\mathcal{M}$ starts with the correct configuration;
  2. if all configurations in the run are of length $p(n)$;
  3. if all final configurations are accepting;
  4. if the part of the configurations around the i-th symbol are coherent.

Intersection problem

Theorem

The following problem is EXPTIME-complete:
Given tree automata $\mathcal{A}_1, \ldots, \mathcal{A}_n$, is $\mathcal{L}(\mathcal{A}_1) \cap \cdots \cap \mathcal{L}(\mathcal{A}_n) \neq \emptyset$?

Proof (sketch):

- Membership: Compute the accessible tuples of states in $\mathcal{A}_1 \times \cdots \times \mathcal{A}_n$.
- Hardness: Simulate an polynomial-space ATM $\mathcal{M}$ with input of length n and space $p(n)$ (using EXPTIME=APSPACE).
  If $\mathcal{M}$ accepts the input, there is an accepting run.
  Encode runs of $\mathcal{M}$ as configuration trees.
  Construct a collection of T-NFTA $\mathcal{A}_i$, for $i = 1, \ldots, p(n)$, such that the intersection of their languages is non-empty iff $\mathcal{M}$ has an accepting run. $\mathcal{A}_i$ checks the following:
  1. if $\mathcal{M}$ starts with the correct configuration;
  2. if all configurations in the run are of length $p(n)$;
  3. if all final configurations are accepting;
  4. if the part of the configurations around the i-th symbol are coherent.

Detailed proof: Veanes, 1997

Congruences on trees

Definition: Congruence

Let $\equiv$ be an equivalence relation on $T(\mathcal{F})$.

- $\equiv$ is called a *congruence* if for any $n \geq 0$ and $f \in \mathcal{F}_n$, $u_1 \equiv v_1, \ldots, u_n \equiv v_n$ we have
$$f(u_1, \ldots, u_n) \equiv f(v_1, \ldots, v_n)$$
- $\equiv$ *saturates* L if $u \equiv v$ implies $u \in L \iff v \in L$.

Congruences on trees

Definition: Congruence

Let $\equiv$ be an equivalence relation on $T(\mathcal{F})$.

- $\equiv$ is called a *congruence* if for any $n \geq 0$ and $f \in \mathcal{F}_n$, $u_1 \equiv v_1, \ldots, u_n \equiv v_n$ we have

$$f(u_1, \ldots, u_n) \equiv f(v_1, \ldots, v_n)$$

- $\equiv$ *saturates* L if $u \equiv v$ implies $u \in L \iff v \in L$.

For $L \subseteq T(\mathcal{F})$, write $u \equiv_L v$ if

$$\forall C \in \mathcal{C}(\mathcal{F}) : C[u] \in L \Leftrightarrow C[v] \in L$$

Congruences on trees

Definition: Congruence

Let $\equiv$ be an equivalence relation on $T(\mathcal{F})$.

- $\equiv$ is called a *congruence* if for any $n \geq 0$ and $f \in \mathcal{F}_n$, $u_1 \equiv v_1, \ldots, u_n \equiv v_n$ we have
$$f(u_1, \ldots, u_n) \equiv f(v_1, \ldots, v_n)$$
- $\equiv$ *saturates* L if $u \equiv v$ implies $u \in L \iff v \in L$.

For $L \subseteq T(\mathcal{F})$, write $u \equiv_L v$ if
$$\forall C \in \mathcal{C}(\mathcal{F}) : C[u] \in L \Leftrightarrow C[v] \in L$$

Myhill-Nerode Theorem for trees

The following are equivalent:

1. $L \subseteq T(\mathcal{F})$ is recognizable.
2. L is saturated by some congruence of finite index.
3. $\equiv_L$ is of finite index.

Myhill-Nerode Theorem

Application:

Consider $L = \{ f(g^i(a), g^i(a)) \mid i \geq 0 \}$.
For any pair $i \neq k$, consider $C = f(x, g^i(a))$.
Then $C[g^i(a)] \in L$ but $C[g^k(a)] \notin L \Rightarrow g^i(a) \not\equiv_L g^k(a)$
Therefore $\equiv_L$ is not of finite index, and L is not recognizable.

Myhill-Nerode Theorem

Application:

Consider $L = \{ f(g^i(a), g^i(a)) \mid i \geq 0 \}$.
For any pair $i \neq k$, consider $C = f(x, g^i(a))$.
Then $C[g^i(a)] \in L$ but $C[g^k(a)] \notin L \Rightarrow g^i(a) \not\equiv_L g^k(a)$
Therefore $\equiv_L$ is not of finite index, and L is not recognizable.

Proof of the theorem (sketch):

- $1 \to 2$: Let $\mathcal{A}$ be DCFTA and let $u \equiv v$ iff $u \to^*_{\mathcal{A}} q \,{}^*_{\mathcal{A}}\!\leftarrow v$.
  Then $\equiv$ is a congruence of finite index and saturates L.

Myhill-Nerode Theorem

Application:

Consider $L = \{ f(g^i(a), g^i(a)) \mid i \geq 0 \}$.
For any pair $i \neq k$, consider $C = f(x, g^i(a))$.
Then $C[g^i(a)] \in L$ but $C[g^k(a)] \notin L \Rightarrow g^i(a) \not\equiv_L g^k(a)$
Therefore $\equiv_L$ is not of finite index, and L is not recognizable.

Proof of the theorem (sketch):

- $1 \to 2$: Let $\mathcal{A}$ be DCFTA and let $u \equiv v$ iff $u \to^*_{\mathcal{A}} q \;{}^*_{\mathcal{A}}\!\leftarrow v$.
  Then $\equiv$ is a congruence of finite index and saturates L.
- $2 \to 3$: Let $\equiv$ be a saturating congruence, $u \equiv v$ implies $u \equiv_L v$ (prove $u \equiv v$ implies $C[u] \equiv C[v]$ for all C, by recurrence over height of position of x in C).

Myhill-Nerode Theorem

Application:

Consider $L = \{ f(g^i(a), g^i(a)) \mid i \geq 0 \}$.
For any pair $i \neq k$, consider $C = f(x, g^i(a))$.
Then $C[g^i(a)] \in L$ but $C[g^k(a)] \notin L \Rightarrow g^i(a) \not\equiv_L g^k(a)$
Therefore $\equiv_L$ is not of finite index, and L is not recognizable.

Proof of the theorem (sketch):

- $1 \rightarrow 2$: Let $\mathcal{A}$ be DCFTA and let $u \equiv v$ iff $u \rightarrow^*_{\mathcal{A}} q \; {}^*_{\mathcal{A}}\!\leftarrow v$.
  Then $\equiv$ is a congruence of finite index and saturates L.
- $2 \rightarrow 3$: Let $\equiv$ be a saturating congruence, $u \equiv v$ implies $u \equiv_L v$
  (prove $u \equiv v$ implies $C[u] \equiv C[v]$ for all C, by recurrence over height of position of x in C).
- $3 \rightarrow 1$: Let $\mathcal{A} = \langle T(\mathcal{F})/\equiv_L, \mathcal{F}, L/\equiv_L, \Delta \rangle$, with

  $$f([u_1], \ldots, [u_n]) \rightarrow [f(u_1, \ldots, u_n)]$$

  for all $n \geq 0$, $f \in \mathcal{F}_n$, $u_1, \ldots, u_n \in T(\mathcal{F})$,
  where $[u]$ is the equivalence class of $u \in T(\mathcal{F})$;

Myhill-Nerode Theorem

Application:

Consider $L = \{ f(g^i(a), g^i(a)) \mid i \geq 0 \}$.
For any pair $i \neq k$, consider $C = f(x, g^i(a))$.
Then $C[g^i(a)] \in L$ but $C[g^k(a)] \notin L \Rightarrow g^i(a) \not\equiv_L g^k(a)$
Therefore $\equiv_L$ is not of finite index, and L is not recognizable.

Proof of the theorem (sketch):

- $1 \to 2$: Let $\mathcal{A}$ be DCFTA and let $u \equiv v$ iff $u \to^*_{\mathcal{A}} q \; {}^*_{\mathcal{A}}\!\leftarrow v$.
  Then $\equiv$ is a congruence of finite index and saturates L.
- $2 \to 3$: Let $\equiv$ be a saturating congruence, $u \equiv v$ implies $u \equiv_L v$ (prove $u \equiv v$ implies $C[u] \equiv C[v]$ for all C, by recurrence over height of position of x in C).
- $3 \to 1$: Let $\mathcal{A} = \langle T(\mathcal{F})/\equiv_L, \mathcal{F}, L/\equiv_L, \Delta \rangle$, with

  $$f([u_1], \ldots, [u_n]) \to [f(u_1, \ldots, u_n)]$$

  for all $n \geq 0$, $f \in \mathcal{F}_n$, $u_1, \ldots, u_n \in T(\mathcal{F})$,
  where $[u]$ is the equivalence class of $u \in T(\mathcal{F})$;

Remark: This can be shown to be the canonical minimal DCFTA.

Path languages

Path languages

Let $t \in T(\mathcal{F})$. The *path language* $\pi(t)$ is defined as follows:

- if $t = a \in \mathcal{F}_0$, then $\pi(t) = \{a\}$;
- if $t = f(t_1, \ldots, t_n)$, for $f \in \mathcal{F}_n$, then $\pi(t) = \{\, fiw \mid w \in \pi(t_i)\,\}$.

We write $\pi(L) = \bigcup\{\, \pi(t) \mid t \in L\,\}$ for $L \subseteq T(\mathcal{F})$.

Example: $L = \{f(a, b), f(b, a)\}$, $\pi(L) = \{f1a, f2b, f1b, f2a\}$.

Path languages

Path languages

Let $t \in T(\mathcal{F})$. The *path language* $\pi(t)$ is defined as follows:

- if $t = a \in \mathcal{F}_0$, then $\pi(t) = \{a\}$;
- if $t = f(t_1, \ldots, t_n)$, for $f \in \mathcal{F}_n$, then $\pi(t) = \{\, fiw \mid w \in \pi(t_i) \,\}$.

We write $\pi(L) = \bigcup\{\, \pi(t) \mid t \in L \,\}$ for $L \subseteq T(\mathcal{F})$.

Example: $L = \{f(a, b), f(b, a)\}$, $\pi(L) = \{f1a, f2b, f1b, f2a\}$.

Path closure

Let $L \subseteq T(\mathcal{F})$ be a tree language.

- The *path closure* of L is $pc(L) = \{\, t \mid \pi(t) \subseteq \pi(L) \,\} \supseteq L$.
- L is called *path-closed* if $L = pc(L)$.

Example: $pc(L) = \{f(a, a), f(a, b), f(b, a), f(b, b)\}$, so L is not path-closed.

Path closure and T-NFTA

Lemma

Let $L \subseteq T(\mathcal{F})$ be a recognizable tree language. Then:

- $\pi(L)$ is a recognizable word language.
- $pc(L)$ is a recognizable tree language.

Path closure and T-NFTA

Lemma

Let $L \subseteq T(\mathcal{F})$ be a recognizable tree language. Then:

- $\pi(L)$ is a recognizable word language.
- $pc(L)$ is a recognizable tree language.

Proof: Let $\mathcal{A} = \langle Q, \mathcal{F}, G, \Delta \rangle$ be a reduced T-NFTA for L.

