(Deterministic) Pushdown Automata

Home assignment to hand in before or on March 9, 2017.

February			1	2	3	4	5
	6	7	8	9	10	11	12
	13	14	15	16	17	18	19
	20	21	22	23	24	25	26
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March			1	2	3	4	5
	6	7	8	9	10	11	12
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	27	28	29	30	31		

Electronic versions (PDF only) can be sent by email to ⟨schmitz@lsv.ens-cachan.fr⟩, paper versions should be handed in on the 9th or put in my mailbox at LSV, ENS Cachan. **No delays**. The numbers in the margins next to exercises are indications of time and difficulty, not necessarily of the points you might earn answering them.

1 Deterministic Pushdown Automata

Pushdown Automata. Recall from the lecture that a *pushdown automaton* (PDA) is syntactically a tuple $\mathcal{A} = \langle Q, \Sigma, \Gamma, \delta, q_0, z_0, F\rangle$ where Q is a finite set of states, Σ and Γ are two finite alphabets of input and stack symbols, $\delta \subseteq Q \times \Gamma \times (\Sigma \cup \{\varepsilon\}) \times Q \times \Gamma^*$ is a finite set of transitions, $q_0 \in Q$ is the initial state, $z_0 \in \Gamma$ the initial stack content, and $F \subseteq Q$ is the set of accepting states.

A *configuration* of $\mathcal{A}$ is a pair $q, \gamma \in Q \times \Gamma^*$ consisting of the current state q and the current stack contents γ; we assume that the top of the stack is to the right. A transition (q, z, a, q', γ') can be applied in q, γ if $\gamma = \gamma'' z$ and leads to the configuration $q', \gamma''\gamma'$, which we write $q, \gamma'' z \xrightarrow{a}_{\mathcal{A}} q', \gamma''\gamma'$. We lift this definition to finite words $w \in \Sigma^*$ and write $q, \gamma \xrightarrow{w}_{\mathcal{A}} q', \gamma'$ if there exists a sequence of transitions labelled by w from q, γ to q', γ'.

The *language* of $\mathcal{A}$ is $L(\mathcal{A}) \overset{\text{def}}{=} \{w \in \Sigma^* \mid \exists q_f \in F, \exists \gamma \in \Gamma^*, q_0, z_0 \xrightarrow{w}_{\mathcal{A}} q_f, \gamma\}$. Its *language by empty stack* is $N(\mathcal{A}) \overset{\text{def}}{=} \{w \in \Sigma^* \mid \exists q \in Q, q_0, z_0 \xrightarrow{w}_{\mathcal{A}} q, \varepsilon\}$.

Exercise 1 (Deterministic Pushdown Automata.)**.** A PDA is *deterministic* (a DPDA) if, for all $q \in Q$ and $z \in \Gamma$,

- either there is an applicable ε-transition $(q, z, \varepsilon, q', \gamma')$ for some q' and γ', and then it is the only applicable transition: $|\{(a, q', \gamma') \mid (q, z, a, q', \gamma') \in \delta\}| = 1$,

- or there is no applicable ε-transition and for all $b \in \Sigma$, there is at most one applicable transition with label b: $|\{(q', \gamma') \mid (q, z, b, q', \gamma') \in \delta\}| \leq 1$.

A context-free language L is *deterministic* if there exists a DPDA $\mathcal{A}$ with $L = L(\mathcal{A})$.

[1] 1. Show that the set of centered palindromes $L_{\text{pal}} \stackrel{\text{def}}{=} \{w\$w^R \mid w \in \{a, b\}^*\}$ is deterministic ($\cdot^R$ denotes the 'mirror', aka 'reversal' operation on finite words).

[2] 2. We can generalise the previous example. Let $h : \Sigma^* \to \Delta^*$ be a homomorphism and let $\$ \notin \Sigma \cup \Delta$ be a symbol. Show that the language $L_h \stackrel{\text{def}}{=} \{w\$(h(w))^R \mid w \in \Sigma^+\}$ is deterministic.

[1] 3. Show that a regular language is also deterministic context-free.

Exercise 2 (Prefix-free Deterministic Languages)**.** A language $L \subseteq \Sigma^*$ is *prefix-free* if no strict prefix of a word from L is in L: $L \cap L\Sigma^+ = \emptyset$.

[2] Let L be a context-free language. Show that there exists a DPDA $\mathcal{A}$ with $L = N(\mathcal{A})$ if and only if L is deterministic and prefix-free.

Exercise 3 (Non-deterministic Language)**.** We want to show that the language $L \stackrel{\text{def}}{=} \{a^n b^n \mid n > 0\} \cup \{a^n b^{2n} \mid n > 0\}$ is not deterministic.

[1] 1. Show that L is context-free.

In order to prove that L is not deterministic, we proceed by contradiction and show that, if there existed a DPDA $\mathcal{A}$ with $L(\mathcal{A}) = L$, then we would be able to construct a PDA recognising $\{a^n b^n c^n \mid n > 0\}$. As the latter language is not context-free, this will contradict the existence of $\mathcal{A}$.

