Algorithmic Aspects of WQO (Well-Quasi-Ordering) Theory
Part I: Basics of WQO Theory

Sylvain Schmitz & Philippe Schnoebelen
LSV, CNRS & ENS Cachan


Lecture notes & exercices available at http://tinyurl.com/esslllil2wqo
MOTIVATIONS FOR THE COURSE

- Well-quasi-orderings (wqo’s) proved to be a powerful tool for decidability/termination in logic, AI, program verification, etc. *NB:* they can be seen as a version of well-founded orderings with more flexibility.

- In program verification, wqo’s are prominent in well-structured transition systems (WSTS’s), a generic framework for infinite-state systems with good decidability properties.

- Analysing the complexity of wqo-based algorithms is still one of the dark arts . . .

- Purposes of these lectures = to disseminate the basic concepts and tools one uses for the complexity analysis of wqo-based algorithms.
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OUTLINE OF THE COURSE

- (This) Lecture 1 = **Basics of Wqo’s.** Rather basic material: explaining and illustrating the definition of wqo’s. Building new wqo’s from simpler ones.

- Lecture 2 = **Algorithmic Applications of Wqo’s.** Well-Structured Transition Systems, Program Termination, Relevance Logic, etc.

- Lecture 3 = **Complexity Classes for Wqo’s.** Fast-growing complexity. Working with subrecursive hierarchies.

- Lecture 4 = **Proving Complexity Lower Bounds.** Simulating fast-growing functions with weak/unreliable computation models.

- Lecture 5 = **Proving Complexity Upper Bounds.** Bounding the length of bad sequences (for Dickson’s and Higman’s Lemmas).
Recalls) Ordered Sets

**Def.** A non-empty \((X, \leq)\) is a quasi-ordering (qo) \(\text{def} \leq\) is a reflexive and transitive relation.

\(\approx\) a partial ordering without requiring antisymmetry, technically simpler but essentially equivalent

**Examples.** \((\mathbb{N}, \leq)\), also \((\mathbb{R}, \leq)\), \((\mathbb{N} \cup \{\omega\}, \leq)\), \ldots

divisibility: \((\mathbb{Z}, \_ | \_)\) where \(x | y \text{def} \exists a : a.x = y\)
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**Def.** \((X, \leq)\) is **linear** if for any \(x, y \in X\) either \(x \leq y\) or \(y \leq x\). (I.e., there is no \(x \# y\).)

**Def.** \((X, \leq)\) is **well-founded** if there is no infinite strictly decreasing sequence \(x_0 > x_1 > x_2 > \cdots\)

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**Well-Quasi-Ordering (WQO)**

**Def1.** $(X, \leq)$ is a wqo $\iff$ any infinite sequence $x_0, x_1, x_2, \ldots$ contains an increasing pair: $x_i \leq x_j$ for some $i < j$.

**Def2.** $(X, \leq)$ is a wqo $\iff$ any infinite sequence $x_0, x_1, x_2, \ldots$ contains an infinite increasing subsequence: $x_{n_0} \leq x_{n_1} \leq x_{n_2} \leq \ldots$

**Def3.** $(X, \leq)$ is a wqo $\iff$ there is no infinite strictly decreasing sequence $x_0 > x_1 > x_2 > \ldots$ —i.e., $(X, \leq)$ is well-founded— and no infinite set $\{x_0, x_1, x_2, \ldots\}$ of mutually incomparable elements $x_i \# x_j$ when $i \neq j$ —we say “$(X, \leq)$ has no infinite antichain”—.

**Fact.** These three definitions are equivalent. Clearly, Def2 $\Rightarrow$ Def1 and Def1 $\Rightarrow$ Def3 (think contrapositively). But the reverse implications are non-trivial.

Recall Infinite Ramsey Theorem: “Let $X$ be some countably infinite set and colour the elements of $X^{(n)}$ (the subsets of $X$ of size $n$) in $c$ different colours. Then there exists some infinite subset $M$ of $X$ s.t. the size $n$ subsets of $M$ all have the same colour.”
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More generally

**Fact.** For linear qo’s: well-founded ⇔ wqo.

**Cor.** Any ordinal is wqo.
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\( (\mathbb{Z}, \mid) \): The prime numbers \( \{2, 3, 5, 7, 11, \ldots\} \) are an infinite antichain.
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<tr>
<td>(\mathbb{N} \cup {\omega}, \leq)</td>
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<td>(\mathbb{N}^3, \leq\times)</td>
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<td>(\Sigma^*, \leq_{\text{pref}})</td>
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<td>(\Sigma^*, \leq_{\text{lex}})</td>
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<td>(\Sigma^*, \leq_{\ast})</td>
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More generally

**(Generalized) Dickson’s lemma.** If \((X_1, \leq_1), \ldots, (X_n, \leq_n)\)’s are wqo’s, then \(\prod_{i=1}^n X_i, \leq_{\times}\) is wqo.