- Construct a finite (word) automaton out of $\mathcal{A}$.
  (Easy, but does require $\mathcal{A}$ to be reduced!)

Path closure and T-NFTA

Lemma

Let $L \subseteq T(\mathcal{F})$ be a recognizable tree language. Then:

- $\pi(L)$ is a recognizable word language.
- $pc(L)$ is a recognizable tree language.

Proof: Let $\mathcal{A} = \langle Q, \mathcal{F}, G, \Delta \rangle$ be a reduced T-NFTA for L.

- Construct a finite (word) automaton out of $\mathcal{A}$.
  (Easy, but does require $\mathcal{A}$ to be reduced!)
- Construct $\mathcal{A}' = \langle Q, \mathcal{F}, G, \Delta' \rangle$ for $pc(L)$ as follows:
  for all $a \in \mathcal{F}_0$:

$$q(a) \rightarrow_{\Delta} \varepsilon \quad \rightarrow \quad q(a) \rightarrow_{\Delta'} \varepsilon$$

  for all $n \geq 1$, $f \in \mathcal{F}_n$:

$$\forall i : q(f) \rightarrow_{\Delta} (q_{i,1}, \ldots, q_{i,n}) \quad \rightarrow \quad q(f) \rightarrow_{\Delta'} (q_{1,1}, \ldots, q_{n,n})$$

Path closure and T-NFTA

Lemma

Let $L \subseteq T(\mathcal{F})$ be a recognizable tree language. Then:

- $\pi(L)$ is a recognizable word language.
- $pc(L)$ is a recognizable tree language.

Proof: Let $\mathcal{A} = \langle Q, \mathcal{F}, G, \Delta \rangle$ be a reduced T-NFTA for L.

- Construct a finite (word) automaton out of $\mathcal{A}$.
  (Easy, but does require $\mathcal{A}$ to be reduced!)
- Construct $\mathcal{A}' = \langle Q, \mathcal{F}, G, \Delta' \rangle$ for $pc(L)$ as follows:
  for all $a \in \mathcal{F}_0$:

  $$q(a) \to_{\Delta} \varepsilon \quad \to \quad q(a) \to_{\Delta'} \varepsilon$$

  for all $n \geq 1$, $f \in \mathcal{F}_n$:

  $$\forall i : q(f) \to_{\Delta} (q_{i,1}, \ldots, q_{i,n}) \quad \to \quad q(f) \to_{\Delta'} (q_{1,1}, \ldots, q_{n,n})$$

  Let $L_q = \mathcal{L}(\langle Q, \mathcal{F}, \{q\}, \Delta \rangle)$ and $L'_q = \mathcal{L}(\langle Q, \mathcal{F}, \{q\}, \Delta' \rangle)$.
  Prove $t \in L'_q \Leftrightarrow \pi(t) \subseteq \pi(L_q)$ for all $q \in Q$, $t \in T(\mathcal{F})$ by induction.

Path closure and T-NFTA

Corollary

It is decidable whether a recognizable tree language is path-closed.

Path closure and T-NFTA

Corollary

It is decidable whether a recognizable tree language is path-closed.

Theorem

Let $L \subseteq T(\mathcal{F})$ be a recognizable tree language.
L is path-closed iff it is recognized by a T-DFTA.

Path closure and T-NFTA

Corollary

It is decidable whether a recognizable tree language is path-closed.

Theorem

Let $L \subseteq T(\mathcal{F})$ be a recognizable tree language.
L is path-closed iff it is recognized by a T-DFTA.

Proof (sketch):

- "$\rightarrow$":
  Let $\mathcal{A} = \langle Q, \mathcal{F}, G, \Delta \rangle$ be a reduced T-NFTA for L.
  Construct a T-DFTA $\mathcal{A}' = \langle 2^Q, \mathcal{F}, G, \Delta' \rangle$ as follows:
  for all $a \in \mathcal{F}_0$, $S(a) \rightarrow_{\Delta'} \varepsilon$ if $\exists q \in S, q(a) \rightarrow_\Delta \varepsilon$;
  for all $n \geq 1, f \in \mathcal{F}_n$, $S(f) \rightarrow_{\Delta'} (S_1, \ldots, S_n)$
  where $S_i = \{\, q_i \mid \exists q \in S, q(f) \rightarrow_\Delta (q_1, \ldots, q_n) \,\}$.

Path closure and T-NFTA

Corollary

It is decidable whether a recognizable tree language is path-closed.

Theorem

Let $L \subseteq T(\mathcal{F})$ be a recognizable tree language.
L is path-closed iff it is recognized by a T-DFTA.

Proof (sketch):

- "$\rightarrow$":
  Let $\mathcal{A} = \langle Q, \mathcal{F}, G, \Delta \rangle$ be a reduced T-NFTA for L.
  Construct a T-DFTA $\mathcal{A}' = \langle 2^Q, \mathcal{F}, G, \Delta' \rangle$ as follows:
  for all $a \in \mathcal{F}_0$, $S(a) \rightarrow_{\Delta'} \varepsilon$ if $\exists q \in S, q(a) \rightarrow_{\Delta} \varepsilon$;
  for all $n \geq 1, f \in \mathcal{F}_n$, $S(f) \rightarrow_{\Delta'} (S_1, \ldots, S_n)$
  where $S_i = \{ q_i \mid \exists q \in S, q(f) \rightarrow_{\Delta} (q_1, \ldots, q_n) \}$.
- "$\leftarrow$":
  Let $\mathcal{A}$ be a complete T-DFTA for L, define L_q as before.
  Prove that $\pi(t) \subseteq \pi(L_q)$ implies $t \in L_q$, for all $q \in Q, t \in T(\mathcal{F})$.

Logic over trees

Alternative specification for sets of trees

E.g., to describe valid HTML documents:

- A `p` tag may only appear inside a `body` tag.
- A `dl` tag must contain pairs of `dt` and `dd` tags.

Logic over trees

Alternative specification for sets of trees

E.g., to describe valid HTML documents:

- A `p` tag may only appear inside a `body` tag.
- A `dl` tag must contain pairs of `dt` and `dd` tags.

Roadmap

- We shall define a logic that defines such properties of trees.
- The sets of trees definable in that language will be recognizable.

Recall: First-/second-order logic

First-order logic (FO)

Let $\sigma = ((R_i)_{1 \leq i \leq n})$ be a relation signature and $\mathcal{X}_1 = \{x_1, x_2, \ldots\}$ a set of variables. The first-order formulas $FO(\sigma)$ are:

$$R_i(x_{j_1}, \ldots, x_{j_i}) \mid x = x' \mid \neg\phi \mid \phi \wedge \phi' \mid \exists x.\phi$$

Recall: First-/second-order logic

First-order logic (FO)

Let $\sigma = ((R_i)_{1 \leq i \leq n})$ be a relation signature and $\mathcal{X}_1 = \{x_1, x_2, \ldots\}$ a set of variables. The first-order formulas $FO(\sigma)$ are:

$$R_i(x_{j_1}, \ldots, x_{j_i}) \mid x = x' \mid \neg\phi \mid \phi \wedge \phi' \mid \exists x.\phi$$

Second-order logic: allow quantifying over relations
Monadic: only quantify over sets

Monadic second-order logic (MSO)

Let σ as before and $\mathcal{X}_1 = \{x_1, x_2, \ldots\}$, $\mathcal{X}_2 = \{X_1, X_2, \ldots\}$ sets of first-/second-order variables. The set of $MSO(\sigma)$ formulae are:

$$R_i(x_{j_1}, \ldots, x_{j_i}) \mid x = x' \mid x \in X \mid \neg\phi \mid \phi \wedge \phi' \mid \exists x.\phi \mid \exists X.\phi$$

Recall: First-/second-order logic

First-order logic (FO)

Let $\sigma = ((R_i)_{1 \le i \le n})$ be a relation signature and $\mathcal{X}_1 = \{x_1, x_2, \ldots\}$ a set of variables. The first-order formulas $FO(\sigma)$ are:

$$R_i(x_{j_1}, \ldots, x_{j_i}) \mid x = x' \mid \neg\phi \mid \phi \wedge \phi' \mid \exists x.\phi$$

Second-order logic: allow quantifying over relations
Monadic: only quantify over sets

Monadic second-order logic (MSO)

Let σ as before and $\mathcal{X}_1 = \{x_1, x_2, \ldots\}$, $\mathcal{X}_2 = \{X_1, X_2, \ldots\}$ sets of first-/second-order variables. The set of $MSO(\sigma)$ formulae are:

$$R_i(x_{j_1}, \ldots, x_{j_i}) \mid x = x' \mid x \in X \mid \neg\phi \mid \phi \wedge \phi' \mid \exists x.\phi \mid \exists X.\phi$$

Weak second-order: only quantify over *finite* sets

WS*k*S (weak MSO over with *k* successors)

$$\text{WS}k\text{S} \;=\; \text{MSO}(<_1, \ldots, <_k)$$

Semantics of MSO

Definition

Let $\mathfrak{M}$ a domain, σ a signature, ν a valuation with

- $\nu(x) \in \mathfrak{M}$ for $x \in \mathcal{X}_1$
- $\nu(X) \subseteq \mathfrak{M}$ for $X \in \mathcal{X}_2$

Semantics of MSO

Definition

Let $\mathfrak{M}$ a domain, σ a signature, ν a valuation with

- $\nu(x) \in \mathfrak{M}$ for $x \in \mathcal{X}_1$
- $\nu(X) \subseteq \mathfrak{M}$ for $X \in \mathcal{X}_2$

$$\begin{array}{lll}
\mathfrak{M},\sigma,\nu \models R_i(x_{j_1},\ldots,x_{j_i}) & \text{if} & (\nu(x_{j_1}),\ldots,\nu(x_{j_i})) \in R_i \\
\mathfrak{M},\sigma,\nu \models x = x' & \text{if} & \nu(x) = \nu(x') \\
\mathfrak{M},\sigma,\nu \models x \in X & \text{if} & \nu(x) \in \nu(X) \\
\mathfrak{M},\sigma,\nu \models \neg\phi & \text{if} & \mathfrak{M},\sigma,\nu \not\models \phi \\
\mathfrak{M},\sigma,\nu \models \phi \wedge \phi' & \text{if} & \mathfrak{M},\sigma,\nu \models \phi \;\wedge\; \mathfrak{M},\sigma,\nu \models \phi' \\
\mathfrak{M},\sigma,\nu \models \exists x.\phi & \text{if} & \exists m \in \mathfrak{M}.\;\; \mathfrak{M},\sigma,\nu[x \mapsto m] \models \phi \\
\mathfrak{M},\sigma,\nu \models \exists X.\phi & \text{if} & \exists M \subseteq \mathfrak{M}.\;\; \mathfrak{M},\sigma,\nu[X \mapsto M] \models \phi
\end{array}$$

Semantics of MSO

Definition

Let $\mathfrak{M}$ a domain, σ a signature, ν a valuation with

- $\nu(x) \in \mathfrak{M}$ for $x \in \mathcal{X}_1$
- $\nu(X) \subseteq \mathfrak{M}$ for $X \in \mathcal{X}_2$

$$
\begin{array}{lll}
\mathfrak{M},\sigma,\nu \models R_i(x_{j_1},\ldots,x_{j_i}) & \text{if} & (\nu(x_{j_1}),\ldots,\nu(x_{j_i})) \in R_i \\
\mathfrak{M},\sigma,\nu \models x = x' & \text{if} & \nu(x) = \nu(x') \\
\mathfrak{M},\sigma,\nu \models x \in X & \text{if} & \nu(x) \in \nu(X) \\
\mathfrak{M},\sigma,\nu \models \neg\phi & \text{if} & \mathfrak{M},\sigma,\nu \not\models \phi \\
\mathfrak{M},\sigma,\nu \models \phi \wedge \phi' & \text{if} & \mathfrak{M},\sigma,\nu \models \phi \;\wedge\; \mathfrak{M},\sigma,\nu \models \phi' \\
\mathfrak{M},\sigma,\nu \models \exists x.\phi & \text{if} & \exists m \in \mathfrak{M}.\;\; \mathfrak{M},\sigma,\nu[x \mapsto m] \models \phi \\
\mathfrak{M},\sigma,\nu \models \exists X.\phi & \text{if} & \exists M \subseteq \mathfrak{M}.\;\; \mathfrak{M},\sigma,\nu[X \mapsto M] \models \phi
\end{array}
$$

We omit $\mathfrak{M},\sigma$ when clear from context.

