Algorithmic Aspects of WQO (Well-Quasi-Ordering) Theory

Part V: Proving Lower Bounds

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Lecture notes & exercices available at `http://tinyurl.com/esslli12wqo`

If you missed the earlier episodes

$(\mathbb{N}^k, \leqslant_\times)$ and $(\Sigma^*, \leqslant_*)$ are well-quasi-orderings: any infinite sequence $\mathbf{x} = x_0, x_1, x_2, \ldots$ contains an increasing pair $x_i \leqslant x_j$ ("is good")

If a sequence like $\mathbf{x}$ cannot grow too quickly —we say it is controlled— then the position i,j of the first increasing pair in $\mathbf{x}$ can be bounded by some length function $L_{X,\text{control}}(|x_0|)$

This gave us upper bounds for the complexity of wqo-based algorithms. Furthermore, these length functions can be precisely pinned down inside elegant subrecursive hierarchies

For example, it gave $\mathbf{F}_\omega$ upper-bounds for the verification —e.g., termination and/or safety— of monotonic counter machines, and $\mathbf{F}_{\omega^\omega}$ upper bounds for lossy channel systems

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That was just the EASY part!!!

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Today we consider the "hardness" question: are these upper bounds optimal? or do we have matching lowing bounds?

—the answer is often "positive" (?)

OUTLINE FOR PART V

- What is the question exactly? And why isn't obvious?
- A strategy for proving hardness
- Hardness for Lossy Counter Machines
- Hardness for Lossy Channel Systems

Problem Statement

We have upper bounds on the complexity of verification for lossy counter machines and lossy channel systems

Do we have matching lower bounds?

Could be for the simple-minded algorithms we presented in Part II

No for the underlying decision problems (witness: VASS's)

Exercise. Give a decision problem solvable in Ackermannian time of its input that requires Ackermannian time (where $Ack(n) \stackrel{\text{def}}{=} A(n,n)$ and A is the usual binary Ackermann function).

Pb 1. Input: x,y,z. Question: Does $A(x,y) = z$?

Pb 2. Input: x,y,x',y'. Question: Is $A(x,y) < A(x',y')$?

Pb 3. Input: x,y. Question: Is $A(x,y)$ prime?

Pb 4. Input: x,y. Question: Is $A(x,y)$ a sum $\sum_{i \in K} p_i^{F_i}$? where p_i and F_i are the ith prime (resp., Fibonacci) number

Pb 5. Input: x. Question: Does the Universal TM halts on x, and furthermore halts in time $Ack(x)$?

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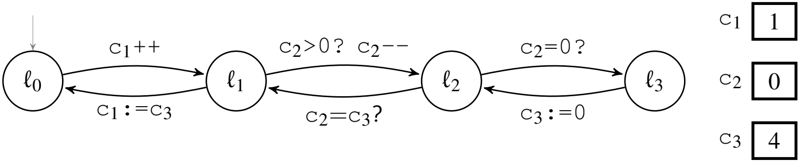
Proving lower bounds for unreliable models

We shall adopt the following strategy:

1. Compute unreliably functions in the Hardy hierarchy
2. Use the result as an unreliable computational ressource
3. “Check” in the end that nothing was lost
4. Need computing unreliably the inverses of Hardy functions

CM = Counter Machines



A run of M: $(\ell_0,0,1,4) \rightarrow_{\text{rel}} (\ell_1,1,1,4) \rightarrow_{\text{rel}} (\ell_2,1,0,4) \rightarrow_{\text{rel}} (\ell_3,1,0,4)$

Ordering states: $(\ell_1,0,0,0) \leqslant (\ell_1,0,1,2)$ but $(\ell_1,0,0,0) \not\leqslant (\ell_2,0,1,2)$.

NB. A counter machine like M above is not monotonic.

Can test that a counter is zero $\Rightarrow$ steps not compatible with ordering

(And we allow other guards/updates that break compatibility).

In fact, the ordering is used to model unreliability.