**Proof.** Easy with Def2. Otherwise, an application of the Infinite Ramsey Theorem.

**(Usual) Dickson’s Lemma.** \((\mathbb{N}^k, \leq_{\times})\) is wqo for any \(k\).
**Spot the WQO's**

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<thead>
<tr>
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<th>linear?</th>
<th>well-founded?</th>
<th>wqo?</th>
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<td>( \Sigma^*, \leq_{\text{lex}} )</td>
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<tr>
<td>( \Sigma^*, \leq_{\ast} )</td>
<td>x</td>
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</tbody>
</table>

\((\Sigma^*, \leq_{\text{pref}})\) has an infinite antichain

\(bb, bab, baab, baaab, \ldots\)

\((\Sigma^*, \leq_{\text{lex}})\) is not well-founded:

\(b >_{\text{lex}} ab >_{\text{lex}} aab >_{\text{lex}} aaab >_{\text{lex}} \ldots\)
**Spot the wqo’s**

<table>
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<td>✔️</td>
<td>x</td>
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<tr>
<td>(\Sigma^<em>, \leq_</em>)</td>
<td>x</td>
<td>✔️</td>
<td>✔️</td>
</tr>
</tbody>
</table>

\((\Sigma^*, \leq_*)\) is wqo by Higman’s Lemma (see next slide).

We can get some feeling by trying to build a bad sequence, i.e., some \(w_0, w_1, w_2, \ldots\) without an increasing pair \(w_i \leq_* w_j\).
**Higman’s Lemma**

**Def.** The sequence extension of a qo \((X, \leq)\) is the qo \((X^*, \leq_*)\) of finite sequences over \(X\) ordered by embedding:

\[
w = x_1 \ldots x_n \leq_* y_1 \ldots y_m = v \iff x_1 \leq y_{l_1} \land \ldots \land x_n \leq y_{l_n}
\]

for some \(1 \leq l_1 < l_2 < \ldots < l_n \leq m\)

\[
\iff w \leq_\times v' \text{ for a length-}n \text{ subsequence } v' \text{ of } v
\]

**Higman’s Lemma.** \((X^*, \leq_*)\) is a wqo iff \((X, \leq)\) is.

With \((\Sigma^*, \leq_*)\), we are considering the sequence extension of \((\Sigma, =)\) which is finite, hence necessarily wqo.

Later we’ll consider the sequence extension of more complex wqo’s, e.g., \(\mathbb{N}^2\):

\[
\begin{array}{ccc}
0 & 2 & 0 \\
1 & 0 & 2
\end{array}
\leq_* ?
\begin{array}{ccc}
2 & 0 & 0 \\
2 & 2 & 0
\end{array}
\leq_\times
\begin{array}{ccc}
2 & 2 & 0 \\
0 & 1
\end{array}
\]
**Higman’s Lemma**

**Def.** The sequence extension of a qo \((X, \leq)\) is the qo \((X^*, \leq^*)\) of finite sequences over \(X\) ordered by embedding:

\[
w = x_1 \ldots x_n \leq^* y_1 \ldots y_m = v \iff x_1 \leq y_{l_1} \land \ldots \land x_n \leq y_{l_n} \text{ for some } 1 \leq l_1 < l_2 < \ldots < l_n \leq m\]

\[
\iff w \leq^* v' \text{ for a length-}n \text{ subsequence } v' \text{ of } v
\]

**Higman’s Lemma.** \((X^*, \leq^*)\) is a wqo iff \((X, \leq)\) is.

With \((\Sigma^*, \leq^*)\), we are considering the sequence extension of \((\Sigma, =)\) which is finite, hence necessarily wqo.

