Home Assignment 2: Fairness and Petri Nets

To hand in before or on February 17, 2013.

Electronic versions (PDF only) can be sent by email to \(\text{schmitz@lsv.ens-cachan.fr}\), paper versions should be handed in on the 17th or put in my mailbox at LSV, ENS Cachan.

The following formula will be applied to late assignments: \(g - 3d + 1\), where \(0 \leq g \leq 20\) is the original grade and \(d > 0\) is the number of days of delay, rounded up onward from the 13th at midnight.

This assignment is concerned with fairness in Petri nets. Fairness properties are employed to rule out behaviours where a process might wait indefinitely before being activated.

The numbers in the margins next to exercises are indications of time and difficulty.

1 The Fairness Fragment

We restrict ourselves to a fragment \(\text{TL}(\text{AP}, \text{GF})\) where the only temporal modality is \(\text{GF}\) (sometimes also written \(\text{F}^\infty\)), i.e. with syntax

\[
\varphi ::= p \mid \top \mid \neg \varphi \mid \varphi \land \varphi \mid \text{GF} \varphi
\]

where \(p\) ranges over \(\text{AP}\).

Let us consider the case of state-based LTL model-checking for Petri nets. In this framework, we are checking infinite sequences of markings of a Petri net \(\mathcal{N} = \langle P, T, W, m_0 \rangle\), i.e. infinite sequences \(m_0 m_1 \cdots m_i \cdots \) in \((\text{NP})^\omega\) such that, for all \(i \in \mathbb{N}\), \(m_i \rightarrow_{\mathcal{N}} m_{i+1}\) is a transition step of \(\mathcal{N}\) according to some \(t\) in \(T\), thus verifying for all \(p\) in \(P\) that \(m_i(p) \geq W(p, t)\) and \(m_{i+1}(p) = m_i(p) - W(p, t) + W(t, p)\). More generally, the effect \(\Delta(u)\) of a transition sequence \(u\) in \(T^*\) is defined by \(\Delta(\varepsilon) = 0^P\) and \(\Delta(ut) = \Delta(u) - W(P, t) + W(t, P)\).

The atomic propositions in \(\text{AP} \subseteq P\) represent places, and are interpreted over such a sequence by \(m_i \models p\) iff \(m_i(p) > 0\). Put differently, a Petri net \(\mathcal{N}\) gives rise to an infinite Kripke structure \(\mathfrak{M}_\mathcal{N} = \langle \text{NP}, \rightarrow_{\mathcal{N}}, \{m_0\}, \text{AP}, \ell \rangle\) where \(\ell(m) = \{p \in \text{AP} \mid m(p) > 0\}\).
Recall (from TD 6, Exercise 3) that the model-checking problem for state-based TL(AP,X,U) is undecidable in general and PSPACE-complete for safe nets. In the following we restrict ourselves to the model-checking problem for ordinary Petri nets, which verify $W(t,p) \leq 1$ and $W(p,t) \leq 1$ for all $p$ in $P$ and $t$ in $T$ (note that this does not imply that the net is safe).

Exercise 1 (Fairness in Petri Nets). Let $\mathcal{N}$ be an ordinary Petri net, $m$ a marking in $\{0,1\}^P$, and $t$ be a transition in $T$. Reduce the following problems to MC $\exists$ (AP,GF) model-checking instances on an ordinary Petri net $\mathcal{N}'$:

repeated coverability (RC$^3$): there exists an infinite execution where $m$ is covered infinitely often, i.e. such that $m_i \geq m$ for an infinite number of indices $i$,

weak fairness (WF$^\forall$): every infinite execution either fires $t$ infinitely often, or $t$ is infinitely often not firable,

strong fairness (SF$^\forall$): in every infinite execution, if $t$ is firable infinitely often, then it is actually fired infinitely often.

One can also consider the existential questions WF$^\exists$ or SF$^\exists$, which ask whether there exist some fair infinite execution. We will see in sections 2 and 3 that RC$^3$ and WF$^3$ are decidable.

It turns out that SF$^3$ is not decidable. Because the proof of this result is a bit involved, we rather look at a decidable case:

Exercise 2 (Strong Fairness in Safe Nets). We consider the SF$^3$ problem when the Petri net $\mathcal{N}$ is known to be safe, i.e. verifying $m(p) \leq 1$ for all reachable $m$ and all places $p$ in $P$.

Show that SF$^3$ is PSPACE-complete when $\mathcal{N}$ is safe. Hint: For hardness, reduce from reachability in safe nets.

2 Repeated Coverability

We show in this section that RC$^3$ is decidable, relying for this on the properties of the coverability graph seen during TD 7.

Exercise 3 (Decidability). We prove in this exercise that repeated coverability is decidable. Let $\mathcal{N}$ be Petri net, $G$ be its coverability graph and $m$ some marking in $\mathcal{N}^P$.

Show that there exists an infinite computation s.t. $m \leq m_i$ for infinitely many indices $i$ if there exists an accessible loop $m' \xrightarrow{\Delta G} m'$ in $G$ s.t. $m \leq m'$ and $\Delta(v) \geq 0^P$. Conclude. Hint: Use Exercise 2 from TD 7.

Exercise 4 (Action-Based LTL). Recall from Exercise 3 of TD 6 that action-based LTL considers labeled Petri nets $\langle \mathcal{N}, \lambda \rangle$ where $\lambda$ is a labelling from $T$ to $2^{AP}$. The model-checking problem then considers the infinite sequences $\lambda(t_1)\lambda(t_2)\cdots$ of transition labels along an execution $m_0 \xrightarrow{t_1} \mathcal{N} m_1 \xrightarrow{t_2} \mathcal{N} m_2 \cdots$; alternatively, we can consider the Kripke
structure associated with $\mathcal{N}$ to be $\mathfrak{M}_\mathcal{N}^\lambda = \langle \mathbb{N}^P \times T, T', I, \ell \rangle$ where $T' = \{(m, t), (m', t')\}$ | $m \xrightarrow{\lambda} m' \land m' \geq W(P, t')\}$, $I = \{m_0\} \times \{t \in T \mid m_0 \geq W(P, t)\}$, and $\ell(m, t) = \lambda(t)$.

3 Weak Fairness

We examine in this section the relationship between the existential weak fairness problem $\text{WF}^3$ and the reachability problem (RP) in Petri nets: given a Petri net $\mathcal{N}$ and a marking $m$ in $\mathbb{N}^P$, does there exist a finite run $m_0 \xrightarrow{\star} m'$? This problem is known to be decidable [3, 1, 2], although its exact complexity is still open.

Exercise 5 (Lower Bounds). We want to exhibit a reduction from RP to $\text{WF}^3$.

1. Show that RP can be reduced to the following problem: given a Petri net $\mathcal{N}$ and a place $p$, does there exist a reachable marking $m$ (i.e. verifying $m_0 \xrightarrow{\star} m$) such that $m(p) = 0$?

2. Show that RP can be reduced to $\text{WF}^3$.

Exercise 6 (Decidability). We want to show that $\text{WF}^3$ is decidable. Let us first concentrate on the subcase where there exists an infinite execution where $t$ is infinitely often not fireable.

1. Reduce this case to the question whether there exists a place $p$ in $P$ and an execution $m_0 \xrightarrow{\star} m \xrightarrow{\star} m'$ with $m \leq m'$ and $m(p) = m'(p) = 0$.

2. Reduce the previous question to an instance of the reachability problem.

3. Deduce that $\text{WF}^3$ is decidable.

References

