TD 7: BDDs

Exercise 1 (Some BDDs). Draw the reduced BDDs for the following functions, using the order of your choice on the variables \( \{x_1, x_2, x_3\} \):

1. \((x_1 \iff x_2) \lor (x_1 \iff x_3)\),
2. the majority function \(m(x_1, x_2, x_3)\): its value is 1 iff the majority of the input bits are 1’s,
3. the constant sum function \(s_c(x_1, x_2, x_3)\) for \(c = 1\): its value is 1 iff \(c = \sum_{i=1}^{3} x_i\),
4. the hidden weighted bit function \(h(x_1, x_2, x_3)\): its value is that of variable \(x_s\), where \(s = \sum_{i=1}^{3} x_i\) and \(x_0\) is defined as 0.

Exercise 2 (Symmetric Functions). A symmetric function of \(n\) variables has the same value for all permutations of the same \(n\) tuple of arguments. Clearly, all variable orderings lead to the same reduced BDD size for symmetric functions.

Show that a reduced BDD for a symmetric function has at most \(\binom{n+2}{2}\) nodes.

Exercise 3 (Counting Solutions). Write a linear time algorithm for counting the number of solutions of a boolean function \(f\) represented by a reduced BDD, i.e. of the number of valuations \(\nu\) s.t. \(\nu | = f\).

Exercise 4 (Shared BDDs). When dealing with several boolean functions at once, with a fixed order on the variables, one can share the reduced BDDs for identical subfunctions. A shared BDD between \(m\) functions is a reduced BDD with \(m\) root pointers assigning a root node to each of the functions.

Let \(x_1, \ldots, x_{2n}\) be the ordered set of variables. We want to compute the \(n+1\) bits \(f_{n+1}f_n \cdots f_1\) of the sum of two \(n\) bits numbers \(x_1x_3 \cdots x_{2n-1}\) and \(x_2x_4 \cdots x_{2n}\). Represent the shared BDD for the functions \(f_3, \ldots, f_1\), i.e. for \(n = 2\).

Exercise 5 (An Upper Bound on the Size of BDDs). The size \(B(f)\) of a reduced BDD for a function \(f\) is defined as the number of its nodes. Consider an arbitrary boolean function \(f\) on the ordered set \(x_1 \cdots x_n\), and consider a variable \(x_k\).

1. Show that we can bound the number of nodes labeled by \(\{x_1, \ldots, x_k\}\) by \(2^k - 1\).
2. How many different subfunctions on the ordered set of variables \(x_{k+1} \cdots x_n\) exist? Deduce another bound for the number of nodes labeled by \(\{x_{k+1}, \ldots, x_n\}\).
3. What global bound do you obtain for \( k = n - \log_2(n - \log_2 n) \)?

**Exercise 6** (Finding the Optimal Order). There are in general \( n! \) different orders for the variables \( \{x_1, \ldots, x_n\} \), and building the ORBDD for each of these is computationally expensive. One can nevertheless design an exponential time algorithm for finding the optimal order. Indeed, an optimal ordering on a subset \( X \) of variables does not depend on the order in which \( \{x_1, \ldots, x_n\}\setminus X \) has been accessed.

1. Given a subset \( X \) of \( \{x_1, \ldots, x_n\} \) and a variable \( x \) in \( X \), how many nodes labeled by \( x \) does a BDD have if it first treats \( \{x_1, \ldots, x_n\}\setminus X \), then \( x \), and last \( X\setminus\{x\} \)?

2. Reduce the optimal order problem to the search of a path of minimal weight in a weighted graph with subsets of \( \{x_1, \ldots, x_n\} \) as vertices.

3. Find the optimal order for the functions of Exercises 1.1 and 1.4.

**Exercise 7** (Quasi Reduced BDDs). An ordered BDD for a boolean function \( f \) on \( \{x_1, \ldots, x_n\} \) is *complete* if all paths from the root to a sink are of length \( n \). A BDD is *quasi reduced* if it is complete and no two nodes define the same subfunction.

1. Give the quasi reduced BDD for the majority function of Exercise 1.2.

2. Show that a quasi reduced BDD is unique up to isomorphism for an ordered set of variables \( x_1 \cdots x_n \).

3. Let \( Q(f) \) be the size of the quasi reduced BDD for the boolean function \( f \) on the ordered set of variables \( x_1 \cdots x_n \). Show that \( Q(f) \leq (n + 1)B(f) \).

**Exercise 8** (Minimal DFAs). A deterministic finite automaton \( A \) recognizes a boolean function \( f \) on the ordered set of variables \( x_1 \cdots x_n \) if \( L(A) = \{ \nu \in \{0,1\}^n \mid \nu \models f \} \), i.e. \( A \) recognizes exactly the solutions of \( f \).

What are the relations between the reduced BDD, the quasi reduced BDD, and the minimal DFA recognizing the same boolean function \( f \) on the ordered set of variables \( x_1 \cdots x_n \)?