

TD 5: EF Games, Separation

1 Separation

Exercise 1 (Expressiveness and Separation). Consider the FO(AP, <) formula

$$\psi(x) = P_a(x) \wedge \forall y. P_a(y) \rightarrow \exists z. (y < x \rightarrow P_b(z) \wedge y < z < x) \\ \wedge (y > x \rightarrow P_c(z) \wedge z > y) .$$

1. Separate $\psi(x)$, i.e. provide pure formulæ $\psi_i(x)$ such that $\psi(x)$ is equivalent to a boolean combination of the $\psi_i(x)$, and each $\psi_i(x)$ only contains separated subformulæ.
2. Provide equivalent TL(AP, SS, SU) formulæ φ_i for the $\psi_i(x)$.
3. Provide Büchi transducers for each $\varphi_i(x)$.

Exercise 2 (Deciding Semantic Purity). Let us consider time flows in $(\mathbb{N}, <)$. Show that the problem whether a TL(AP, SS, SU) formula φ is *semantically* pure future is in PSPACE.

2 EF Games

Exercise 3 (Non-Strict Until).

1. Show that SU is not expressible in TL(AP, S, U) over $(\mathbb{R}, <)$.
2. Show that SU is not expressible in TL(AP, S, U) over $(\mathbb{N}, <)$.

Exercise 4 (Periodic Properties).

1. Show that the fact that a finite temporal time flow is of “even length” cannot be expressed in TL(AP, SS, SU).
2. Recall Exercise 3 of TD 2: Show that the set $(\{p\}\Sigma)^\omega$ cannot be expressed in TL($\{p\}$, SS, SU) over $(\mathbb{N}, <)$.

3 LTL with Past

Exercise 5 (Succinctness of Past Formulæ). Consider the time flow $(\mathbb{N}, <)$. Let $\text{AP}_{n+1} = \{p_0, \dots, p_n\} = \text{AP}_n \cup \{p_n\}$ be a set of atomic propositions, defining the alphabet $\Sigma_{n+1} = 2^{\text{AP}_{n+1}}$. We want to show the existence of an $O(n)$ -sized LTL formula with past such that any equivalent pure future LTL formula is of size $\Omega(2^n)$.

First consider the following LTL formula of exponential size:

$$\bigwedge_{S \subseteq \text{AP}_n} \left(\begin{array}{l} (\bigwedge_{p_i \in S} p_i \wedge \bigwedge_{p_j \notin S} \neg p_j \wedge p_n) \Rightarrow \mathbf{G}((\bigwedge_{p_i \in S} p_i \wedge \bigwedge_{p_j \notin S} \neg p_j) \Rightarrow p_n) \\ \wedge (\bigwedge_{p_i \in S} p_i \wedge \bigwedge_{p_j \notin S} \neg p_j \wedge \neg p_n) \Rightarrow \mathbf{G}((\bigwedge_{p_i \in S} p_i \wedge \bigwedge_{p_j \notin S} \neg p_j) \Rightarrow \neg p_n) \end{array} \right) \quad (\varphi_n)$$

1. Describe ‘intuitively’ which words of Σ_{n+1}^ω are the models of φ_n .
2. Can an LTL formula with past modalities check whether it is at the initial position of a word?
3. Provide an LTL formula with past ψ_n of size $O(n)$ *initially* equivalent to φ_n .
4. Consider the language $L_n = \{\sigma \in \Sigma_{n+1}^\omega \mid \sigma \models \mathbf{G} \varphi_n\}$. We want to prove that any generalized Büchi automaton that recognizes L_n requires at least 2^{2^n} states.

For this we fix a permutation $a_0 \cdots a_{2^n-1}$ of the symbols in Σ_n and we consider all the different subsets $K \subseteq \{0, \dots, 2^n - 1\}$. For each K we consider the word

$$w_K = b_0 \cdots b_{2^n-1}$$

in $\Sigma_{n+1}^{2^n}$, defined for each i in $\{0, \dots, 2^n - 1\}$ by

$$\begin{array}{ll} b_i = a_i & \text{if } i \in K \\ b_i = a_i \cup \{p_n\} & \text{otherwise.} \end{array}$$

Thus K is the set of positions of w_K where p_n does not hold.