Recall: Common abbreviations

- $\forall x$, $\forall X$, $\vee$, etc can be expressed in the usual way.
- $X \subseteq Y$:
$$\forall x.(x \in X \to x \in Y)$$
- $Z = X \cup Y$:
$$\forall x.(x \in Z \leftrightarrow x \in X \vee x \in Y)$$
- $Partition(X, X_1, \ldots, X_m)$:
$$\left(\forall x.\left(x \in X \leftrightarrow \bigvee_{i=1}^{m} x \in X_i\right)\right) \wedge \left(\bigwedge_{i=1}^{m} \bigwedge_{j \neq i} \forall x.(x \notin X_i \vee x \notin X_j)\right)$$
- Similarly, $X = \emptyset$, $X = \{x\}$, $X = Y$,...

WSkS and trees

Let $\mathfrak{M} = N^*$, we fix $<_i$ to be the relation $<_i = \{ \langle p, pip' \rangle \mid p, p' \in N^* \}$.
We define $< = \bigcup_{i=1}^{k} <_i$ and $\leq$ as usual, and ε for the minimal element.
We write xi to denote the least q s.t. $\nu(x) <_i q$.

WSkS and trees

Let $\mathfrak{M} = N^*$, we fix $<_i$ to be the relation $<_i = \{ \langle p, pip' \rangle \mid p, p' \in N^* \}$.
We define $< = \bigcup_{i=1}^{k} <_i$ and $\leq$ as usual, and ε for the minimal element.
We write xi to denote the least q s.t. $\nu(x) <_i q$.

Coding of a tree

Let $t \in T(\mathcal{F})$ and k the maximal arity in $\mathcal{F}$.
As a shorthand, define $S_{\mathcal{F}} := (S_f)_{f \in \mathcal{F}}$.
We note $C(t) := (S, S_{\mathcal{F}})$, where:

- $S = \bigcup_{f \in \mathcal{F}} S_f$;
- for all $f \in \mathcal{F}$, $S_f = \{ p \in Pos_t \mid t(p) = f \}$.

WSkS and trees

Let $\mathfrak{M} = N^*$, we fix $<_i$ to be the relation $<_i = \{ \langle p, pip' \rangle \mid p, p' \in N^* \}$.
We define $< = \bigcup_{i=1}^{k} <_i$ and $\leq$ as usual, and ε for the minimal element.
We write xi to denote the least q s.t. $\nu(x) <_i q$.

Coding of a tree

Let $t \in T(\mathcal{F})$ and k the maximal arity in $\mathcal{F}$.
As a shorthand, define $S_{\mathcal{F}} := (S_f)_{f \in \mathcal{F}}$.
We note $C(t) := (S, S_{\mathcal{F}})$, where:

- $S = \bigcup_{f \in \mathcal{F}} S_f$;
- for all $f \in \mathcal{F}$, $S_f = \{ p \in Pos_t \mid t(p) = f \}$.

$(S, S_{\mathcal{F}})$ encodes a tree if $Tree(S, S_{\mathcal{F}})$ holds:

$$\begin{aligned} Tree(S, S_{\mathcal{F}}) \quad := \quad & S \neq \emptyset \wedge Partition(S, S_{\mathcal{F}}) \\ & \wedge \forall x. \forall y. (x \in S \wedge y < x) \rightarrow y \in S \\ & \wedge \bigwedge_{n=1}^{k} \bigwedge_{f \in \mathcal{F}_n} \bigwedge_{i=1}^{n} (x \in S_f \rightarrow xi \in S) \\ & \wedge \bigwedge_{n=1}^{k} \bigwedge_{f \in \mathcal{F}_n} \bigwedge_{i=n+1}^{k} (x \in S_f \rightarrow xi \notin S) \end{aligned}$$

Semantics of WSkS on trees

Coded valuation

Let $\mathcal{F}' := \mathcal{F} \times 2^{\mathcal{X}_1 \cup \mathcal{X}_2}$. The arity of (f, τ) is n if $f \in \mathcal{F}_n$.
Let $t \in T(\mathcal{F})$ and ν a valuation. The tuple $\langle t, \nu \rangle$ is *coded* by a tree $t' \in T(\mathcal{F}')$, as follows, for all $p \in Pos$ and $t'(p) = \langle f, \tau \rangle$:

- if $x \in \mathcal{X}_1$ then $\tau(x) = 1$ iff $p = \nu(x)$;
- if $X \in \mathcal{X}_2$ then $\tau(X) = 1$ iff $p \in \nu(X)$.

A tree $t' \in T(\mathcal{F}')$ is *valid* ($t' \in T_v(\mathcal{F}')$) if it codes some $\langle t, \nu \rangle$.

Semantics of WSkS on trees

Coded valuation

Let $\mathcal{F}' := \mathcal{F} \times 2^{\mathcal{X}_1 \cup \mathcal{X}_2}$. The arity of (f, τ) is n if $f \in \mathcal{F}_n$.
Let $t \in T(\mathcal{F})$ and ν a valuation. The tuple $\langle t, \nu \rangle$ is *coded* by a tree $t' \in T(\mathcal{F}')$, as follows, for all $p \in Pos$ and $t'(p) = \langle f, \tau \rangle$:

- if $x \in \mathcal{X}_1$ then $\tau(x) = 1$ iff $p = \nu(x)$;
- if $X \in \mathcal{X}_2$ then $\tau(X) = 1$ iff $p \in \nu(X)$.

A tree $t' \in T(\mathcal{F}')$ is *valid* ($t' \in T_v(\mathcal{F}')$) if it codes some $\langle t, \nu \rangle$.

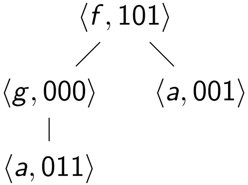
Semantics of WSkS

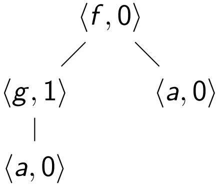
Let ϕ be a formula of WSkS and $V \subseteq (\mathcal{X}_1 \cup \mathcal{X}_2) \uplus (\{S\} \cup S_\mathcal{F})$ its free variables.

$$\mathcal{L}(\phi) := \{ \langle t, \nu \rangle \in T_v(\mathcal{F}') \mid \nu[(S, S_\mathcal{F}) \mapsto C(t)] \models \phi \}$$

Examples

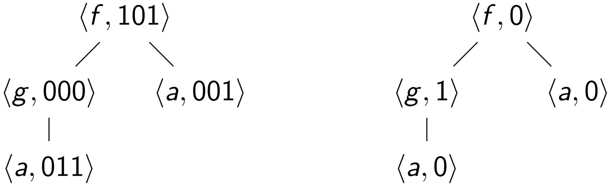
- Let $t = f(g(a), a)$.
  Left: $\langle t, \nu \rangle$ with $\nu(x) = \varepsilon$, $\nu(y) = 11$, and $\nu(Z) = \{\varepsilon, 11, 2\}$.
  Right: $\langle t, \nu' \rangle$ with $\nu'(x) = 1$





Examples

- Let $t = f(g(a), a)$.
  Left: $\langle t, \nu \rangle$ with $\nu(x) = \varepsilon$, $\nu(y) = 11$, and $\nu(Z) = \{\varepsilon, 11, 2\}$.
  Right: $\langle t, \nu' \rangle$ with $\nu'(x) = 1$



- We have $C(t) = (S, S_f, S_g, S_a)$ with $S = \{\varepsilon, 1, 11, 2\}$, $S_f = \{\varepsilon\}$, $S_g = \{1\}$, $S_a = \{11, 2\}$.
- $\nu'[(S, S_{\mathcal{F}}) \mapsto C(t)] \models x \in S_g$, thus $\langle t, \nu' \rangle \in \mathcal{L}(x \in S_g)$
- $t \in \mathcal{L}(\exists x . x \in S_g)$

WSkS and recognizability

Theorem

A tree language $L \subseteq T(\mathcal{F})$ is recognizable
iff $L = \mathcal{L}(\phi)$ for some formula $\phi(S, S_{\mathcal{F}})$ of WSkS.

WSkS and recognizability

Theorem

A tree language $L \subseteq T(\mathcal{F})$ is recognizable
iff $L = \mathcal{L}(\phi)$ for some formula $\phi(S, S_{\mathcal{F}})$ of WSkS.

Proof: (sketch)

- DCFTA $\mathcal{A} \rightarrow$ WSkS: Construct formula ϕ that
  (i) verifies that the structure is a tree;
  (ii) guesses a computation of $\mathcal{A}$, i.e. partitioning of S onto states;
  (iii) verifies that the computation is locally correct;
  (iv) verifies that the root is labelled by an accepting state.

WSkS and recognizability

Theorem

A tree language $L \subseteq T(\mathcal{F})$ is recognizable
iff $L = \mathcal{L}(\phi)$ for some formula $\phi(S, S_{\mathcal{F}})$ of WSkS.

Proof: (sketch)

- DCFTA $\mathcal{A} \to$ WSkS: Construct formula ϕ that
  (i) verifies that the structure is a tree;
  (ii) guesses a computation of $\mathcal{A}$, i.e. partitioning of S onto states;
  (iii) verifies that the computation is locally correct;
  (iv) verifies that the root is labelled by an accepting state.
- WSkS $\phi \to$ NFTA $\mathcal{A}$: Proceed by recurrence on ϕ,
  show that all subformulae of ϕ are recognizable.

Example: DCFTA → WSkS

- Let $Q := \{q_0, q_1, q_f\}$, $\mathcal{F} = \{f(2), g(1), a\}$, $G := \{q_f\}$, and rules
$$a \to q_0 \quad g(q_0) \to q_1 \quad g(q_1) \to q_1 \quad f(q_1, q_1) \to q_f$$
(automate à compléter !)