Later we’ll consider the sequence extension of more complex wqo’s, e.g., \(\mathbb{N}^2\):

\[
|0|2|0|2 \leq^*? |2|0|2|2|2|0|0|1
\]
Proof of Higman’s Lemma

Let \((X, \leq)\) be wqo and assume by way of contradiction that \((X^*, \leq_*)\) admits bad sequences (sequences with no increasing pairs).

Let \(w_0 \in X^*\) be the shortest word that can start a bad sequence.
Let \(w_1 \in X^*\) be the shortest word that can continue, i.e., such that there is a bad sequence starting with \(w_0, w_1\)
Continue. This way we pick an infinite sequence \(S = w_0, w_1, w_2, w_3, \ldots\)

Claim. \(S\) too is bad (easy with Def1)

Write \(w_i\) under the form \(w_i = x_i v_i\). Since \(X\) is wqo, there is an infinite increasing sequence \(x_{n_0} \leq x_{n_1} \leq x_{n_2} \leq \cdots\) (here we use Def2)

Now consider \(S' \overset{\text{def}}{=} w_0, w_1, \ldots, w_{n_0-1}, v_{n_0}, v_{n_1}, v_{n_2}, \ldots\)
It cannot be bad (otherwise \(w_{n_0}\) would not have been shortest).
But an increasing pair \(v_n \leq_* v_m\) leads to \(x_n v_n \leq_* x_m v_m\), i.e., \(w_n \leq_* w_m\), a contradiction.
Proof of Higman’s Lemma

Let $(X, \leq)$ be wqo and assume by way of contradiction that $(X^*, \leq^*)$ admits bad sequences (sequences with no increasing pairs).

Let $w_0 \in X^*$ be the shortest word that can start a bad sequence.

Let $w_1 \in X^*$ be the shortest word that can continue, i.e., such that there is a bad sequence starting with $w_0, w_1$.

Continue. This way we pick an infinite sequence $S = w_0, w_1, w_2, w_3, \ldots$

Claim. $S$ too is bad (easy with Def1)

Write $w_i$ under the form $w_i = x_i v_i$. Since $X$ is wqo, there is an infinite increasing sequence $x_{n_0} \leq x_{n_1} \leq x_{n_2} \leq \cdots$ (here we use Def2)

Now consider $S' \overset{\text{def}}{=} w_0, w_1, \ldots, w_{n_0-1}, v_{n_0}, v_{n_1}, v_{n_2}, \ldots$

It cannot be bad (otherwise $w_{n_0}$ would not have been shortest).

But an increasing pair $v_n \leq^* v_m$ leads to $x_n v_n \leq^* x_m v_m$, i.e., $w_n \leq^* w_m$, a contradiction.
**Proof of Higman’s Lemma**

Let \((X, \leq)\) be wqo and assume by way of contradiction that \((X^*, \leq^*)\) admits **bad** sequences (sequences with no increasing pairs).

Let \(w_0 \in X^*\) be the **shortest** word that can start a bad sequence.

Let \(w_1 \in X^*\) be the **shortest word that can continue**, i.e., such that there is a bad sequence starting with \(w_0, w_1\).

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PROOF OF HIGMAN’S LEMMA

Let \((X, \leq)\) be wqo and assume by way of contradiction that \((X^*, \leq_*)\) admits bad sequences (sequences with no increasing pairs).
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Let \((X, \leq)\) be wqo and assume by way of contradiction that \((X^*, \leq_*)\) admits bad sequences (sequences with no increasing pairs).

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Proof of Higman’s Lemma

Let \((X, \leq)\) be wqo and assume by way of contradiction that \((X^*, \leq^*)\) admits bad sequences (sequences with no increasing pairs).

Let \(w_0 \in X^*\) be the shortest word that can start a bad sequence.

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Continue. This way we pick an infinite sequence \(S = w_0, w_1, w_2, w_3, \ldots\)

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Write \(w_i\) under the form \(w_i = x_i v_i\). Since \(X\) is wqo, there is an infinite increasing sequence \(x_{n_0} \leq x_{n_1} \leq x_{n_2} \leq \cdots\) (here we use Def2)

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Proof of Higman’s Lemma

Let \((X, \preceq)\) be wqo and assume by way of contradiction that \((X^*, \preceq_*)\) admits bad sequences (sequences with no increasing pairs).
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Continue. This way we pick an infinite sequence \(S = w_0, w_1, w_2, w_3, \ldots\)

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Write \(w_i\) under the form \(w_i = x_i v_i\). Since \(X\) is wqo, there is an infinite increasing sequence \(x_{n_0} \preceq x_{n_1} \preceq x_{n_2} \preceq \cdots\) (here we use Def2)

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MORE WQO’S

- Finite Trees ordered by embeddings (Kruskal’s Tree Theorem)
**Proof of Kruskal’s Tree Theorem**

Let \((X, \leq)\) be wqo and assume, b.w.o.c., that \((T(X), \sqsubseteq)\) is not wqo.

