Algorithmic Aspects of WQO (Well-Quasi-Ordering) Theory
Part V: Proving Lower Bounds

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Lecture notes & exercices available at http://tinyurl.com/essllli12wqo
IF YOU MISSED THE EARLIER EPISODES

\[(\mathbb{N}^k, \leq_x)\text{ and } (\Sigma^*, \leq_\ast)\text{ are well-quasi-orderings: any infinite sequence } x = x_0, x_1, x_2, \ldots \text{ contains an increasing pair } x_i \leq x_j \text{ ("is good")}

If a sequence like \(x\) cannot grow too quickly —we say it is controlled— then the position \(i, j\) of the first increasing pair in \(x\) can be bounded by some length function \(L_{X, \text{control}}(|x_0|)\)

This gave us upper bounds for the complexity of wqo-based algorithms. Furthermore, these length functions can be precisely pinned down inside elegant subrecursive hierarchies

For example, it gave \(F_\omega\) upper-bounds for the verification —e.g., termination and/or safety— of monotonic counter machines, and \(F_\omega \omega\) upper bounds for lossy channel systems
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That was just the EASY part!!!
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Today we consider the "hardness" question: are these upper bounds optimal? or do we have matching lowing bounds? — the answer is often "positive" (?)
OUTLINE FOR PART V

- What is the question exactly? And why isn’t obvious?
- A strategy for proving hardness
- Hardness for Lossy Counter Machines
- Hardness for Lossy Channel Systems
**PROBLEM STATEMENT**

We have upper bounds on the complexity of verification for lossy counter machines and lossy channel systems.
Do we have matching lower bounds?

Could be for the simple-minded algorithms we presented in Part II

No for the underlying decision problems (witness: VASS’s)

**Exercise.** Give a decision problem solvable in Ackermannian time of its input that requires Ackermannian time (where $\text{Ack}(n) \overset{\text{def}}{=} A(n,n)$ and $A$ is the usual binary Ackermann function).

**Pb 1.** Input: $x, y, z$. Question: Does $A(x, y) = z$?

**Pb 2.** Input: $x, y, x', y'$. Question: Is $A(x, y) < A(x', y')$?

**Pb 3.** Input: $x, y$. Question: Is $A(x, y)$ prime?

**Pb 4.** Input: $x, y$. Question: Is $A(x, y)$ a sum $\sum_{i \in K} p_i^{F_i}$? where $p_i$ and $F_i$ are the $i$th prime (resp., Fibonacci) number.

**Pb 5.** Input: $x$. Question: Does the Universal TM halts on $x$, and furthermore halts in time $\text{Ack}(x)$?
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We shall adopt the following strategy:

1. Compute unreliably functions in the Hardy hierarchy
2. Use the result as an unreliable computational resource
3. “Check” in the end that nothing was lost
4. Need computing unreliably the inverses of Hardy functions
A run of $M$: $(\ell_0, 0, 1, 4) \rightarrow_{\text{rel}} (\ell_1, 1, 1, 4) \rightarrow_{\text{rel}} (\ell_2, 1, 0, 4) \rightarrow_{\text{rel}} (\ell_3, 1, 0, 4)$

Ordering states: $(\ell_1, 0, 0, 0) \leq (\ell_1, 0, 1, 2)$ but $(\ell_1, 0, 0, 0) \nleq (\ell_2, 0, 1, 2)$.

NB. A counter machine like $M$ above is not monotonic. Can test that a counter is zero $\implies$ steps not compatible with ordering (And we allow other guards/updates that break compatibility).

In fact, the ordering is used to model unreliability.
**CM = COUNTER MACHINES**

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LCM = Lossy COUNTER MACHINES

\[
\begin{align*}
(\ell, a) & \rightarrow (\ell', b) \overset{\text{def}}{\iff} (\ell, a) \succeq (\ell, x) \rightarrow_{\text{rel}} (\ell', y) \succeq (\ell', b) \text{ for some } x, y
\end{align*}
\]

A run of \(M\): \((\ell_0, 0, 1, 4) \rightarrow (\ell_1, 1, 1, 2) \rightarrow (\ell_2, 1, 0, 2) \rightarrow (\ell_1, 1, 0, 0)\)

The unreliable counter machine is a WSTS

**Paradox:** It does much more than its reliable twin but can compute much less

**NB:** These lossy counter machines are not a toy!!!
LCM = *Lossy Counter Machines*

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RECALL: HARDY HIERARCHY

\[ H^0(n) \overset{\text{def}}{=} n \quad H^\alpha+1(n) \overset{\text{def}}{=} H^\alpha(n + 1) \quad H^\lambda(n) \overset{\text{def}}{=} H^\lambda(n) \]