- Corresponding formula:

$$\begin{aligned}
\phi \quad = \quad & Tree(S, S_{\mathcal{F}}) \\
& \wedge \exists Q_0, Q_1, Q_f. Partition(S, Q_0, Q_1, Q_f) \\
& \qquad \wedge \forall x.(x \in S_a \to x \in Q_0) \\
& \qquad \wedge \forall x.((x \in S_g \wedge x1 \in Q_0) \to x \in Q_1) \\
& \qquad \wedge \forall x.((x \in S_g \wedge x1 \in Q_1) \to x \in Q_1) \\
& \qquad \wedge \forall x.((x \in S_f \wedge x1 \in Q_1 \wedge x2 \in Q_1) \to x \in Q_f) \\
& \qquad \wedge \ \cdots \\
& \qquad \wedge \ \varepsilon \in Q_f
\end{aligned}$$

Example: WSkS $\rightarrow$ NFTA

Consider $\mathcal{F} = \{f(2), g(1), a\}$.

- $\phi = x \in S_g$
  $\mathcal{A}_\phi = \langle \{q, q'\}, \mathcal{F} \times 2^{\{x\}}, \{q'\}, \Delta \rangle$ with transitions

$$\begin{array}{lll} \langle a, 0 \rangle \rightarrow q & & \\ \langle g, 1 \rangle(q) \rightarrow q' & \langle g, 0 \rangle(q) \rightarrow q & \langle g, 0 \rangle(q') \rightarrow q' \\ \langle f, 0 \rangle(q, q) \rightarrow q & \langle f, 0 \rangle(q, q') \rightarrow q' & \langle f, 0 \rangle(q', q) \rightarrow q' \end{array}$$

  accepts $\mathcal{L}(x \in S_g)$ (scans for a single g-position with $\tau(x) = 1$).

Example: WSkS $\to$ NFTA

Consider $\mathcal{F} = \{f(2), g(1), a\}$.

- $\phi = x \in S_g$
  $\mathcal{A}_\phi = \langle \{q, q'\}, \mathcal{F} \times 2^{\{x\}}, \{q'\}, \Delta \rangle$ with transitions

  $$\begin{array}{lll} \langle a, 0\rangle \to q & & \\ \langle g, 1\rangle(q) \to q' & \langle g, 0\rangle(q) \to q & \langle g, 0\rangle(q') \to q' \\ \langle f, 0\rangle(q, q) \to q & \langle f, 0\rangle(q, q') \to q' & \langle f, 0\rangle(q', q) \to q' \end{array}$$

  accepts $\mathcal{L}(x \in S_g)$ (scans for a single g-position with $\tau(x) = 1$).

- $\phi' = \exists x.\phi$
  Obtain $\mathcal{A}_{\phi'}$ from $\mathcal{A}_\phi$ by stripping $\tau(x)$:

  $\mathcal{A}_{\phi'} = \langle \{q, q'\}, \mathcal{F}, \{q'\}, \Delta \rangle$

  $$\begin{array}{lll} a \to q & & \\ g(q) \to q' & g(q) \to q & g(q') \to q' \\ f(q, q) \to q & f(q, q') \to q' & f(q', q) \to q' \end{array}$$

Unranked trees

We now consider *finite ordered unranked* trees.

- *ordered* : internal nodes have children $1 \dots n$
- *unranked* : nodes may have an arbitrary number of children

Unranked trees

We now consider *finite ordered unranked* trees.

- *ordered* : internal nodes have children $1 \ldots n$
- *unranked* : nodes may have an arbitrary number of children

Motivation: e.g., XML documents

- "A `html` tag contains an optional `head` and an obligatory `body`."
- "A `div` tag contains an unlimited number of `p`, `ol`, `ul`, ... tags."

Unranked trees

We now consider *finite ordered unranked* trees.

- *ordered* : internal nodes have children $1 \ldots n$
- *unranked* : nodes may have an arbitrary number of children

Motivation: e.g., XML documents

- "A `html` tag contains an optional `head` and an obligatory `body`."
- "A `div` tag contains an unlimited number of `p`, `ol`, `ul`, . . . tags."

Definition: Tree (recall)

A (finite, ordered) *tree* is a non-empty, finite, prefix-closed set $Pos \subseteq N^*$.

Hedge automata

Definition: (Bottom-up) hedge automaton

A *hedge automaton* (NHA) is a tuple $\mathcal{A} = \langle Q, \Sigma, G, \Delta \rangle$, where:

- Q is a finite set of *states*;
- Σ a finite alphabet;
- $G \subseteq Q$ are the *final states*;
- Δ is a finite set of rules of the form

$$a(R) \to q$$

for $a \in \Sigma$, $q \in Q$, and R a regular (word) language over Q.

Hedge automata

Definition: (Bottom-up) hedge automaton

A *hedge automaton* (NHA) is a tuple $\mathcal{A} = \langle Q, \Sigma, G, \Delta \rangle$, where:

- Q is a finite set of *states*;
- Σ a finite alphabet;
- $G \subseteq Q$ are the *final states*;
- Δ is a finite set of rules of the form

$$a(R) \to q$$

for $a \in \Sigma$, $q \in Q$, and R a regular (word) language over Q.

Example: $Q := \{q_x, q_h, q_b, q_p\}$, $\Sigma = \{x, h, b, p\}$, $G := \{q_x\}$, and rules

$$x(q_h^? q_b) \to q_x \quad h(\varepsilon) \to q_h \quad b(q_p^*) \to q_b \quad p(\varepsilon) \to q_p$$

This accepts trees of the form $x(h, b(p, \ldots, p))$ and $x(b(p, \ldots, p))$.

Semantics of hedge automata

Remark:

- The R in $a(R) \to q$ are called *horizontal languages*.
- They are (finitely) represented by regular expressions or finite automata.

Semantics of hedge automata

Remark:

- The R in $a(R) \to q$ are called *horizontal languages*.
- They are (finitely) represented by regular expressions or finite automata.

Computation of NHA

Let $t \in T(\Sigma)$ be a tree. A *run* or *computation* of $\mathcal{A}$ on t is a tree $t' \in T(Q)$, i.e. for all $p \in Pos$:

- if $t(p) = a \in \Sigma$, $t'(p) = q \in Q$, and $Pos \cap pN = \{p1, \dots, pn\}$, there exists $a(R) \to q \in \Delta$ such that $t'(p1) \cdots t'(pn) \in R$.

Acceptance condition: $t'(\varepsilon) \in G$

Semantics of hedge automata

Remark:

- The R in $a(R) \to q$ are called *horizontal languages*.
- They are (finitely) represented by regular expressions or finite automata.

Computation of NHA

Let $t \in T(\Sigma)$ be a tree. A *run* or *computation* of $\mathcal{A}$ on t is a tree $t' \in T(Q)$, i.e. for all $p \in Pos$:

- if $t(p) = a \in \Sigma$, $t'(p) = q \in Q$, and $Pos \cap pN = \{p1, \ldots, pn\}$, there exists $a(R) \to q \in \Delta$ such that $t'(p1) \cdots t'(pn) \in R$.

Acceptance condition: $t'(\varepsilon) \in G$

$L \subseteq T(\Sigma)$ is called *hedge-recognizable* if $L = \mathcal{L}(\mathcal{A})$ for some NHA $\mathcal{A}$.

Complete / normalized / deterministic HA

An NHA is ...

- *complete* if for all $t \in T(\Sigma)$, $t \rightarrow^*_{\mathcal{A}} q$ for some q;
- *full* if for all $a \in \Sigma$, $q \in Q$, there is some $a(R) \rightarrow q$;
- *reduced* if $a(R_1) \rightarrow q, a(R_2) \rightarrow q \in \Delta$ implies $R_1 = R_2$;
- *deterministic* (DHA) if $a(R_1) \rightarrow q_1, a(R_2) \rightarrow q_2 \in \Delta$ implies $R_1 \cap R_2 = \emptyset$ or $q_1 = q_2$.

Complete / normalized / deterministic HA

An NHA is ...

- *complete* if for all $t \in T(\Sigma)$, $t \rightarrow^*_{\mathcal{A}} q$ for some q;
- *full* if for all $a \in \Sigma$, $q \in Q$, there is some $a(R) \rightarrow q$;
- *reduced* if $a(R_1) \rightarrow q, a(R_2) \rightarrow q \in \Delta$ implies $R_1 = R_2$;
- *deterministic* (DHA) if $a(R_1) \rightarrow q_1, a(R_2) \rightarrow q_2 \in \Delta$ implies $R_1 \cap R_2 = \emptyset$ or $q_1 = q_2$.

Any NHA has an equivalent complete / full / reduced / deterministic NHA.

- complete: add garbage state, as usual
- full: add rules $a(\emptyset) \rightarrow q$ where necessary
- reduced: replace $a(R_1) \rightarrow q$ and $a(R_2) \rightarrow q$ with $a(R_1 \cup R_2) \rightarrow q$ where necessary

Determinization

Determinization of NHA

Let $\mathcal{A} = \langle Q, \Sigma, G, \Delta \rangle$ be a complete, full, reduced NHA. The complete, full, reduced DHA $\mathcal{A}' = \langle 2^Q, \Sigma, G', \Delta' \rangle$ is equivalent to $\mathcal{A}$ where:

- $G' = \{ S \subseteq Q \mid S \cap G \neq \emptyset \}$;
- let $R_{a,q}$ denote the (unique) language s.t. $a(R_{a,q}) \to q \in \Delta$;
- $R'_{a,q} := R_{a,q}[\, q' \to (S \cup \{q'\}) \mid q' \in Q, S \subseteq Q \,]$
- for all $a \in \Sigma$, $S \subseteq Q$, we have $a(R_{a,S}) \to S \in \Delta'$;

$$R_{a,S} := \left(\bigcap_{q \in S} R'_{a,q} \right) \setminus \left(\bigcup_{q \notin S} R'_{a,q} \right)$$

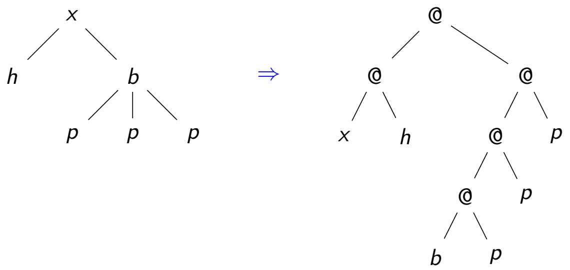
Encoding unranked trees

Bijective encoding of unranked into ranked trees

- Let Σ an alphabet; $\mathcal{F}_\Sigma := \{@(2)\} \cup \{ a(0) \mid a \in \Sigma \}$.
- Define the coding $C_@(t) \in T(\mathcal{F}_\Sigma)$ of $t \in T(\Sigma)$ as

$$C_@(a(t_1, \dots, t_n)) = \underbrace{@(@(\dots(@(}_{n} a, C_@(t_1)), C_@(t_2)), \dots), C_@(t_n))$$

Example:



Recognizing encoded trees

Theorem

A language $L \subseteq T(\Sigma)$ is hedge-recognizable iff $C_{@}(L)$ is recognizable.

- NHA $\rightarrow$ NFTA:
  Let $\mathcal{A} = \langle Q, \Sigma, G, \Delta \rangle$ an NHA; $\Delta = \{a_1(R_1) \rightarrow q_1, \ldots, a_n(R_n) \rightarrow q_n\}$;
  R_i represented by det.compl. FA $\mathcal{A}_i = \langle S_i, Q, s_0^{(i)}, F_i, \delta_i \rangle$.