We pick a “minimal” bad sequence \(S = t_0, t_1, t_2, \ldots\) —Def1

Write every \(t_i\) under the form \(t_i = f_i(u_{i,1}, \ldots, u_{i,k_i})\).

Claim. The set \(U = \{u_{i,j}\}\) of the immediate subterms is wqo.
(Indeed, an infinite bad sequence \(u_{i_0,j_0}, u_{i_1,j_1}, \ldots\) could be used to show that \(t_{i_0}\) was not shortest).

Since \(U\) is wqo, and using Higman’s Lemma on \(U^*\), there is some
\((u_{n_1,1}, \ldots, u_{n_1,k_{n_1}}) \leq^* (u_{n_2,1}, \ldots, u_{n_2,k_{n_2}}) \leq^* (u_{n_3,1}, \ldots, u_{n_3,k_{n_3}}) \leq^* \cdots\) —Def2

Further extracting some \(f_{n_{i_1}} \leq f_{n_{i_2}} \leq \cdots\) exhibits an infinite increasing subsequence \(t_{n_{i_1}} \sqsubseteq t_{n_{i_2}} \sqsubseteq \cdots\) in \(S\), a contradiction.
PROOF OF KRUSKAL’S TREE THEOREM

Let \((X, \leq)\) be wqo and assume, b.w.o.c., that \((T(X), \sqsubseteq)\) is not wqo.

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**MORE WQO’S**

- Finite Trees ordered by embeddings (Kruskal’s Tree Theorem)

  ![Tree Diagram]

- Finite Graphs ordered by embeddings (Robertson-Seymour Theorem)

  \[ C_n \leq_{\text{minor}} K_n \text{ and } C_n \leq_{\text{minor}} C_{n+1} \]

- \((X^{\omega}, \leq_*)\) for \(X\) linear wqo.

- \((\mathcal{P}_f(X), \sqsubseteq_H)\) for \(X\) wqo, where

  \[ U \sqsubseteq_H V \overset{\text{def}}{\iff} \forall x \in U : \exists y \in V : x \leq y \]
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\[ U \sqsubseteq_H V \iff \forall x \in U : \exists y \in V : x \leq y \]
FINITE-BASES CHARACTERIZATION

Defn. \((X, \leq)\) is a \(wqo\) if every non-empty subset \(V\) of \(X\) has at least one and at most finitely many (non-equivalent) minimal elements.

Say \(V \subseteq X\) is \text{upward-closed} if \(x \geq y \in V\) implies \(x \in V\). (There is a similar notion of \text{downward-closed} sets).

For \(B \subseteq X\), the \text{upward-closure} \(\uparrow B\) of \(B\) is \(\{x \mid x \geq b \text{ for some } b \in B\}\).

Note that \(\uparrow (\bigcup_i B_i) = \bigcup_i \uparrow B_i\), and that \(V\) is \text{upward-closed} iff \(V = \uparrow V\).

Cor1. Any \text{upward-closed} \(U \subseteq X\) has a \text{finite basis}, i.e., \(U\) is some \(\uparrow \{m_1, \ldots, m_k\}\).

Cor2. Any \text{downward-closed} \(V \subseteq X\) can be defined by a finite set of excluded minors:

\[ x \in V \iff m_1 \not\preceq x \land \cdots \land m_k \not\preceq x \]
**FINITE-BASIS CHARACTERIZATION**

**Defn.** $(X, \leq)$ is a wqo $\iff$ every non-empty subset $V$ of $X$ has at least one and at most finitely many (non-equivalent) minimal elements.

Say $V \subseteq X$ is **upward-closed** if $x \geq y \in V$ implies $x \in V$. (There is a similar notion of downward-closed sets).

For $B \subseteq X$, the **upward-closure** $\uparrow B$ of $B$ is $\{x \mid x \geq b \text{ for some } b \in B\}$.