Recall:
\[ F_\alpha(n) = H^{\omega_\alpha}(n) \quad H^\alpha(n) \leq H^\alpha(n + 1) \]
\[ \alpha \sqsubseteq \alpha' \text{ implies } H^\alpha(n) \leq H^{\alpha'}(n) \]

Nb. \( H^\alpha(n) \) can be evaluated by transforming a pair
\[ \alpha, n = \alpha_0, n_0 \overset{H}{\rightarrow} \alpha_1, n_1 \overset{H}{\rightarrow} \alpha_2, n_2 \overset{H}{\rightarrow} \cdots \overset{H}{\rightarrow} \alpha_k, n_k \text{ with } \alpha_0 > \alpha_1 > \alpha_2 > \cdots \]
until eventually \( \alpha_k = 0 \) and \( n_k = H^\alpha(n) \) % tail-recursion!!

Below we compute fast-growing functions and their inverses
by encoding \( \alpha, n \overset{H}{\rightarrow} \alpha', n' \) and \( \alpha', n' \overset{H}{\rightarrow}^{-1} \alpha, n \)
**Recall: Hardy Hierarchy**

\[
H^0(n) \overset{\text{def}}{=} n \quad H^{\alpha+1}(n) \overset{\text{def}}{=} H^\alpha(n+1) \quad H^\lambda(n) \overset{\text{def}}{=} H^{\lambda(n)}(n)
\]

Recall:

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Nb. \( H^\alpha(n) \) can be evaluated by transforming a pair \( \alpha, n = \alpha_0, n_0 \to H \to \alpha_1, n_1 \to H \to \alpha_2, n_2 \to H \to \cdots \to H \to \alpha_k, n_k \) with \( \alpha_0 > \alpha_1 > \alpha_2 > \cdots \)
until eventually \( \alpha_k = 0 \) and \( n_k = H^\alpha(n) \) \% tail-recursion!!

Below we compute fast-growing functions and their inverses by encoding \( \alpha, n \overset{H}{\to} \alpha', n' \) and \( \alpha', n' \overset{H}{\to}^{-1} \alpha, n \)
ENCODING ORDINALS $< \omega \omega$ IN TUPLES OF NUMBERS

Write $\alpha$ in CNF with coefficients $\alpha = \omega^m.a_m + \omega^{m-1}.a_{m-1} + \cdots + \omega^0a_0$

Encoding of $\alpha,n$ is $\langle a_m,\ldots,a_0;n \rangle \in \mathbb{N}^{m+2}$.

$$\langle a_m,\ldots,a_0+1;n \rangle \xrightarrow{H} \langle a_m,\ldots,a_0;n+1 \rangle \quad \% H^{\alpha+1}(n) = H^\alpha(n+1)$$

$$\langle a_m,\ldots,a_{k+1},0,\ldots,0;n \rangle \xrightarrow{H} \langle a_m,\ldots,a_{k},n,0,\ldots,0;n \rangle \quad \% H^{\lambda}(n) = H^{\lambda_n}(n)$$
ENCODING ORDINALS $< \omega^\omega$ IN TUPLES OF NUMBERS

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$\langle a_m, \ldots, a_0 + 1; n \rangle \xrightarrow{H} \langle a_m, \ldots, a_0; n + 1 \rangle$ \hspace{1cm} $\% H^{\alpha+1}(n) = H^\alpha(n + 1)$

$\langle a_m, \ldots, a_k + 1, 0, \ldots, 0; n \rangle \xrightarrow{H} \langle a_m, \ldots, a_k, n, 0, \ldots, 0; n \rangle$ \hspace{1cm} $\% H^\lambda(n) = H^\lambda_n(n)$

Recall: $(\gamma + \omega^{k+1})_n \overset{\text{def}}{=} \gamma + \omega^k \cdot n$
ENCODING ORDINALS $< \omega^\omega$ IN TUPLES OF NUMBERS

Write $\alpha$ in CNF with coefficients $\alpha = \omega^m a_m + \omega^{m-1} a_{m-1} + \cdots + \omega^0 a_0$

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\[
\langle a_m, \ldots, a_0+1; n \rangle \quad \overset{H}{\longrightarrow} \quad \langle a_m, \ldots, a_0; n+1 \rangle \quad \% H^{\alpha+1}(n) = H^\alpha(n+1)
\]

