TD 5: EF Games, Separation

1 Separation

Exercise 1 (Expressiveness and Separation). Consider the FO(AP, <) formula

\[ \psi(x) = P_a(x) \land \forall y.P_a(y) \rightarrow \exists z.(y < x \rightarrow P_b(z) \land y < z < x) \]
\[ \land (y > x \rightarrow P_c(z) \land z > y) . \]

1. Separate \( \psi(x) \), i.e. provide pure formulæ \( \psi_i(x) \) such that \( \psi(x) \) is equivalent to a boolean combination of the \( \psi_i(x) \), and each \( \psi_i(x) \) only contains separated subformulae.

2. Provide equivalent TL(AP, SS, SU) formulæ \( \varphi_i \) for the \( \psi_i(x) \).

Exercise 2 (Deciding Semantic Purity). Let us consider time flows in \((\mathbb{N}, <)\). Show that the problem whether a TL(AP, SS, SU) formula \( \varphi \) is semantically pure future is in PSPACE.

2 EF Games

Exercise 3 (Non-Strict Until).

1. Show that SU is not expressible in TL(AP, S, U) over \((\mathbb{R}, <)\).

2. Show that SU is not expressible in TL(AP, S, U) over \((\mathbb{N}, <)\).

Exercise 4 (Periodic Properties).

1. Show that the fact that a finite temporal time flow is of “even length” cannot be expressed in TL(AP, SS, SU).

2. Recall Exercise 3 of TD 2: Show that the set \((\{p\} \Sigma)^\omega\) cannot be expressed in TL((\{p\}, SS, SU) over \((\mathbb{N}, <)\).

3 LTL with Past

Exercise 5 (Succinctness of Past Formulæ). Consider the time flow \((\mathbb{N}, <)\). Let AP_{n+1} = \{p_0, \ldots, p_n\} = AP_n \cup \{p_n\} be a set of atomic propositions, defining the alphabet \( \Sigma_{n+1} = 2^{AP_{n+1}} \). We want to show the existence of an \( O(n) \)-sized LTL formula with past such that any equivalent pure future LTL formula is of size \( \Omega(2^n) \).
First consider the following LTL formula of exponential size:

\[ \bigwedge_{S \subseteq AP_n} \left( \left( \bigwedge_{p_i \in S} p_i \land \bigwedge_{p_j \not\in S} \neg p_j \right) \Rightarrow G \left( \left( \bigwedge_{p_i \in S} p_i \land \bigwedge_{p_j \not\in S} \neg p_j \right) \Rightarrow p_n \right) \right) \land \left( \left( \bigwedge_{p_i \in S} p_i \land \bigwedge_{p_j \not\in S} \neg p_j \right) \Rightarrow G \left( \left( \bigwedge_{p_i \in S} p_i \land \bigwedge_{p_j \not\in S} \neg p_j \right) \Rightarrow \neg p_n \right) \right) \]

(\varphi_n)

1. Describe ‘intuitively’ which words of \( \Sigma_{n+1}^\omega \) are the models of \( \varphi_n \).

2. Can an LTL formula with past modalities check whether it is at the initial position of a word?

3. Provide an LTL formula with past \( \psi_n \) of size \( O(n) \) initially equivalent to \( \varphi_n \).

4. Consider the language \( L_n = \{ \sigma \in \Sigma_{n+1}^\omega \mid \sigma \models G \varphi_n \} \). We want to prove that any generalized Büchi automaton that recognizes \( L_n \) requires at least \( 2^{2n} \) states.

For this we fix a permutation \( a_0 \cdots a_{2^n-1} \) of the symbols in \( \Sigma_n \) and we consider all the different subsets \( K \subseteq \{0, \ldots, 2^n-1\} \). For each \( K \) we consider the word

\[ w_K = b_0 \cdots b_{2^n-1} \]

in \( \Sigma_{n+1}^{2^n} \), defined for each \( i \) in \( \{0, \ldots, 2^n-1\} \) by

\[ b_i = a_i \quad \text{if } i \in K \]
\[ b_i = a_i \cup \{p_n\} \quad \text{otherwise} \]

Thus \( K \) is the set of positions of \( w_K \) where \( p_n \) does not hold.