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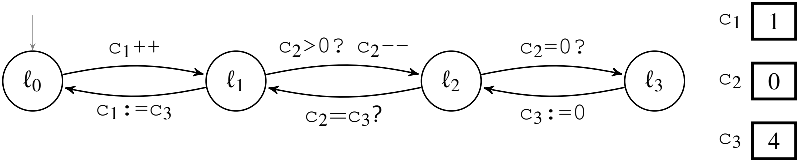

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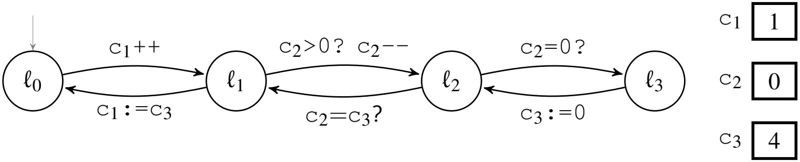

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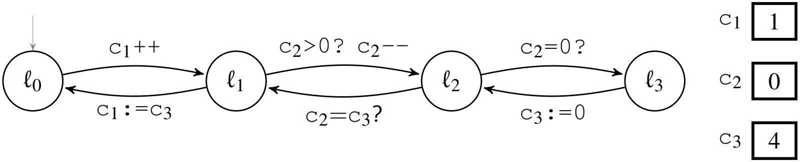

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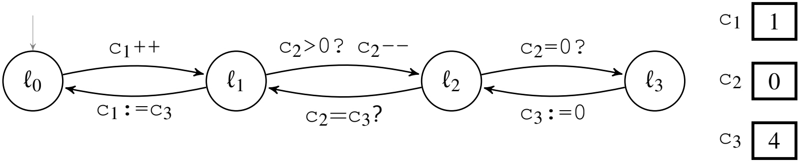

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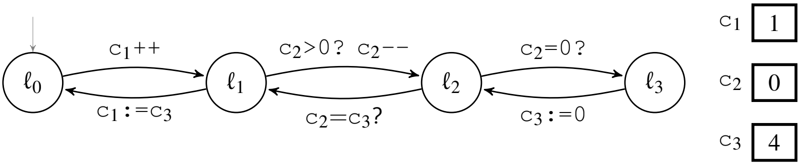

$$(\ell,\mathbf{a}) \to (\ell',\mathbf{b}) \overset{\text{def}}{\Leftrightarrow} (\ell,\mathbf{a}) \geqslant (\ell,\mathbf{x}) \to_{\text{rel}} (\ell',\mathbf{y}) \geqslant (\ell',\mathbf{b}) \text{ for some } \mathbf{x},\mathbf{y}$$

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The unreliable counter machine is a WSTS

Paradox: It does much more than its reliable twin but can compute much less

NB: These lossy counter machines are not a toy!!!

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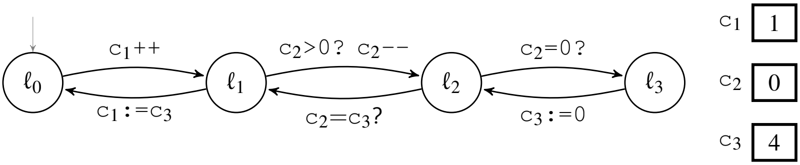

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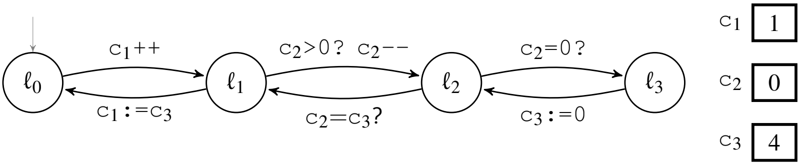

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Recall: Hardy Hierarchy

$$H^0(n) \stackrel{\text{def}}{=} n \qquad H^{\alpha+1}(n) \stackrel{\text{def}}{=} H^{\alpha}(n+1) \qquad H^{\lambda}(n) \stackrel{\text{def}}{=} H^{\lambda_n}(n)$$

Recall: $F_\alpha(n) = H^{\omega^\alpha}(n)$ $\quad H^\alpha(n) \leqslant H^\alpha(n+1)$

$\alpha \sqsubseteq \alpha'$ implies $H^\alpha(n) \leqslant H^{\alpha'}(n)$

Nb. $H^\alpha(n)$ can be evaluated by transforming a pair
$\alpha,n = \alpha_0,n_0 \xrightarrow{H} \alpha_1,n_1 \xrightarrow{H} \alpha_2,n_2 \xrightarrow{H} \cdots \xrightarrow{H} \alpha_k,n_k$ with $\alpha_0 > \alpha_1 > \alpha_2 > \cdots$
until eventually $\alpha_k = 0$ and $n_k = H^\alpha(n)$ % tail-recursion!!