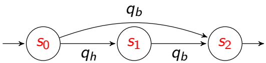
  Construct NFTA $\mathcal{A}' = \langle Q', \mathcal{F}_\Sigma, G, \Delta' \rangle$, where:
  - $Q' = Q \cup \biguplus_{i=1}^{n} S_i$
  - $\Delta' = \bigcup_{i=1}^{n} (\Delta_1^i \cup \Delta_2^i \cup \Delta_3^i)$

$$
\begin{aligned}
\Delta_1^i &= \{\, a_i \rightarrow s_0^{(i)} \,\} \\
\Delta_2^i &= \{\, @(s, q) \rightarrow \delta_i(s, q) \mid s \in S_i, q \in Q \,\} \\
\Delta_3^i &= \{\, s_f \rightarrow q_i \mid s_f \in F_i \,\}
\end{aligned}
$$

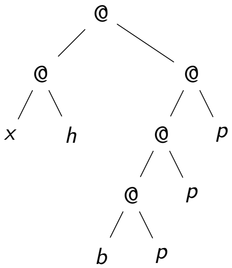
Example: NHA $\to$ NFTA

- $Q := \{q_x, q_h, q_b, q_p\}$, $\Sigma = \{x, h, b, p\}$, $G := \{q_x\}$, and rules

$$x(q_h^? q_b) \to q_x \quad h(\varepsilon) \to q_h \quad b(q_p^*) \to q_b \quad p(\varepsilon) \to q_p$$

- Automaton for first rule: 

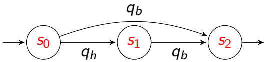
- Single-state automata with s_h, s_b, s_p for the other rules



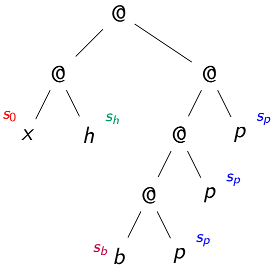
Example: NHA $\to$ NFTA

- $Q := \{q_x, q_h, q_b, q_p\}$, $\Sigma = \{x, h, b, p\}$, $G := \{q_x\}$, and rules

$$x(q_h^? q_b) \to q_x \quad h(\varepsilon) \to q_h \quad b(q_p^*) \to q_b \quad p(\varepsilon) \to q_p$$

- Automaton for first rule: 

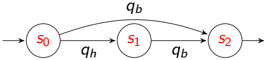
- Single-state automata with s_h, s_b, s_p for the other rules



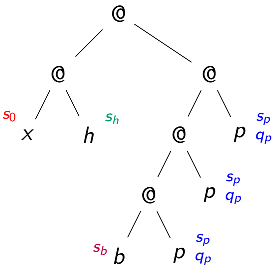
Example: NHA $\to$ NFTA

- $Q := \{q_x, q_h, q_b, q_p\}$, $\Sigma = \{x, h, b, p\}$, $G := \{q_x\}$, and rules

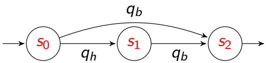
$$x(q_h^? q_b) \to q_x \quad h(\varepsilon) \to q_h \quad b(q_p^*) \to q_b \quad p(\varepsilon) \to q_p$$

- Automaton for first rule: 

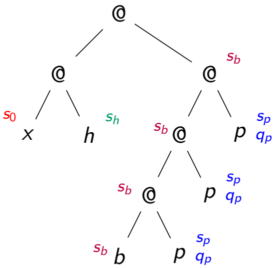
- Single-state automata with s_h, s_b, s_p for the other rules



Example: NHA $\to$ NFTA

- $Q := \{q_x, q_h, q_b, q_p\}$, $\Sigma = \{x, h, b, p\}$, $G := \{q_x\}$, and rules
  $x(q_h^? q_b) \to q_x \quad h(\varepsilon) \to q_h \quad b(q_p^*) \to q_b \quad p(\varepsilon) \to q_p$
- Automaton for first rule: 

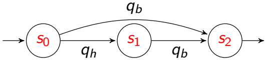
- Single-state automata with s_h, s_b, s_p for the other rules



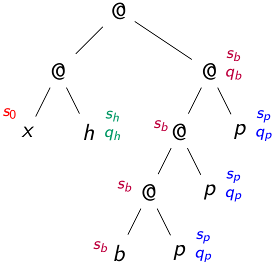
Example: NHA $\to$ NFTA

- $Q := \{q_x, q_h, q_b, q_p\}$, $\Sigma = \{x, h, b, p\}$, $G := \{q_x\}$, and rules

$$x(q_h^? q_b) \to q_x \quad h(\varepsilon) \to q_h \quad b(q_p^*) \to q_b \quad p(\varepsilon) \to q_p$$

- Automaton for first rule: 

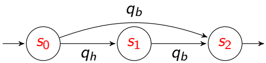
- Single-state automata with s_h, s_b, s_p for the other rules



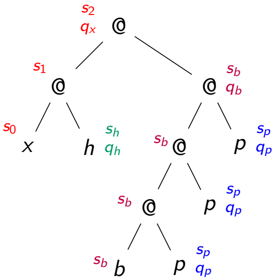
Example: NHA $\to$ NFTA

- $Q := \{q_x, q_h, q_b, q_p\}$, $\Sigma = \{x, h, b, p\}$, $G := \{q_x\}$, and rules

$$x(q_h^? q_b) \to q_x \quad h(\varepsilon) \to q_h \quad b(q_p^*) \to q_b \quad p(\varepsilon) \to q_p$$

- Automaton for first rule: 

- Single-state automata with s_h, s_b, s_p for the other rules



Recognizing encoded trees

Theorem

A language $L \subseteq T(\Sigma)$ is hedge-recognizable iff $C_@(L)$ is recognizable.

- NFTA $\to$ NHA:
  Let $\mathcal{A} = \langle Q, \mathcal{F}_\Sigma, G, \Delta \rangle$ an NFTA (without ε-moves).

  Define $\Delta_R := \{ \langle q_0, q_1, q_2 \rangle \mid @(q_0, q_1) \to_\Delta q_2 \}$
  and $Out := G \cup \{ q \mid \exists q', q'' : @(q', q) \to_\Delta q'' \}$.
  For $q \in Q, q' \in Out$, let $A_{q,q'} := \langle Q, Q, q, \{q'\}, \Delta_R \rangle$ a word automaton.

  Construct NHA $\mathcal{A}' := \langle Q, \Sigma, G, \Delta' \rangle$, where

$$\Delta' = \{ a(\mathcal{L}(\mathcal{A}_{q,q'})) \to q' \mid a \to_\Delta q, q' \in Out \}$$

Recognizing encoded trees

Theorem

A language $L \subseteq T(\Sigma)$ is hedge-recognizable iff $C_{@}(L)$ is recognizable.

- NFTA $\to$ NHA:
  Let $\mathcal{A} = \langle Q, \mathcal{F}_\Sigma, G, \Delta \rangle$ an NFTA (without ε-moves).

  Define $\Delta_R := \{ \langle q_0, q_1, q_2 \rangle \mid @(q_0, q_1) \to_\Delta q_2 \}$
  and $Out := G \cup \{ q \mid \exists q', q'' : @(q', q) \to_\Delta q'' \}$.
  For $q \in Q, q' \in Out$, let $A_{q,q'} := \langle Q, Q, q, \{q'\}, \Delta_R \rangle$ a word automaton.

  Construct NHA $\mathcal{A}' := \langle Q, \Sigma, G, \Delta' \rangle$, where
  $$\Delta' = \{ a(\mathcal{L}(\mathcal{A}_{q,q'})) \to q' \mid a \to_\Delta q, q' \in Out \}$$

Corollary

Hedge-recognizable languages are closed under boolean operations.

Unranked trees and logic

UTL = weak MSO(*child*,*next*) interpreted over $\mathfrak{M} = N^*$, where

- $child(x, y)$ iff $y = xi$ for some $i \in N$
- $next(x, y)$ iff $\exists z, i : x = zi \wedge y = z(i + 1)$

Unranked trees and logic

UTL = weak MSO(*child*,*next*) interpreted over $\mathfrak{M} = N^*$, where

- $child(x, y)$ iff $y = xi$ for some $i \in N$
- $next(x, y)$ iff $\exists z, i : x = zi \wedge y = z(i + 1)$

Further predicates can be defined from this:

- $right(x, y)$ = "y is a right sibling of x"
- $desc(x, y)$ = "y is a descendant of x" = "$x \leq y$"

Unranked trees and logic

UTL = weak MSO(*child*,*next*) interpreted over $\mathfrak{M} = N^*$, where

- $child(x, y)$ iff $y = xi$ for some $i \in N$
- $next(x, y)$ iff $\exists z, i : x = zi \wedge y = z(i + 1)$

Further predicates can be defined from this:

- $right(x, y)$ = "y is a right sibling of x"
- $desc(x, y)$ = "y is a descendant of x" = "$x \leq y$"

Notions like $\mathcal{L}(\phi)$ are defined in analogy with WSkS.

Theorem: UTL = NHA

A language $L \subseteq T(\Sigma)$ is hedge-recognizable
iff $L = \mathcal{L}(\phi)$ for some formula $\phi(S, S_\Sigma)$ of UTL.

UTL = NHA: Proof sketch

- UTL $\rightarrow$ NHA:
  Let ϕ be an UTL formula. Define ϕ' of WS2S s.t. $\mathcal{L}(\phi') = C_{@}(\mathcal{L}(\phi))$.

UTL = NHA: Proof sketch

- UTL $\to$ NHA:
  Let ϕ be an UTL formula. Define ϕ' of WS2S s.t. $\mathcal{L}(\phi') = C_{@}(\mathcal{L}(\phi))$.

  Define *leftmost*(x, y) as

$$\begin{aligned} \forall X : \quad & \big(x \in X \wedge \forall z, z' : (z \in X \wedge z' = z1 \to z' \in X) \\ & \quad \wedge \forall z : (z \in X \to z = x \vee (\exists z' : z' \in X \wedge z = z'1))\big) \\ & \to \big(y \in X \wedge \forall z : z \in X \to z \leq y\big) \end{aligned}$$

  ("y is the maximal position in $x1^*$")

UTL = NHA: Proof sketch

- UTL $\rightarrow$ NHA:
  Let ϕ be an UTL formula. Define ϕ' of WS2S s.t. $\mathcal{L}(\phi') = C_{@}(\mathcal{L}(\phi))$.

  Define *leftmost*(x, y) as

$$\begin{aligned} \forall X : \quad & \big(x \in X \wedge \forall z, z' : (z \in X \wedge z' = z1 \rightarrow z' \in X) \\ & \qquad \wedge \forall z : (z \in X \rightarrow z = x \vee (\exists z' : z' \in X \wedge z = z'1))\big) \\ & \quad \rightarrow \big(y \in X \wedge \forall z : z \in X \rightarrow z \leq y\big) \end{aligned}$$

  ("y is the maximal position in $x1^*$")

  Then *child* and *next* can be translated as follows:

$$\begin{aligned} \mathit{child}(x, y) &:= \exists z : \mathit{leftmost}(z, x) \wedge \mathit{leftmost}(z2, y) \\ \mathit{next}(x, y) &:= \exists z : \mathit{leftmost}(z12, x) \wedge \mathit{leftmost}(z2, y) \end{aligned}$$

UTL = NHA: Proof sketch

- NHA $\to$ UTL:
  Let $\mathcal{A}$ be a complete, full, normalized, deterministic NHA.