Assume that $\mathcal{A} = \langle Q, \{a, b\}, \Gamma, \delta, q_0, z_0, F\rangle$ is a DPDA with $L = L(\mathcal{A})$. Let $g : \{a, b\}^* \to \{a, c\}^*$ be the homomorphism defined by $g(a) \stackrel{\text{def}}{=} a$ and $g(b) \stackrel{\text{def}}{=} c$. We define $\mathcal{A}'$ as 'running two copies' of $\mathcal{A}$, one before going through an accepting state, and one after: $\mathcal{A}' \stackrel{\text{def}}{=} \langle Q', \{a, b, c\}, \Gamma, \delta', (q_0, 0), z_0, F \times \{1\}\rangle$ where

$$
\begin{aligned}
Q' &\stackrel{\text{def}}{=} Q \times \{0, 1\} \\
\delta' &\stackrel{\text{def}}{=} \{((q, 0), z, d, (q', 0), \gamma) \mid d \in \{a, b, \varepsilon\},\ q \notin F,\ \text{and}\ (q, z, d, q', \gamma) \in \delta\} \\
&\quad \cup \{((q, 0), z, g(d), (q', 1), \gamma) \mid d \in \{b, \varepsilon\},\ q \in F,\ (q, z, d, q', \gamma) \in \delta\} \\
&\quad \cup \{((q, 1), z, g(d), (q', 1), \gamma) \mid d \in \{b, \varepsilon\},\ (q, z, d, q', \gamma) \in \delta\}\ .
\end{aligned}
$$

2. Let $h : \{a, b, c\}^* \to \{a, b\}^*$ be defined by $h(a) \stackrel{\text{def}}{=} a$ and $h(b) \stackrel{\text{def}}{=} h(c) \stackrel{\text{def}}{=} b$.

   [1] (a) Show that, if $w \in L(\mathcal{A}')$, then $h(w) \in L(\mathcal{A})$.

   [1] (b) Show that, if $w \in L(\mathcal{A}')$, then $w = w_1 w_2$ with $w_1 \in L(\mathcal{A})$ and $w_2 \in \{c\}^*$.

[1] (c) Deduce that $L(\mathcal{A}') \subseteq \{a^n b^n c^n \mid n > 0\} \cup L$.

[2] 3. Show that $\{a^n b^n c^n \mid n > 0\} \subseteq L(\mathcal{A}')$.

[1] 4. Derive the final contradiction and conclude.

2 Closure Properties

In the following, we will need two more closure properties of deterministic context-free languages—that will hopefully be proven during the second part of the course:

Theorem 1 (Closure under Complementation)**.** *Given a DPDA $\mathcal{A}$ over input alphabet Σ, we can compute a DPDA $\mathcal{A}'$ over Σ such that $L(\mathcal{A}') = \Sigma^* \setminus L(\mathcal{A})$.*

Theorem 2 (Closure under Left Quotient)**.** *Given a DPDA $\mathcal{A}$ over input alphabet Σ and a word $w \in \Sigma^*$, we can compute a DPDA $\mathcal{A}'$ over Σ such that $L(\mathcal{A}') = w^{-1} \cdot L(\mathcal{A})$.*

3 Undecidability Results

We prove in this section several undecidability results for context-free languages.

Exercise 4 (Basic Undecidable Problems)**.** In the following, we are given as input of the decision problem one or two PDAs $\mathcal{A}_1$ and $\mathcal{A}_2$ over some alphabet Σ.

[2] 1. (Emptiness of Intersection) Show by a reduction from the Post Correspondence Problem that the problem whether $L(\mathcal{A}_1) \cap L(\mathcal{A}_2)$ is empty is undecidable, even if $\mathcal{A}_1$ and $\mathcal{A}_2$ are deterministic.

[1] 2. Show the following corollaries:
   (a) (Inclusion) whether $L(\mathcal{A}_1) \subseteq L(\mathcal{A}_2)$ is undecidable, even if $\mathcal{A}_1$ and $\mathcal{A}_2$ are deterministic.
   (b) (Universality) whether $L(\mathcal{A}_1) = \Sigma^*$ is undecidable.

Exercise 5. Let $P \subseteq 2^{\Sigma^*}$ be a subset of the context-free languages such that
(i) there exists a context-free language $L_0 \notin P$,
(ii) P contains all the regular languages,
(iii) P is closed under left quotients, and
(iv) P is closed under intersection with regular languages.

The *P-membership problem* is then, given as input a PDA $\mathcal{A}$, whether $L(\mathcal{A}) \in P$.

[1] 1. Show by a reduction from the universality problem that, under conditions (i)–(iv) above, P-membership is undecidable.

[3] 2. Show the following corollary: it is undecidable
   (a) (Regularity) whether a context-free language is regular.
   (b) (Determinism) whether a context-free language is deterministic.