Below we compute fast-growing functions and their inverses
by encoding $\alpha,n \xrightarrow{H} \alpha',n'$ and $\alpha',n' \xrightarrow{H}{}^{-1} \alpha,n$

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Encoding ordinals $< \omega^\omega$ in tuples of numbers

Write α in CNF with coefficients $\alpha = \omega^m.a_m + \omega^{m-1}.a_{m-1} + \cdots + \omega^0 a_0$

Encoding of α, n is $\langle a_m, \ldots, a_0; n\rangle \in \mathbb{N}^{m+2}$.

$$\langle a_m, \ldots, a_0{+}1; n\rangle \xrightarrow{H} \langle a_m, \ldots, a_0; n{+}1\rangle \qquad \% H^{\alpha+1}(n) = H^{\alpha}(n+1)$$

$$\langle a_m, \ldots, a_k{+}1, \overbrace{0, \ldots, 0}^{k>0}; n\rangle \xrightarrow{H} \langle a_m, \ldots, a_k, n, \overbrace{0, \ldots, 0}^{k-1}; n\rangle \qquad \% H^{\lambda}(n) = H^{\lambda_n}(n)$$

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Recall: $(\gamma + \omega^{k+1})_n \stackrel{\text{def}}{=} \gamma + \omega^k \cdot n$

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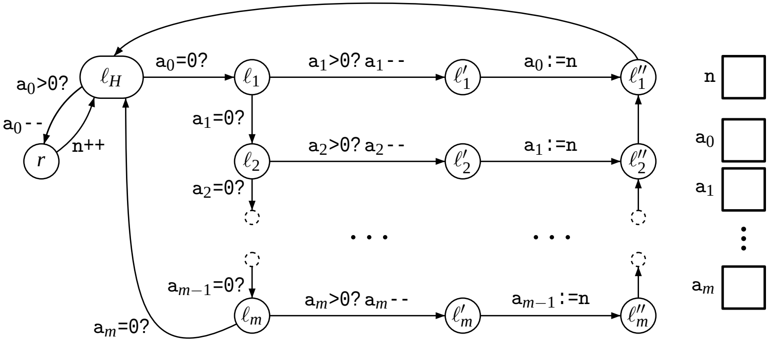

Now for $\xrightarrow{H}^{-1}$

$$\langle a_m,\ldots,a_0;n+1\rangle \xrightarrow{H}^{-1} \langle a_m,\ldots,a_0+1;n\rangle \qquad \% H^{\alpha+1}(n) = H^{\alpha}(n+1)$$

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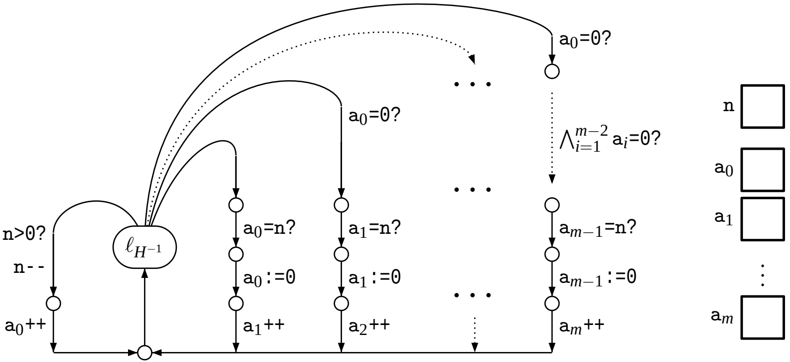
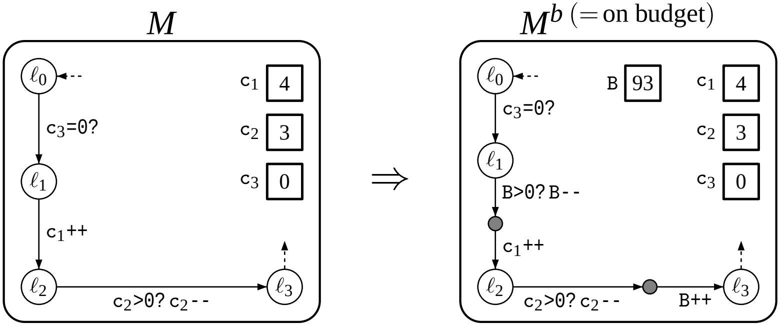


Prop. [Robustness] $\mathbf{a} \leqslant_\times \mathbf{a}'$ and $\mathrm{n} \leqslant \mathrm{n}'$ imply $H^{\alpha}(n) \leqslant H^{\alpha'}(n')$