  Construct formula $\phi(S, S_\Sigma)$ of UTL that
  (i) verifies that the structure is a tree;
  (ii) guesses a computation of $\mathcal{A}$, i.e. partitioning of S onto states;
  (iii) verifies that the computation is locally correct;
  (iv) verifies that the root is labelled by an accepting state.

UTL = NHA: Proof sketch

- NHA $\rightarrow$ UTL:
  Let $\mathcal{A}$ be a complete, full, normalized, deterministic NHA.

  Construct formula $\phi(S, S_\Sigma)$ of UTL that
  (i) verifies that the structure is a tree;
  (ii) guesses a computation of $\mathcal{A}$, i.e. partitioning of S onto states;
  (iii) verifies that the computation is locally correct;
  (iv) verifies that the root is labelled by an accepting state.

  The major difference with the NFTA $\rightarrow$ WSkS construction is (iii):
  (iii): whenever the computation puts q on an a-labelled position p, guess a run of the automaton for $R_{a,q}$ over p and its children

Tuples of trees

Let $t_1, t_2 \in T(\mathcal{F})$ ranked trees. Add a fresh symbol $-$ to $\mathcal{F}_0$ and let

$$\mathcal{F}' := \{ \langle f, g \rangle(k) \mid f \in \mathcal{F}_m, g \in \mathcal{F}_n, k = \max\{m, n\} \}.$$

$\langle t_1, t_2 \rangle$ denotes the ranked tree $t \in T(\mathcal{F}')$ as follows:

- $Pos_t = Pos_{t_1} \cup Pos_{t_2}$
- for all $p \in Pos_t$,

$$t(p) = \begin{cases} \langle f, g \rangle & \text{if } t \in Pos_{t_1} \cap Pos_{t_2}, t_1(p) = f, t_2(p) = g \\ \langle f, - \rangle & \text{if } t \in Pos_{t_1} \setminus Pos_{t_2}, t_1(p) = f \\ \langle -, g \rangle & \text{if } t \in Pos_{t_2} \setminus Pos_{t_1}, t_2(p) = g \end{cases}$$

Tuples of trees

Let $t_1, t_2 \in T(\mathcal{F})$ ranked trees. Add a fresh symbol $-$ to $\mathcal{F}_0$ and let

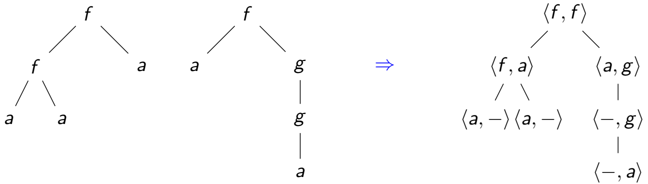
$$\mathcal{F}' := \{\, \langle f, g\rangle(k) \mid f \in \mathcal{F}_m, g \in \mathcal{F}_n, k = \max\{m, n\}\,\}.$$

$\langle t_1, t_2\rangle$ denotes the ranked tree $t \in T(\mathcal{F}')$ as follows:

- $Pos_t = Pos_{t_1} \cup Pos_{t_2}$
- for all $p \in Pos_t$,

$$t(p) = \begin{cases} \langle f, g\rangle & \text{if } t \in Pos_{t_1} \cap Pos_{t_2}, t_1(p) = f, t_2(p) = g \\ \langle f, -\rangle & \text{if } t \in Pos_{t_1} \setminus Pos_{t_2}, t_1(p) = f \\ \langle -, g\rangle & \text{if } t \in Pos_{t_2} \setminus Pos_{t_1}, t_2(p) = g \end{cases}$$

Example:



Tree relations

We consider (binary) relations $R \subseteq T(\mathcal{F})^2$.

- Let $\mathfrak{R}_2$ be the class of recognizable relations (= recognizable languages over $\mathcal{F}'$).
- Let $\mathfrak{X}_2$ be the class of *finite unions of cross products* $R \in \mathfrak{X}_2$ iff $R = \bigcup_{i=1}^{n} \left(L_1^{(i)} \times L_2^{(i)} \right)$, for some $n \geq 0$ and $L_1^{(i)}, L_2^{(i)}$ recognizable for all i
- Let $\mathfrak{T}_2$ be the class of relations recognizable by GTT.

Tree relations

We consider (binary) relations $R \subseteq T(\mathcal{F})^2$.

- Let $\mathfrak{R}_2$ be the class of recognizable relations (= recognizable languages over $\mathcal{F}'$).
- Let $\mathfrak{X}_2$ be the class of *finite unions of cross products* $R \in \mathfrak{X}_2$ iff $R = \bigcup_{i=1}^{n} \left(L_1^{(i)} \times L_2^{(i)} \right)$, for some $n \geq 0$ and $L_1^{(i)}, L_2^{(i)}$ recognizable for all i
- Let $\mathfrak{T}_2$ be the class of relations recognizable by GTT.

Definition: Ground Tree Transducer

A *ground tree transducer* (GTT) is pair $\mathcal{G} = \langle \mathcal{A}_1, \mathcal{A}_2 \rangle$ of bottom-up NFTA over $\mathcal{F}$. (The states of $\mathcal{A}_1$ and $\mathcal{A}_2$ may overlap.)
The relation accepted by $\mathcal{G}$ is

$$\begin{aligned} \{ \langle t, u \rangle \quad | \quad & \exists n \geq 0,\ C \in \mathcal{C}^n(\mathcal{F}), \\ & \quad t_1, \ldots, t_n \in T(\mathcal{F}),\ u_1, \ldots, u_n \in T(\mathcal{F}),\ q_1, \ldots, q_n : \\ & t = C[t_1, \ldots, t_n] \wedge u = C[u_1, \ldots, u_n] \\ & \quad \wedge \forall i : t_i \rightarrow^*_{\mathcal{A}_1} q_i \ {}_{\mathcal{A}_2}^{\ *}\!\leftarrow u_i \} \end{aligned}$$

Relations between $\mathfrak{R}_2, \mathfrak{X}_2, \mathfrak{T}_2$

Propositions

1. $\mathfrak{R}_2 \not\subseteq \mathfrak{X}_2$ and $\mathfrak{T}_2 \not\subseteq \mathfrak{X}_2$
2. $\mathfrak{R}_2 \not\subseteq \mathfrak{T}_2$ and $\mathfrak{X}_2 \not\subseteq \mathfrak{T}_2$
3. $\mathfrak{X}_2 \subseteq \mathfrak{R}_2$
4. $\mathfrak{T}_2 \subseteq \mathfrak{R}_2$
5. $\mathfrak{X}_2 \cup \mathfrak{T}_2 \subsetneq \mathfrak{R}_2$

Proofs:

1. $\{ \langle t, t \rangle \mid t \in T(\mathcal{F}) \}$ is in $\mathfrak{T}_2 \cap \mathfrak{R}_2$ but not $\mathfrak{X}_2$
2. $\emptyset$ is in $\mathfrak{X}_2 \cap \mathfrak{R}_2$ but not $\mathfrak{T}_2$
3. see next slides
4. see next slides
5. see next slides

Proof of $\mathfrak{X}_2 \subseteq \mathfrak{R}_2$

3. Let $A_i = \langle Q_i, \mathcal{F}, G_i, \Delta_i \rangle$ (for $i = 1, 2$) be NFTA and let $R = \mathcal{L}(\mathcal{A}_1) \times \mathcal{L}(\mathcal{A}_2) \in \mathfrak{X}_2$.

   Construct NFTA $\mathcal{A} = \langle Q, \mathcal{F}', G_1 \times G_2, \Delta \rangle$ with $\mathcal{L}(\mathcal{A}) = R$:

   - $Q = (Q_1 \cup \{-\}) \times (Q_2 \cup \{-\})$
   - for every $f \in \mathcal{F}_m$, $g \in \mathcal{F}_n$, $m \geq n$, $\neg(f = g = -)$
     Δ contains
     - $\langle f, g \rangle(\langle q_1, q_1' \rangle, \ldots, \langle q_n, q_n' \rangle, \langle q_{n+1}, - \rangle, \ldots, \langle q_m, - \rangle) \to \langle q, q' \rangle$ if $f(q_1, \ldots, q_m) \to q \in \Delta_1$ and $g(q_1', \ldots, q_n') \to q' \in \Delta_2$
     - $\langle g, f \rangle(\langle q_1, q_1' \rangle, \ldots, \langle q_n, q_n' \rangle, \langle -, q_{n+1}' \rangle, \ldots, \langle -, q_m \rangle) \to \langle q, q' \rangle$ if $f(q_1', \ldots, q_m') \to q \in \Delta_2$ and $g(q_1, \ldots, q_n) \to q' \in \Delta_1$

   (reminder: we assume that $-$ is a fresh symbol in $\mathcal{F}_0$)

   Intuition: Modified cross-product construction.

Proof of $\mathfrak{T}_2 \subseteq \mathfrak{R}_2$

4. Let $\mathcal{G} = \langle \mathcal{A}_1, \mathcal{A}_2 \rangle$, $\mathcal{A}_i = \langle Q_i, \mathcal{F}, G_i, \Delta_i \rangle$ (for $i = 1, 2$).
   We construct NFTA $\mathcal{A}' = \langle Q', \mathcal{F}', \{q_f\}, \Delta' \rangle$ with $\mathcal{L}(\mathcal{A}') = \mathcal{L}(\mathcal{G})$.

   Construct NFTA $\mathcal{A} = \langle Q, \mathcal{F}', G, \Delta \rangle$ from $\mathcal{A}_1, \mathcal{A}_2$ as in previous proof.
   Then:

   - $Q' = Q \uplus \{q_f\}$
   - $\Delta' = \Delta \cup \Delta_1 \cup \Delta_2$
     $\Delta_1 = \{ \langle q, q \rangle \to q_f \mid q \in Q_1 \cap Q_2 \}$
     $\Delta_2 = \{ \langle f, f \rangle(q_f, \ldots, q_f) \to q_f \mid f \in \mathcal{F}_n,\ f \neq - \}$

   Intuition:
   Δ reads pairs of trees from $\mathcal{A}_1, \mathcal{A}_2$;
   Δ_1 allows to plug pairs of subtrees into some context C;
   Δ_2 reads the remaining context C.

Proof of $\mathfrak{X}_2 \cup \mathfrak{T}_2 \subsetneq \mathfrak{R}_2$

5. Let $\mathcal{F} = \{f(1), g(1), a\}$.
   Let $R = \{\, \langle t_1, t_2 \rangle \mid \exists C \in \mathcal{C}(\mathcal{F}), t \in T(\mathcal{F}) : t_1 = C[t] \wedge t_2 = C[f(t)] \,\}$.
   - $R \notin \mathfrak{X}_2$:
     By pigeonhole principle using $\langle f^i(a), f^{i+1}(a) \rangle$, $i \geq 0$.

Proof of $\mathfrak{X}_2 \cup \mathfrak{T}_2 \subsetneq \mathfrak{R}_2$

5. Let $\mathcal{F} = \{f(1), g(1), a\}$.
   Let $R = \{\, \langle t_1, t_2 \rangle \mid \exists C \in \mathcal{C}(\mathcal{F}), t \in T(\mathcal{F}) : t_1 = C[t] \wedge t_2 = C[f(t)] \,\}$.
   - $R \notin \mathfrak{X}_2$:
     By pigeonhole principle using $\langle f^i(a), f^{i+1}(a) \rangle$, $i \geq 0$.
   - $R \notin \mathfrak{T}_2$:
     Suppose that R is accepted by GTT $\langle \mathcal{A}_1, \mathcal{A}_2 \rangle$ with n states in common.
     For all $i \geq 0$, let q_i such that $g^i(a) \rightarrow^*_{\mathcal{A}_1} q_i$ and $f(g^i(a)) \rightarrow^*_{\mathcal{A}_2} q_i$.
     Contradiction follows from pigeon-hole principle.