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\]
NOW FOR \( \frac{H}{→} -1 \)

\[
\langle a_m, \ldots, a_0; n+1 \rangle \xrightarrow{H} -1 \langle a_m, \ldots, a_0+1; n \rangle
\]

\[
\langle a_m, \ldots, a_k, n, 0, \ldots, 0; n \rangle \xrightarrow{H} -1 \langle a_m, \ldots, a_k+1, 0, \ldots, 0; n \rangle
\]

\[
\%H^{\alpha+1}(n) = H^{\alpha}(n + 1)
\]

\[
\%H^{\lambda}(n) = H^{\lambda_n}(n)
\]
NOW FOR $\mathcal{H}^{-1}$

$$\langle a_m, \ldots, a_0; n+1 \rangle \xrightarrow{H}^{-1} \langle a_m, \ldots, a_0+1; n \rangle$$

$\% H^{\alpha+1}(n) = H^{\alpha}(n + 1)$

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$\% H^{\lambda}(n) = H^{\lambda n}(n)$

Prop. [Robustness] $a \preceq_{\times} a'$ and $n \preceq n'$ imply $H^{\alpha}(n) \preceq H^{\alpha'}(n')$
Ensures:
1. $M^b \vdash (\ell, B, a) \xrightarrow{\ast_{\text{rel}}} (\ell', B', a')$ implies $B + |a| = B' + |a'|$
2. $M^b \vdash (\ell, B, a) \xrightarrow{\ast_{\text{rel}}} (\ell', B', a')$ implies $M \vdash (\ell, a) \xrightarrow{\ast_{\text{rel}}} (\ell', a')$
3. If $M \vdash (\ell, a) \xrightarrow{\ast_{\text{rel}}} (\ell', a')$ then $\exists B, B': M^b \vdash (\ell, B, a) \xrightarrow{\ast_{\text{rel}}} (\ell', B', a')$
4. If $M^b \vdash (\ell, B, a) \xrightarrow{\ast} (\ell', B', a')$
   then $M^b \vdash (\ell, B, a) \xrightarrow{\ast_{\text{rel}}} (\ell', B', a')$ iff $B + |a| = B' + |a'|$
**Counter Machines on a Budget**

\[ \ell_0 \quad \ell_1 \quad \ell_2 \quad \ell_3 \]

\[ c_1 = 0? \]

\[ c_2 > 0? \]

\[ c_3 = 0? \]

\[ c_1 \quad c_2 \quad c_3 \]

\[ \ell_0 \rightarrow \ell_1 \]

\[ \ell_1 \rightarrow \ell_2 \]

\[ \ell_2 \rightarrow \ell_3 \]

\[ \ell_1 \rightarrow \ell_0 \]

\[ \ell_2 \rightarrow \ell_1 \]

\[ \ell_3 \rightarrow \ell_2 \]

\[ \ell_3 \rightarrow \ell_3 \]

\[ \begin{array}{c}
M
M^b (= on budget)
\end{array} \]

\[ B > 0? B-- \]

\[ c_1 ++ \]

\[ c_2 > 0? c_2-- \]

\[ c_3 = 0? \]

\[ B > 0? B-- \]

\[ c_1 ++ \]

\[ c_2 > 0? c_2-- \]

\[ c_3 = 0? \]

\[ B > 0? B-- \]

\[ c_1 ++ \]

\[ c_2 > 0? c_2-- \]

\[ B++ \]

\[ \ell_0 \quad \ell_1 \quad \ell_2 \quad \ell_3 \]

\[ c_1 \quad 4 \]

\[ c_2 \quad 3 \]

\[ c_3 \quad 0 \]

\[ c_1 \quad 4 \]

\[ c_2 \quad 3 \]

\[ c_3 \quad 0 \]

**Ensures:**

1. \( M^b \vdash (\ell, B, a) \xrightarrow{*} \text{rel} (\ell', B', a') \) implies \( B + |a| = B' + |a'| \)
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**Prop.** $M(m)$ has a lossy run
\[(\ell_H, a_m : 1, 0, \ldots, n : m, 0, \ldots) \xrightarrow{*} (\ell_{H^{-1}}, 1, 0, \ldots, m, 0, \ldots)\]
iff $M(m)$ has a reliable run
\[(\ell_H, a_m : 1, 0, \ldots, n : m, 0, \ldots) \xrightarrow{*_{\text{rel}}} (\ell_{H^{-1}}, a_m : 1, 0, \ldots, n : m, 0, \ldots)\]
iff $M$ has a reliable run from $\ell_{\text{ini}}$ to $\ell_{\text{fin}}$ that is bounded by $H^{\omega m}(m)$, i.e., by $\text{Ackermann}(m)$

