**TD 8: Pushdown Systems**

**Exercise 1** (Regular Valuations). The course notes prove the decidability of LTL model checking with *simple valuations* \( \nu : P \times \Gamma^* \rightarrow 2^{AP} \) satisfying \( \nu(qZ\gamma) = \nu(qZ) \) for all \( q \) in \( P \), \( Z \) in \( \Gamma \) and \( \gamma \) in \( \Gamma^* \).

A *regular valuation* is defined through a collection of finite complete deterministic automata \( A_p = \langle Q_p, \Gamma \sqcup P, \delta_p, q_{0,p}, F_p \rangle \) for each \( p \in AP \), s.t.

\[
\nu(qZ\gamma) = \{ p \in AP \mid \delta_p(q_{0,p}, \gamma^R Zq) \in F_p \},
\]

i.e. each \( A_p \) is run bottom-to-top on the stack and pushdown state, and \( p \) holds if we reach a final state in \( F_p \).

Show that the LTL model-checking problem with regular valuations for PDS can be reduced to the LTL model-checking problem with simple valuations.

**Exercise 2** (CTL* Model Checking). Show that CTL* model checking with regular valuations can be reduced to LTL model checking with simple valuations.

**Exercise 3** (Multi-Pushdown Systems). A \( n \)-dimensional *multi-pushdown system* is a tuple \( M = \langle P, \Gamma, (\Delta_i)_{0 \leq i \leq n} \rangle \) where \( n \geq 1 \) is the number of stacks, \( P \) a finite set of states, \( \Gamma \) a finite stack alphabet, and each \( \Delta_i \subseteq P \times \Gamma \times P \times \Gamma^* \) is a finite transition relation. A configuration of a \( n \)-MPDS is a tuple \( c = (q, \gamma_1, \ldots, \gamma_n) \) in \( P \times (\Gamma^*)^n \). The *transition* relation \( \Rightarrow \) on configurations is defined as \( \Rightarrow = \bigcup_{0 \leq i \leq n} \Rightarrow_i \), where

\[
(q, \gamma_1, \ldots, Z\gamma_i, \ldots, \gamma_n) \Rightarrow_i (q', \gamma_1, \ldots, \gamma'_i \gamma_i, \ldots, \gamma_n)
\]

iff \( qZ \hookrightarrow_i q' \gamma'_i \) is in \( \Delta_i \).

1. Show that the *control state reachability problem*, i.e. given an initial configuration \( c \) in \( P \times \Gamma^n \) and a control state \( p \in P \), whether there exist \( \gamma_1, \ldots, \gamma_n \) s.t. \( c \Rightarrow^* (q, \gamma_1, \ldots, \gamma_n) \), is undecidable as soon as \( n \geq 2 \).

2. Let us consider a restriction on \( \Rightarrow^* \): \( k \)-bounded runs between are defined as the \( k \)th iterates \( c \rightarrow^k c' \) of the relation

\[
c \rightarrow c' \text{ iff } \exists i.c \Rightarrow_i^* c'
\]

i.e. a \( k \)-bounded run can be decomposed into \( k \) subruns where a single PDS is running.

Show that the \( k \)-bounded control-state reachability problem, i.e. given an initial configuration \( c \) in \( P \times \Gamma^n \) and a control state \( p \in P \), whether there exist \( \gamma_1, \ldots, \gamma_n \) s.t. \( c \rightarrow^k (q, \gamma_1, \ldots, \gamma_n) \), is decidable.