Counter machines on a budget



Ensures:

1. $M^b \vdash (\ell, B, \mathbf{a}) \xrightarrow{*}_{\mathsf{rel}} (\ell', B', \mathbf{a}')$ implies $B + |\mathbf{a}| = B' + |\mathbf{a}'|$
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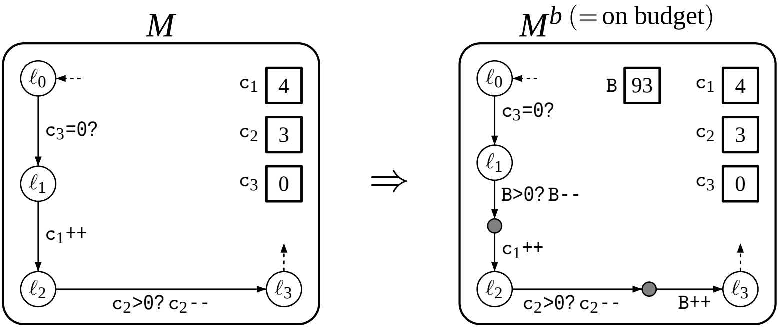

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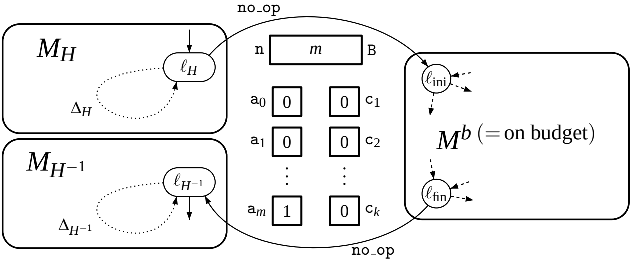
$M(m)$: Wrapping it up



Prop. $M(m)$ has a lossy run

$$(\ell_H, a_m:1, 0, \ldots, n:m, 0, \ldots) \xrightarrow{*} (\ell_{H-1}, 1, 0, \ldots, m, 0, \ldots)$$

iff $M(m)$ has a reliable run

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Cor. LCM verification is Ackermann-hard (hence ...-complete)

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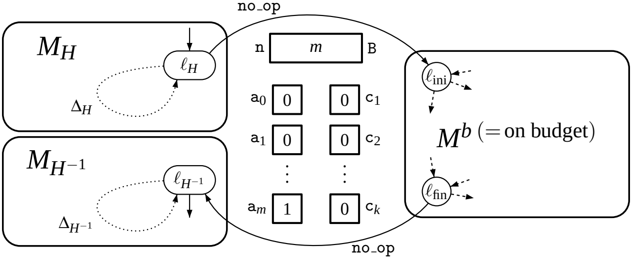

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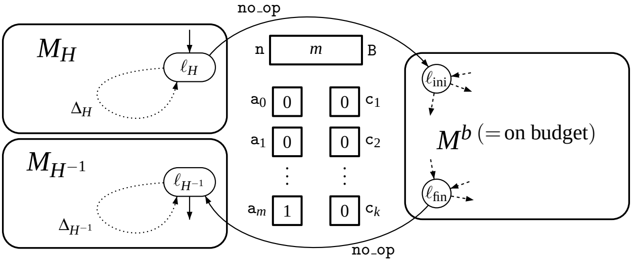

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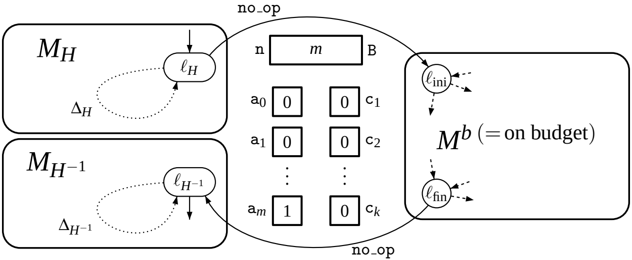

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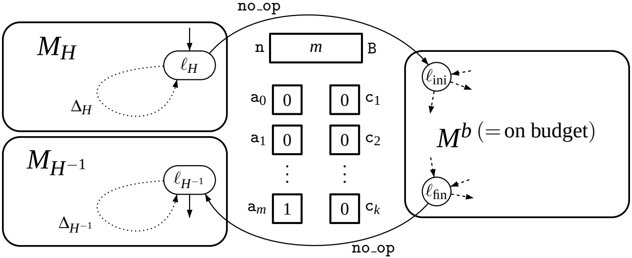