**Cor.** LCM verification is Ackermann-hard (hence ...-complete)
**Prop.** \( M(m) \) has a lossy run

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(\ell_H, a_m : 1, 0, \dots, n : m, 0, \dots) \xrightarrow{*} (\ell_{H-1}, 1, 0, \dots, m, 0, \dots)
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**Cor.** LCM verification is Ackermann-hard (hence \( \ldots \)-complete)
**M(m): WRAPPING IT UP**

Prop. $M(m)$ has a lossy run

$$(\ell_H, a_m : 1, 0, \ldots, n : m, 0, \ldots) \mapsto (\ell_{H-1}, 1, 0, \ldots, m, 0, \ldots)$$

iff $M(m)$ has a **reliable** run

$$(\ell_H, a_m : 1, 0, \ldots, n : m, 0, \ldots) \mapsto_{\text{rel}} (\ell_{H-1}, a_m : 1, 0, \ldots, n : m, 0, \ldots)$$

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**Cor.** LCM verification is Ackermann-hard (hence \( \ldots \)-complete)
Prop. $M(m)$ has a lossy run

$$(\ell_H,a_m : 1,0,\ldots,n : m,0,\ldots) \xrightarrow{*} (\ell_{H-1},1,0,\ldots,m,0,\ldots)$$

iff $M(m)$ has a reliable run

$$(\ell_H,a_m : 1,0,\ldots,n : m,0,\ldots) \xrightarrow{\text{rel}} (\ell_{H-1},a_m : 1,0,\ldots,n : m,0,\ldots)$$

iff $M$ has a reliable run from $\ell_{\text{ini}}$ to $\ell_{\text{fin}}$ that is bounded by $H^{\omega^m}(m)$, i.e., by Ackermann$(m)$

Cor. LCM verification is Ackermann-hard (hence ...-complete)
Recall: LCS / Lossy Channel Systems

A configuration $\sigma = (\ell_1, \ell_2, w_1, w_2)$ with $w_i \in \Sigma^*$.
E.g., $w_1 = \text{hup.ack.ack}$.

Reliable steps: $\sigma \rightarrow_{\text{rel}} \rho$ read in front of channels, write at end (FIFO)

Lossy steps: messages may be lost nondeterministically

\[ \sigma \rightarrow \sigma' \iff \sigma \sqsubseteq \rho \rightarrow_{\text{rel}} \rho' \sqsubseteq \sigma' \text{ for some } \rho, \rho' \]

where $(S, \sqsubseteq)$ is the wqo $(\text{Loc}_1, =) \times (\text{Loc}_2, =) \times (\Sigma^*, \leq_{*})^{\{c_1, c_2\}}$

A model useful for concurrent protocols but also timed automata, metric temporal logic, products of modal logics, ...
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A model useful for concurrent protocols but also timed automata, metric temporal logic, products of modal logics, ...
Encoding Ordinals $< \omega^\omega$ in Channels

We use $\Sigma = \{a_0, \ldots, a_m\} \cup \{I\}$ to encode ordinals $\alpha < \omega^{m+1}$.

Two-level “differential” encoding:

$\beta : \{a_0, \ldots, a_m\}^* \rightarrow \omega^{m+1}$

$\beta(a_{r_1} \ldots a_{r_k}) \overset{\text{def}}{=} \omega^{r_1} + \cdots + \omega^{r_k}$

E.g. $\beta(\epsilon) = 0$, $\beta(a_3a_0a_0) = \omega^3 + 2$

$\alpha : \Sigma^* \rightarrow \omega^{\omega^{m+1}}$

$\alpha(a_1|a_2|\ldots|a_l) \overset{\text{def}}{=} \omega^{\beta(a_1a_2\ldots a_l)} + \cdots + \omega^{\beta(a_1a_2)} + \omega^{\beta(a_1)}$

E.g. $\alpha(\underline{III}) = \omega^0 + \omega^0 + \omega^0 = 3$

$\alpha(a_1a_0|a_1|) = \omega^{\omega \cdot 2} + \omega^{\omega + 1 \cdot 2}$

Property: $w \preceq_* w'$ implies $\alpha(w) \leq \alpha(w')$

Difficulty. $\alpha(w)$ is not always a CNF
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E.g. $\alpha(\|\|\|) = \omega^0 + \omega^0 + \omega^0 = 3$ \hspace{1cm} $\alpha(a_1 a_0 \| a_1 \|) = \omega^{\omega \cdot 2} + \omega^{\omega + 1} \cdot 2$