Note that $\uparrow (\bigcup_i B_i) = \bigcup_i \uparrow B_i$, and that $V$ is upward-closed iff $V = \uparrow V$.

**Cor1.** Any upward-closed $U \subseteq X$ has a **finite basis**, i.e., $U$ is some $\uparrow \{m_1, \ldots, m_k\}$.

**Cor2.** Any downward-closed $V \subseteq X$ can be defined by a finite set of excluded minors:

$$x \in V \iff m_1 \not\preceq x \land \cdots \land m_k \not\preceq x$$
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**Defn.** \((X, \leq)\) is a wqo \(\overset{\text{def}}{\iff}\) every non-empty subset \(V\) of \(X\) has at least one and at most finitely many (non-equivalent) minimal elements.

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For \(B \subseteq X\), the **upward-closure** \(\uparrow B\) of \(B\) is \(\{x \mid x \geq b \text{ for some } b \in B\}\). Note that \(\uparrow (\bigcup_i B_i) = \bigcup_i \uparrow B_i\), and that \(V\) is upward-closed iff \(V = \uparrow V\).

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**Cor2.** Any downward-closed \(V \subseteq X\) can be defined by a finite set of excluded minors:

\[
x \in V \iff m_1 \not\leq x \land \cdots \land m_k \not\leq x
\]

E.g, **Kuratowski Theorem**: a graph is planar iff it does not contain \(K_5\) or \(K_{3,3}\).

Gives polynomial-time characterization of closed sets.
**FINITE-BASIS CHARACTERIZATION**

**Defn.**  \((X, \leq)\) is a wqo \(\text{def} \) every non-empty subset \(V\) of \(X\) has at least one and at most finitely many (non-equivalent) minimal elements.

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**Cor2.** Any downward-closed \(V \subseteq X\) can be defined by a finite set of excluded minors:

\[
x \in V \iff m_1 \not\succeq x \land \cdots \land m_k \not\succeq x
\]

**Cor3.** Any sequence \(\uparrow V_0 \subseteq \uparrow V_1 \subseteq \uparrow V_2 \subseteq \cdots\) of upward-closed subsets converges in finite-time: \(\exists m : (\bigcup_i \uparrow V_i) = \uparrow V_m = \uparrow V_{m+1} = \cdots\)
Beyond wqo’s

For \((X, \leq)\), we consider \((\mathcal{P}(X), \subseteq_S)\) defined with

\[
U \subseteq_S V \iff \forall y \in V : \exists x \in U : x \leq y \quad (\iff \uparrow U \supseteq \uparrow V)
\]

Fact. \(\mathcal{P}(X)\) is well-founded iff \(X\) is wqo

---Defn’

NB. \(X\) well-founded \(\not\Rightarrow\) \(\mathcal{P}(X)\) well-founded

Question. Does \(X\) wqo \(\Rightarrow\) \(\mathcal{P}(X)\) wqo? (Equivalently \(\mathcal{P}_f(X)\) wqo?)
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**BEYOND WQO’S**

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\[
X \overset{\text{def}}{=} \{(a, b) \in \mathbb{N}^2 | a < b\}
\]

\[ (a, b) < (a', b') \overset{\text{def}}{\iff} \begin{cases} a = a' \text{ and } b < b' \\ \text{or } b < a' \end{cases} \]

**Fact.** \((X, \leq)\) is WQO
BEYOND WQO’S

For \((X, \leq)\), we consider \((\mathcal{P}(X), \subseteq_S)\) defined with

\[ U \subseteq_S V \iff \forall y \in V : \exists x \in U : x \leq y \quad (\iff \uparrow U \supseteq \uparrow V) \]

**Fact.** \(\mathcal{P}(X)\) is well-founded iff \(X\) is wqo

---Defn’

**NB.** \(X\) well-founded \(\not\Rightarrow\) \(\mathcal{P}(X)\) well-founded

**Question.** Does \(X\) wqo \(\Rightarrow\) \(\mathcal{P}(X)\) wqo? (Equivalently \(\mathcal{P}_f(X)\) wqo?)

**Thm. 1.** \((\mathcal{P}_f(X), \subseteq_S)\) is not wqo: rows are incomparable

**Thm. 2.** \((\mathcal{P}(Y), \subseteq_S)\) is wqo iff \(Y\) does not contain \(X\)