Proof of $\mathfrak{X}_2 \cup \mathfrak{T}_2 \subsetneq \mathfrak{R}_2$

5. Let $\mathcal{F} = \{f(1), g(1), a\}$.
   Let $R = \{\, \langle t_1, t_2 \rangle \mid \exists C \in \mathcal{C}(\mathcal{F}), t \in T(\mathcal{F}) : t_1 = C[t] \wedge t_2 = C[f(t)] \,\}$.
   - $R \notin \mathfrak{X}_2$:
     By pigeonhole principle using $\langle f^i(a), f^{i+1}(a) \rangle$, $i \geq 0$.
   - $R \notin \mathfrak{T}_2$:
     Suppose that R is accepted by GTT $\langle \mathcal{A}_1, \mathcal{A}_2 \rangle$ with n states in common.
     For all $i \geq 0$, let q_i such that $g^i(a) \to^*_{\mathcal{A}_1} q_i$ and $f(g^i(a)) \to^*_{\mathcal{A}_2} q_i$.
     Contradiction follows from pigeon-hole principle.
   - $R \in \mathfrak{R}_2$:
     Let $\mathcal{A} = \langle \{q_a, q_f, q_g, q\}, \mathcal{F}', \{q\}, \Delta \rangle$ with:

     $$\langle -, a \rangle \to q_a \quad \langle x, y \rangle (q_x) \to q_y \quad q_f \to q \quad \langle x, x \rangle (q) \to q$$

     for $x, y \in \{f, g, a\}$

Closure properties

Boolean closure

$\mathfrak{X}_2$ and $\mathfrak{R}_2$ are closed under boolean operations.

Transitive closure

If $R \in \mathfrak{T}_2$, then $R^* \in \mathfrak{T}_2$.

Closure properties

Boolean closure

$\mathfrak{X}_2$ and $\mathfrak{R}_2$ are closed under boolean operations.

Transitive closure

If $R \in \mathfrak{T}_2$, then $R^* \in \mathfrak{T}_2$.

Proof: Let $\langle \mathcal{A}_1, \mathcal{A}_2 \rangle$ with states Q_1, Q_2 a GTT accepting R.
We construct $\langle \mathcal{B}_1, \mathcal{B}_2 \rangle$ accepting R^* by adding transitions to $\mathcal{A}_1$ and $\mathcal{A}_2$ using the following saturation rule:

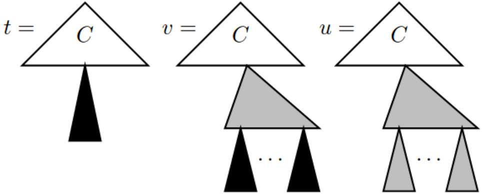
- For $i \neq j$ and all $q \in Q_1 \cap Q_2$, $q' \in Q_j$, if there exists a tree t s.t.

  $$t \rightarrow^*_{\mathcal{B}_i} q \quad \text{and} \quad t \rightarrow^*_{\mathcal{B}_j} q'$$

  then add $q \rightarrow q'$ to $\mathcal{B}_j$.

Transitive closure: Intuition

Suppose that $\langle t, v\rangle, \langle v, u\rangle \in R$. The interesting case is illustrated below:

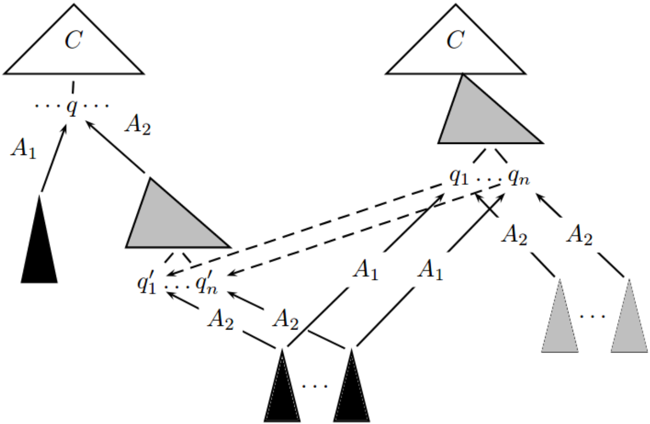


Suppose that $\langle t, v\rangle$ differ in a position p
and $\langle v, u\rangle$ in positions $pp_1, \ldots, pp_n$.

Then in $\mathcal{A}_2$ we want the subtrees of u at $pp_1, \ldots, pp_n$ to be substitutable for the corresponding subtrees in v.

Transitive closure: Intuition

Consider the runs of t, v, u in $\langle \mathcal{A}_1, \mathcal{A}_2 \rangle$:



Adding $q_i \to q'_i$ to the right-hand side automaton achieves the objective.

Transitive closure: $R^* \subseteq \mathcal{L}(\langle \mathcal{B}_1, \mathcal{B}_2 \rangle)$

Proof by induction: Let $\langle t, u \rangle \in R^i$, for $i \geq 0$.

- $i = 0$: trivial
- $i \to i + 1$: Let v s.t. $\langle t, v \rangle \in R^i$ and $\langle v, u \rangle \in R$.
  Then (by induction) $\langle t, v \rangle$ is accepted by $\langle \mathcal{B}_1, \mathcal{B}_2 \rangle$.
  Let P be the positions in which $\langle t, v \rangle$ differ
  and P' be the positions in which $\langle v, u \rangle$ differ.
  All incomparable pairs in $P \times P'$ are handled by the definition of GTT.
  For $p \in P$ and $pp1, \ldots, pp_n \in P'$ consider the previous drawings.
  The case $pp1, \ldots, pp_n \in P$ and $p \in P'$ is symmetric.

Transitive closure: $R^* \supseteq \mathcal{L}(\langle \mathcal{B}_1, \mathcal{B}_2 \rangle)$

Let $\langle \mathcal{B}_1^i, \mathcal{B}_2^i \rangle$ denote the GTT after adding i transitions and show that its language is included in R^*.

- $i = 0$: trivial
- $i \to i+1$: Let $q \to q'$ be the transition added in the $(i+1)$-th step (to $\mathcal{B}_1$, say) and let $q \to q'$ be used j times in accepting some $\langle t, u \rangle$.

  If $j = 0$, then $\langle t, u \rangle \in R^*$ by induction hypothesis. Otherwise:

  1. there exist $n \geq 0$, $C \in \mathcal{C}^n(\mathcal{F})$ etc such that $t = C[t_1, \ldots, t_n]$, $u = C[u_1, \ldots, u_n]$ and $\forall k : t_k \to^*_{\mathcal{B}_1^{i+1}} q_k \;{}_{\mathcal{B}_2^{i+1}}\!\!{}^* \leftarrow u_k$.
  2. Suppose $t_k = C'[t'] \to^*_{\mathcal{B}_1^{i+1}} C'[q] \to C'[q'] \to^*_{\mathcal{B}_1^{i+1}} q_k$ for some k, C', t'.
  3. There must be some $v \in T(\mathcal{F})$ with $v \to^*_{\mathcal{B}_2^i} q$ and $v \to^*_{\mathcal{B}_1^i} q'$.
  4. From (2) et (3) we have $C'[v] \to^*_{\mathcal{B}_1^{i+1}} q_k$.
  5. Replacing t_k by $C'[v]$ in (1) we get $\langle t[t'/v], u \rangle \in \mathcal{L}(\langle \mathcal{B}_1^{i+1}, \mathcal{B}_2^{i+1} \rangle)$ with fewer than j times $q \to q'$, thus by ind.hyp. $\langle t[t'/v], u \rangle \in R^*$.
  6. From (2) and (3), $t' \to^*_{\mathcal{B}_1^{i+1}} q \;{}_{\mathcal{B}_2^{i}}\!\!{}^* \leftarrow v$, with fewer than j times $q \to q'$.
  7. From (6) by ind.hyp. $\langle t, t[t'/v] \rangle \in R^*$.

Application: XML

XML = Extensible Markup Language

- Conceived for platform-independent exchange of *structured data*
- An XML document consists of *tags* with *attributes* and text (parsed character data, *pcdata*)

Application: XML

XML = Extensible Markup Language

- Conceived for platform-independent exchange of *structured data*
- An XML document consists of *tags* with *attributes* and text (parsed character data, *pcdata*)

Example:

```
<html><head><meta charset="UTF-8"/>
<title>My web page</title></head>
<body><p>Bonne ann&eacute;e !</p></body></html>
```

Application: XML

XML = Extensible Markup Language

- Conceived for platform-independent exchange of *structured data*
- An XML document consists of *tags* with *attributes* and text (parsed character data, *pcdata*)

Example:

```
<html><head><meta charset="UTF-8"/>
<title>My web page</title></head>
<body><p>Bonne ann&eacute;e !</p></body></html>
```

- A *well-formed* XML document forms a tree (balanced tags, one single root tag)
- Testing for well-formedness / generating tree from document: visibly pushdown automaton, LL/LR parser

Valid XML documents

- Languages of XML documents defined by *schemas* (DTD, XML Schema, Relax NG)
- Schemas define permissible tags (+attributes) and their nesting
- Examples of XML languages: HTML, SVG, KML, ...

Valid XML documents

- Languages of XML documents defined by *schemas*
  (DTD, XML Schema, Relax NG)
- Schemas define permissible tags (+attributes) and their nesting
- Examples of XML languages: HTML, SVG, KML, ...

- *Valid* XML document: well-formed document satisfying a schema
- Example: XML-Schema for KML

DTD for XML

DTD = Document Type Definition

DTD define a (restricted) subclass of XML languages.
Essentially, defines a regular language of child tags for each tag type.

Example (from Wikipedia):

```
<!ELEMENT html (head,body)>
<!ELEMENT hr EMPTY>
<!ELEMENT div (#PCDATA | p | ul | | table | pre | hr |
          h1|h2|h3|h4|h5|h6 | blockquote | ...)*>
<!ELEMENT dl (dt|dd)+>
```

Validity checking of DTD

The language of XML documents defined by DTD is accepted by NHA.

Restrictions on DTD

Expressivity of DTD

There are hedge-recognizable languages that cannot be defined by DTD.

Example: $\{f(g(a)), f'(g(b))\}$

Restrictions on DTD

Expressivity of DTD

There are hedge-recognizable languages that cannot be defined by DTD.

Example: $\{f(g(a)), f'(g(b))\}$

DTD contain another restriction:

> *It is an error if the content model allows an element to match more than one occurrence of an element type in the content model.*

Restrictions on DTD

Expressivity of DTD

There are hedge-recognizable languages that cannot be defined by DTD.

Example: $\{f(g(a)), f'(g(b))\}$

DTD contain another restriction:

> *It is an error if the content model allows an element to match more than one occurrence of an element type in the content model.*

E.g., $(ab|ac)$ is not allowed (but $a(b|c)$ is).

Deterministic regular expressions

Definition: Marked RE

Let e be a RE over Σ. The *marked RE* $\bar{e}$ is a RE over $\Sigma \times \mathbb{N}$ obtained by adding a unique subscript to each letter in e.