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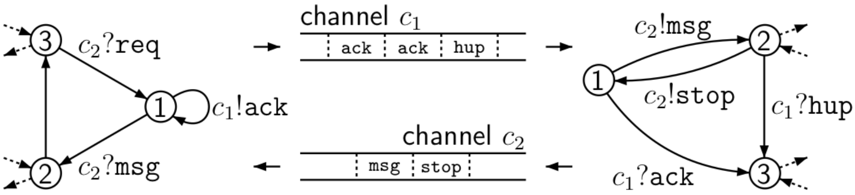
RECALL: LCS / LOSSY CHANNEL SYSTEMS



A configuration $\sigma = (\ell_1, \ell_2, w_1, w_2)$ with $w_i \in \Sigma^*$.
E.g., $w_1 = \mathtt{hup.ack.ack}$.

Reliable steps: $\sigma \to_{\text{rel}} \rho$ read in front of channels, write at end (FIFO)

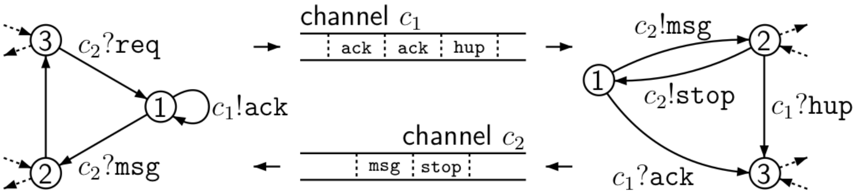
Lossy steps: messages may be lost nondeterministically

$$\sigma \to \sigma' \overset{\text{def}}{\Leftrightarrow} \sigma \sqsupseteq \rho \to_{\text{rel}} \rho' \sqsupseteq \sigma' \text{ for some } \rho, \rho'$$

where $(S, \sqsubseteq)$ is the wqo $(Loc_1, =) \times (Loc_2, =) \times (\Sigma^*, \leqslant_*)^{\{c_1, c_2\}}$

A model useful for concurrent protocols but also timed automata, metric temporal logic, products of modal logics, ...

Recall: LCS / Lossy Channel Systems



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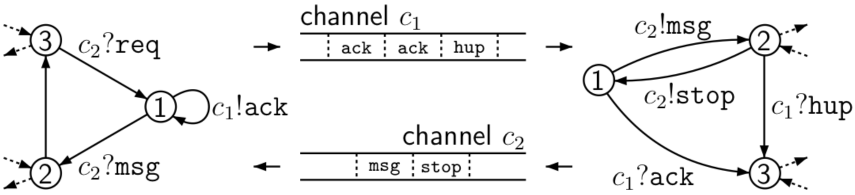

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ENCODING ORDINALS $< \omega^{\omega^\omega}$ IN CHANNELS

We use $\Sigma = \{\mathtt{a}_0, \ldots, \mathtt{a}_m\} \cup \{\mathsf{I}\}$ to encode ordinals $\alpha < \omega^{\omega^{m+1}}$.

Two-level "differential" encoding:

$\beta : \{\mathtt{a}_0, \ldots, \mathtt{a}_m\}^* \to \omega^{m+1}$

$\beta(\mathtt{a}_{r_1} \ldots \mathtt{a}_{r_k}) \stackrel{\text{def}}{=} \omega^{r_1} + \cdots + \omega^{r_k}$ E.g. $\beta(\epsilon) = 0$, $\beta(\mathtt{a}_3\mathtt{a}_0\mathtt{a}_0) = \omega^3 + 2$

$\alpha : \Sigma^* \to \omega^{\omega^{m+1}}$

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Difficulty. $\alpha(w)$ is not always a CNF

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Weakly computing $\xrightarrow{H}$ with LCS's

$$(\mathsf{I}w,n) \xrightarrow{H} (w,n+1) \qquad \% H^{\alpha+1}(n) = H^{\alpha}(n+1)$$

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$(\cdots$ similar rules for $\xrightarrow{H}^{-1}$ $\cdots)$

Prop. [Robustness]
$w \leqslant_* w'$ and $n \leqslant n'$ and w' pure imply $H^{\alpha(w)}(n) \leqslant H^{\alpha(w')}(n')$

where purity means that w' has no superfluous symbols
(a regular condition that can be enforced by LCS's)