Using the \( w_K \) for different values of \( K \), prove that any generalized Büchi automaton for \( G \varphi_n \) requires at least \( 2^{2^n} \) states.

5. Conclude using the fact that any pure future LTL formula \( \varphi \) can be given a generalized Büchi automaton with at most \( 2|\varphi| \) states.

4 Stavi Connectives

Exercise 6 (Linear Orders with Gaps). In this exercise we assume \((T, <)\) to be a linear time flow.

1. Let us define a new unary “gap” modality \( \text{gap} \):

\[ w, i \models \text{gap} \varphi \text{ iff } \forall k. k > i \rightarrow (\exists \ell. k < \ell \land \forall j. i < j < \ell \rightarrow w, j \models \varphi) \]
\[ \lor \left( \exists j. i < j < k \land w, j \models \neg \varphi \right) \]
\[ \land \exists k_1. k_1 > i \land \forall j. i < j \leq k_1 \rightarrow w, j \models \varphi \]
\[ \land \exists k_2. k_2 > i \land w, k_2 \models \neg \varphi \]
The intuition behind \( \text{gap} \) is that \( \varphi \) should hold for some time until a gap occurs in the time flow, after which \( \neg \varphi \) holds at points arbitrarily close to the gap.

(a) Express \( \text{gap} \varphi \) using the standard \( SU \) modality.

(b) Show that, if \( (T, <) \) is Dedekind-complete, then \( \text{gap} p \) for \( p \in \text{AP} \) cannot be satisfied.

2. Consider the temporal flow \( (\{0\} \times \mathbb{Z}_{<0} \times \mathbb{Z} \cup \{1\} \times \mathbb{Z} \times \mathbb{Z}, <) \) where \( < \) is the lexicographic ordering and \( \text{AP} = \{p\} \). Let \( n \) be an even integer in \( \mathbb{Z} \), and define

\[
\begin{align*}
h_0(p) &= \{(0, i, j) \in T \mid i \text{ is odd}\} \cup \{(1, i, j) \in T \mid i \text{ is odd}\} \\
h_1(p) &= \{(0, i, j) \in T \mid i \text{ is odd}\} \cup \{(1, i, j) \in T \mid i > n \text{ is odd}\}.
\end{align*}
\]

(a) Show that \( w_0, (x, i, j) \models \text{gap} p \) for any \( x \in \{0, 1\} \), odd \( i \), and \( j \).

(b) Show that no \( \text{TL}(\{p\}, SS, SU) \) formula can distinguish between \( (w_0, (0, -1, 0)) \) and \( (w_1, (0, -1, 0)) \).

(c) Here is the definition of the Stavi “until” modality:

\[
w, i \models \varphi U \psi \text{ iff } \exists \ell, i < \ell\]

\[
\land \forall k, i < k < \ell \rightarrow [\exists j_1, k < j_1 \land \forall j, i < j < j_1 \rightarrow w, j \models \varphi]
\lor [\forall j_2, k < j_2 < \ell \rightarrow w, j_2 \models \psi]
\land (\exists j_3, i < j_3 < k \land w, j_3 \models \neg \varphi)]
\land \exists k_1, i < k_1 < \ell \land w, k_1 \models \neg \varphi
\land \exists k_2, i < k_2 < \ell \land \forall j, i < j < k_2 \rightarrow w, j \models \varphi
\]

This modality is quite similar to \( \text{gap} \varphi \), but further requires \( \psi \) to hold for some time after the gap (the “\( j_2 \)” condition above).

Show that \( w_1, (0, -1, 0) \models p U \neg \text{gap} p \) but \( w_0, (0, -1, 0) \not\models p U \neg \text{gap} p \).