Using the w_K for different values of K , prove that any generalized Büchi automaton for $\mathbf{G} \varphi_n$ requires at least 2^{2^n} states.

5. Conclude using the fact that any pure future LTL formula φ can be given a generalized Büchi automaton with at most $2^{|\varphi|}$ states.

4 Stavi Connectives

Exercise 6 (Linear Orders with Gaps). In this exercise we assume $(\mathbb{T}, <)$ to be a linear time flow.

1. Let us define a new unary “gap” modality **gap**:

$$\begin{aligned}
w, i \models \mathbf{gap}\varphi \text{ iff } & \forall k. k > i \rightarrow (\exists \ell. k < \ell \wedge \forall j. i < j < \ell \rightarrow w, j \models \varphi) \\
& \vee (\exists j. i < j < k \wedge w, j \models \neg\varphi) \\
& \wedge \exists k_1. k_1 > i \wedge \forall j. i < j \leq k_1 \rightarrow w, j \models \varphi \\
& \wedge \exists k_2. k_2 > i \wedge w, k_2 \models \neg\varphi .
\end{aligned}$$

The intuition behind **gap** is that φ should hold for some time until a gap occurs in the time flow, after which $\neg\varphi$ holds at points arbitrarily close to the gap.

- (a) Express **gap** φ using the standard **SU** modality.
(b) Show that, if $(\mathbb{T}, <)$ is Dedekind-complete, then **gap** p for $p \in \text{AP}$ cannot be satisfied.
2. Consider the temporal flow $(\{0\} \times \mathbb{Z}_{<0} \times \mathbb{Z} \cup \{1\} \times \mathbb{Z} \times \mathbb{Z}, <)$ where $<$ is the lexicographic ordering and $\text{AP} = \{p\}$. Let n be an even integer in \mathbb{Z} , and define

$$\begin{aligned}
h_0(p) &= \{(0, i, j) \in \mathbb{T} \mid i \text{ is odd}\} \cup \{(1, i, j) \in \mathbb{T} \mid i \text{ is odd}\} \\
h_1(p) &= \{(0, i, j) \in \mathbb{T} \mid i \text{ is odd}\} \cup \{(1, i, j) \in \mathbb{T} \mid i > n \text{ is odd}\} .
\end{aligned}$$

- (a) Show that $w_0, (x, i, j) \models \mathbf{gap}p$ for any $x \in \{0, 1\}$, odd i , and j .
(b) Show that no **TL**($\{p\}$, **SS**, **SU**) formula can distinguish between $(w_0, (0, -1, 0))$ and $(w_1, (0, -1, 0))$.
(c) Here is the definition of the Stavi “until” modality:

$$\begin{aligned}
w, i \models \varphi \bar{\mathbf{U}} \psi \text{ iff } & \exists \ell. i < \ell \\
& \wedge \forall k. i < k < \ell \rightarrow [\exists j_1. k < j_1 \wedge \forall j. i < j < j_1 \rightarrow w, j \models \varphi] \\
& \vee [(\forall j_2. k < j_2 < \ell \rightarrow w, j_2 \models \psi) \\
& \quad \wedge (\exists j_3. i < j_3 < k \wedge w, j_3 \models \neg\varphi)] \\
& \wedge \exists k_1. i < k_1 < \ell \wedge w, k_1 \models \neg\varphi \\
& \wedge \exists k_2. i < k_2 < \ell \wedge \forall j. i < j < k_2 \rightarrow w, j \models \varphi
\end{aligned}$$

This modality is quite similar to **gap** φ , but further requires ψ to hold for some time after the gap (the “ j_2 ” condition above).

Show that $w_1, (0, -1, 0) \models p \bar{\mathbf{U}} \neg \mathbf{gap}p$ but $w_0, (0, -1, 0) \not\models p \bar{\mathbf{U}} \neg \mathbf{gap}p$.