Example: $e = (ab|ac)$, then $\bar{e} = (a_1 b_2 | a_3 c_4)$

Deterministic regular expressions

Definition: Marked RE

Let e be a RE over Σ. The *marked RE* $\bar{e}$ is a RE over $\Sigma \times \mathbb{N}$ obtained by adding a unique subscript to each letter in e.

Example: $e = (ab|ac)$, then $\bar{e} = (a_1 b_2 | a_3 c_4)$

Definition: Deterministic RE

Let e a RE over Σ. We call e *deterministic* if $\bar{e}$ satisfies the following: for all $u, v, w \in (\Sigma \times \mathbb{N})^*$ and $a \in \Sigma$, if $ua_i v, ua_j w \in L(\bar{e})$ then $i = j$.

Example: $e = (ab|ac)$, $\bar{e} = (a_1 b_2 | a_3 c_4)$, not deterministic because $a_1 b_2, a_3 c_4 \in L(\bar{e})$

Parsing deterministic RE

Parsing det. RE

Let e be a deterministic RE. A DFA for e can be constructed in polynomial (linear) time. [Brüggemann-Klein 1993, Groz et al 2012]

Proof (sketch): Construction of Glushkov automaton from e.

Parsing deterministic RE

Parsing det. RE

Let e be a deterministic RE. A DFA for e can be constructed in polynomial (linear) time. [Brüggemann-Klein 1993, Groz et al 2012]

Proof (sketch): Construction of Glushkov automaton from e.

Expressivity of det. RE

Not every regular language can be defined by a deterministic RE.

XML Schema

XML Schema can define more expressive XML languages.
Example:

```
<xsd:complexType name="track">
 <xsd:sequence minOccurs="1" maxOccurs="unbounded">
  <xsd:choice>
   <xsd:element name="invSession" type="invSession"
     minOccurs="1" maxOccurs="1"/>
   <xsd:element name="conSession" type="conSession"
     minOccurs="1" maxOccurs="1"/>
  </xsd:choice>
  <xsd:element name="break" type="xsd:string"
    minOccurs="0" maxOccurs="1"/>
 </xsd:sequence>
</xsd:complexType>
```

XML Schema and Hedge Automata

XML Schema = NHA

XML Schema (restricted to occurrence and nesting conditions) correspond to the class of hedge-recognizable languages.

Moreover, XML Schema also permit non-hedge-recognizable features:

- constraints on data types in attributes and pcdata
- consistency constraints (e.g., unique keys)

XSL Transformation

- XSLT allows to transform XML documents into other documents (incl. non XML)
- XQuery used to specify nodes on which to apply a transformation

Example (from Wikipedia):

```
<xsl:template match="//title">
  <em>
    <xsl:apply-templates/>
  </em>
</xsl:template>
<xsl:for-each select="book">
  <xsl:sort select="price" order="ascending" />
</xsl:for-each>
```

Tree transducers

Definition: Bottom-up tree transducer

A *(finite bottom-up) tree transducer* (NUTT) is a tuple $\mathcal{U} = \langle Q, \mathcal{F}, \mathcal{F}', G, \Delta \rangle$, where:

- Q is a finite set of *states* and $G \subseteq Q$ are *final* states;
- $\mathcal{F}, \mathcal{F}'$ are finite ranked alphabets;
- Δ is a finite set of rules of the form

  $$f(q_1(x_1), \ldots, q_n(x_n)) \to q(u)$$

  for $f \in \mathcal{F}_n$ and $q, q_1, \ldots, q_n \in Q$, $u \in T(\mathcal{F}', \mathcal{X}_n)$, or

  $$q(x_1) \to q'(u)$$

  for $q, q' \in Q$, $u \in T(\mathcal{F}', \mathcal{X}_1)$ (ε-rule).

Example: $\mathcal{F} = \{f(1), a\}$, $\mathcal{F}' = \mathcal{F} \cup \{h(2), g(1)\}$;
$\mathcal{U}_1 = \langle \{q, q_f\}, \mathcal{F}, \mathcal{F}', \{q_f\}, \Delta \rangle$, with rules

$$a \to q(a) \qquad f(q(x_1)) \to q(f(x_1)) \mid q(g(x_1)) \mid q_f(h(x_1, x_1))$$

NUTT move relation

Move relation

Let $t, t' \in T(\mathcal{F}, \mathcal{F}', Q)$. We write $t \to_{\mathcal{U}} t'$ if the following are satisfied:

- $t = C[f(q_1(u_1), \ldots, q_n(u_n))]$ for some context C and $u_1, \ldots, u_n \in T(\mathcal{F}')$;
- $t' = C[q(u\{x_1 \leftarrow u_1, \ldots, x_n \leftarrow u_n\}]$ for some rule $f(q_1(x_1), \ldots, q_n(x_1)) \to q(u)$ of $\mathcal{U}$.

Idea: Like an NFTA, but can additionally reorder/copy/delete subtrees and "explode" symbols into subtrees like a homomorphism.

A NUTT $\mathcal{U}$ defines the relation $\mathcal{R}(\mathcal{U}) = \{ \langle t, t' \rangle \mid t \to^*_{\mathcal{U}} q(t'),\ q \in G \}$.

Relations of NUTT

We write $\mathcal{U}(t)$ for $\{ t' \mid \langle t, t' \rangle \in \mathcal{R}(\mathcal{U}) \}$.

Examples:

- Example 1: $\mathcal{U}_1(fffa) = \{h(ffa, ffa), h(fga, fga), h(gfa, gfa), h(gga, gga)\}$
- Example 2: $\mathcal{F} = \{f(2), g(1), a\}$, $\mathcal{F}' = \mathcal{F}$;
  $\mathcal{U}_2 = \langle \{q, q', q''\}, \mathcal{F}, \mathcal{F}', \{q''\}, \Delta \rangle$, with rules

  $$a \to q(a) \qquad g(q(x_1)) \to q(g(x_1)) \qquad f(q(x_1), q(x_2)) \to q(f(x_1, x_2))$$

  $$a \to q'(a) \qquad g(q'(x_1)) \to q'(g(x_1))$$

  $$f(q(x_1), q'(x_2)) \to q''(g(x_1))$$

  $\mathcal{R}(\mathcal{U}_2) = \{ \langle f(t, g^m(a)), g(t) \rangle \mid t \in T(\mathcal{F}), m \geq 0 \}$

Properties of NUTT

A NUTT $\mathcal{U}$ is

- *ε-free* if it contains no ε-rule;
- *linear* if in rules of Δ, u is linear;
- *non-erasing* if in every rule, $\mathcal{H}(u) > 0$ (not just a variable);
- *complete* if for every rule with $f \in \mathcal{F}_n$ on the left-hand side, u on the right-hand side contains all of $\mathcal{X}_n$;
- *deterministic* (DUTT) if it is ε-free and no two rules have the same left-hand side.

Examples:

- $\mathcal{U}_1$ is non-deterministic, non-linear, complete.
- $\mathcal{U}_2$ is non-deterministic, linear, non-complete.

NUTT and other relation classes

(Linear) NUTT and $\mathfrak{R}_2$ are incomparable

- $\mathcal{R}(\mathcal{U}_2)^{-1}$ is in $\mathfrak{R}_2$, accepted by the following (with q'' final):

$$\langle a, a\rangle \to q \qquad \langle g, g\rangle(q) \to q \qquad \langle f, f\rangle(q, q) \to q$$

$$\langle a, -\rangle \to q' \qquad \langle g, -\rangle(q') \to q' \qquad \langle f, g\rangle(q, q') \to q''$$

  But $\mathcal{R}(\mathcal{U}_2)^{-1}$ is in $\mathfrak{R}_2$ is not definable by NUTT: Suppose that such a NUTT existed with k rules.
- $\mathfrak{R}_2$ is incapable of copying or reordering subtrees.

Top-down transducers

Definition: Top-down tree transducer

A *top-down tree transducer* (NDTT) is a tuple $\mathcal{D} = \langle Q, \mathcal{F}, \mathcal{F}', I, \Delta \rangle$, where:

- Q is a finite set of *states* and $I \subseteq Q$ are *initial* states;
- $\mathcal{F}, \mathcal{F}'$ are finite ranked alphabets;
- Δ is a finite set of rules of the form

$$q(f) \to u[q_1(x_{i_1}), \ldots, q_k(x_{i_k})]$$

for $f \in \mathcal{F}_n$, $q, q_1, \ldots, q_k \in Q$, $u \in \mathcal{C}^k(\mathcal{F}')$, $x_{i_1}, \ldots, x_{i_k} \in \mathcal{X}_n$, or

$$q(x_1) \to u[q_1(x_1), \ldots, q_k(x_1)]$$

for $q, q' \in Q$ and $u \in \mathcal{C}^k(\mathcal{F}')$ (ε-rule).

NDTT move relation

Move relation

Let $t, t' \in T(\mathcal{F}, \mathcal{F}', Q)$. We write $t \to_{\mathcal{D}} t'$ if the following are satisfied:

- $t = C[q(f(u_1, \ldots, u_n))]$ for some context C and $u_1, \ldots, u_n \in T(\mathcal{F})$;
- $t' = C[u[q_1(u_{i_1}), \ldots, q_k(u_{i_k})]]$ for some rule $q(f) \to u[q_1(x_{i_1}), \ldots, q_k(x_{i_k})]$ of $\mathcal{D}$.

The relation defined by $\mathcal{D}$ is $\mathcal{R}(\mathcal{D}) = \{ \langle t, t' \rangle \mid q(t) \to t', q \in I \}$.

NDTT move relation

Move relation

Let $t, t' \in T(\mathcal{F}, \mathcal{F}', Q)$. We write $t \to_{\mathcal{D}} t'$ if the following are satisfied:

- $t = C[q(f(u_1, \ldots, u_n))]$ for some context C and $u_1, \ldots, u_n \in T(\mathcal{F})$;
- $t' = C[u[q_1(u_{i_1}), \ldots, q_k(u_{i_k})]]$ for some rule $q(f) \to u[q_1(x_{i_1}), \ldots, q_k(x_{i_k})]$ of $\mathcal{D}$.

The relation defined by $\mathcal{D}$ is $\mathcal{R}(\mathcal{D}) = \{\, \langle t, t' \rangle \mid q(t) \to t', q \in I \,\}$.

Example: $\mathcal{F} = \{f(1), a\}$, $\mathcal{F}' = \mathcal{F} \cup \{f(1), g(1), h(2), a\}$;
$\mathcal{D}_1 = \langle \{q, q'\}, \mathcal{F}, \mathcal{F}', \{q\}, \Delta \rangle$, with rules

$$q(f(x)) \to h(q'(x), q'(x)) \qquad q'(f(x)) \to f(q'(x)) \mid g(q'(x)) \qquad q'(a) \to a$$

Then $\mathcal{D}_1(ffa) = \{h(fa, fa), h(fa, ga), h(ga, fa), h(ga, ga)\}$.

Closure properties

Properties of NUTT and DUTT

- There exist relations expressible by NUTT but not NDTT.
- There exist relations expressible by NDTT but not NUTT.
- NUTT are closed under union, but not intersection.
- NUTT are not closed under composition, but linear NUTT are.
- Linear complete NUTT and NDTT are equivalent.