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Weakly computing $H \rightarrow$ with LCS’s

$$(|w,n) \xrightarrow{H} (w,n+1)$$

$$(ua_0|w,n) \xrightarrow{H} (u|^{n+1}a_0w,n)$$

$$(ua_r+1|w,n) \xrightarrow{H} (u^{a_r+1}|a_rw,n)$$

(⋯ similar rules for $H \rightarrow^{-1}$ ⋯)

% $H^{\alpha+1}(n) = H^{\alpha}(n+1)$

% $H^{\gamma+\omega^{k+1}}(n) = H^{\gamma+\omega^k \cdot (n+1)}(n)$

% $H^{\gamma+\omega^{\beta+\omega^k \cdot (n+1)}}(n) = H^{\gamma+\omega^{\beta+\omega^k \cdot (n+1)}}(n)$

Prop. [Robustness]

$w \preceq w'$ and $n \preceq n'$ and $w'$ pure imply $H^{\alpha(w)}(n) \leq H^{\alpha(w')}(n')$

where purity means that $w'$ has no superfluous symbols

(a regular condition that can be enforced by LCS’s)
Weakly computing $\xrightarrow{H}$ with LCS’s

$$(|w, n) \xrightarrow{H} (w, n + 1)$$

$$((ua_0)_w, n) \xrightarrow{H} ((u|^{n+1}a_0w, n)$$

$$(ua_r|w, n) \xrightarrow{H} (ua_{r}^{n+1}|aw, n)$$

(\cdots \text{similar rules for } \xrightarrow{H} - 1 \cdots)

\begin{align*}
\%H^{\alpha+1}(n) &= H^{\alpha}(n + 1) \\
\%H^{\gamma + \omega^{k+1}}(n) &= H^{\gamma + \omega^{k}.(n+1)}(n) \\
\%H^{\gamma + \omega^{\beta + \omega^{k+1}}}(n) &= H^{\gamma + \omega^{\beta + \omega^{k}.(n+1)}}(n)
\end{align*}

Prop. [Robustness]

$w \leq_* w'$ and $n \leq n'$ and $w'$ pure imply $H^{\alpha(w)}(n) \leq H^{\alpha(w')}(n')$

where purity means that $w'$ has no superfluous symbols
(a regular condition that can be enforced by LCS’s)
We now store $u$ and $I^n$ as two strings (with endmarker #) on two channels $p$ and $d$.

The diagram illustrates the process:
- $p$ receives $|u#$ and $d$ receives $|n#$.
- There is a process marked by $*$.
- The output is $u#$ and $|^{n+1}$#.

The diagram shows the flow of messages:
- $p!x$ and $d?x$ connect to the `copy` node.
- $p?x$ and $d!x$ also connect to `copy`.
- The `copy` node sends messages to `beg`, `wrap`, and `end`.
- `beg` sends $p!l$ and receives $p?l$.
- `wrap` sends $p?#$ and $p!#$ connection to `end`.
- `end` receives $d?#$, $d!#$.
Computing $H \rightarrow$ with LCS’s: second rule

$p: a_{i_1} \ldots a_{i_p} a_0 | u#$

$d: |^n#$

$\rightarrow$

$p: a_{i_1} \ldots a_{i_p} |^{n+1} a_0 u#$

$d: |^n#$
As we did for lossy counter machines, this time with channels

**Bottom line:** a LCS with $|\Sigma| = m + 3$
— can build a workspace of size $H^{\omega \omega^{m+1}}(m) = H^{\omega \omega}(m) = F_{\omega \omega}(m)$,
— use this as a computational resource,
— and fold back the workspace by computing the inverse of $H$

Checking that the above computation is performed reliably can be stated as (reduces to) a reachability (or termination) question

**Cor.** LCS verification is hard for $F_{\omega \omega}$ (hence ..-complete)

**Confirms:** the main parameter for complexity is the size of the message alphabet
WRAPPING IT UP (SKETCHILY)

As we did for lossy counter machines, this time with channels

**Bottom line:** a LCS with $|\Sigma| = m + 3$
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CONCLUSION FOR THE COURSE

Length of bad sequences is key to bounding the complexity of WQO-based algorithms.

Here computer scientists need results/theories from other fields: proof-theory and combinatorics.

Proving matching lower bounds is not necessarily tricky (and is easy for LCM’s or LCS’s) but we still lack:
— a tutorial/textbook on subrecursive hierarchies (like fast-growing and Hardy hierarchies)
— a toolkit of coding tricks for computing with ordinals
— a large enough user community

The approach is workable: we could characterize the complexity of Timed-Arc Petri nets and Data Petri Nets at level $F_{\omega^\omega}$.
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Thanks for your participation!