WEAKLY COMPUTING $\xrightarrow{H}$ WITH LCS'S

$$(|w,n) \xrightarrow{H} (w,n+1) \qquad \%H^{\alpha+1}(n) = H^{\alpha}(n+1)$$

$$(u\mathtt{a}_0|w,n) \xrightarrow{H} (u|^{n+1}\mathtt{a}_0 w,n) \qquad \%H^{\gamma+\omega^{k+1}}(n) = H^{\gamma+\omega^{k}\cdot(n+1)}(n)$$

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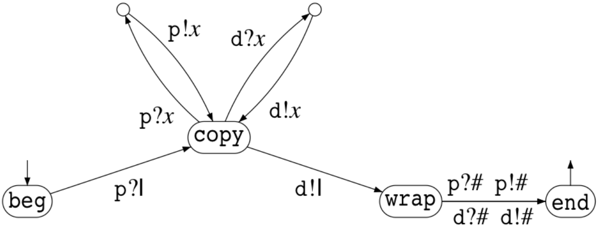
COMPUTING $\xrightarrow{H}$ WITH LCS'S: FIRST RULE

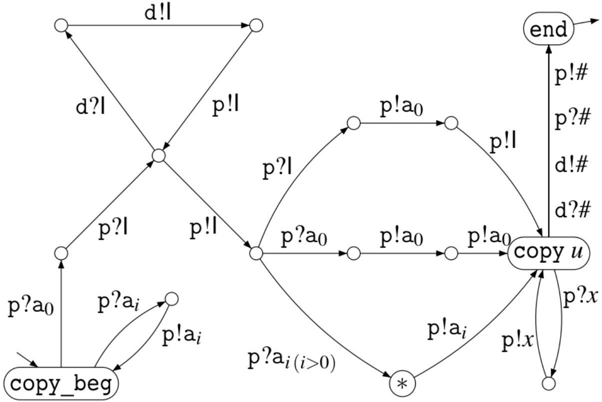
We now store u and I^n as two strings (with endmarker #) on two channels `p` and `d`.

$$\begin{array}{ll} \texttt{p:} & \mathsf{I}u\# \\ \texttt{d:} & \mathsf{I}^n\# \end{array} \quad \xrightarrow{*} \quad \begin{array}{l} u\# \\ \mathsf{I}^{n+1}\# \end{array}$$



COMPUTING $\xrightarrow{H}$ WITH LCS'S: SECOND RULE

$$\begin{array}{ll} \hline \texttt{p:} & \texttt{a}_{i_1}\ldots\texttt{a}_{i_p}\texttt{a}_0\,|\,u\# \\ \hline \\ \hline \texttt{d:} & |^n\# \\ \hline \end{array} \quad \xrightarrow{*} \quad \begin{array}{l} \hline \texttt{a}_{i_1}\ldots\texttt{a}_{i_p}|^{n+1}\texttt{a}_0 u\# \\ \hline \\ \hline |^n\# \\ \hline \end{array}$$



WRAPPING IT UP (SKETCHILY)

As we did for lossy counter machines, this time with channels

Bottom line: a LCS with $|\Sigma| = m+3$

— can build a workspace of size $H^{\omega^{\omega^{m+1}}}(m) = H^{\omega^{\omega^{\omega}}}(m) = F_{\omega^\omega}(m)$,
— use this as a computational resource,
— and fold back the workspace by computing the inverse of H

Checking that the above computation is performed reliably can be stated as (reduces to) a reachability (or termination) question

Cor. LCS verification is hard for $\mathbf{F}_{\omega^\omega}$ (hence ..-complete)

Confirms: the main parameter for complexity is the size of the message alphabet

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Conclusion for the course

Length of bad sequences is key to bounding the complexity of WQO-based algorithms

Here computer scientists need results/theories from other fields: proof-theory and combinatorics

Proving matching lower bounds is not necessarily tricky (and is easy for LCM's or LCS's) but we still lack:

— a collection of hard problems: Post Embedding Problem, ...

— a tutorial/textbook on subrecursive hierarchies (like fast-growing and Hardy hierarchies)

— a toolkit of coding tricks for computing with ordinals

— a large enough user community

The approach is workable: we could characterize the complexity of Timed-Arc Petri nets and Data Petri Nets at level $\mathbf{F}_{\omega^{\omega^{\omega}}}$

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Thanks for your participation!