TD 7: Simulation & Bisimulation

Exercise 1 (Bisimulations). Consider the following Kripke structures:

For each couple of structures, exhibit a bisimulation relation if they are bisimilar, or a CTL* formula allowing to distinguish between them if they are not bisimilar.

Exercise 2 (Computing the Coarsest Bisimulation). Computing $\equiv$ on a single Kripke structure is very similar to the computation of a minimal DFA.

1. Design a partition refinement algorithm for computing $\equiv$, i.e. an algorithm that computes an initial relation $\equiv_0$ and refines it successively until $\equiv_k = \equiv$ for some $k$. Prove that your algorithm terminates and computes $\equiv$.

2. Apply your algorithm to the union of two bisimilar systems from the previous exercise and draw the quotiented system.

Exercise 3 (Simulations). Consider the following two systems:
1. Exhibit a simulation to prove $t_0 \preceq s_0$.

2. Show that $s_0 \not\preceq t_0$.

3. Let $M = \langle S, T, I, AP, l \rangle$ be a single Kripke structure. Show that $\preceq$ is reflexive and transitive on $S$. Is it symmetric?

4. Propose an algorithm for computing $\preceq$ on a single structure $M$.

**Exercise 4** (Simulation Quotienting). Two Kripke structures $M_1$ and $M_2$ are *simulation equivalent*, noted $M_1 \simeq M_2$ if $M_1 \preceq M_2$ and $M_2 \preceq M_1$. The lecture notes provide an example of two simulation equivalent but not bisimilar structures. Consider now the two following structures $M_s$ and $M_t$:

1. Which of the following relations hold: $M_s \preceq M_t$, $M_t \preceq M_s$, $M_s \simeq M_t$?

2. Construct the quotient of $(M_s \cup M_t)$ by $\simeq$. Is the resulting system bisimilar to $(M_s \cup M_t)$?

3. Prove that if $M/\simeq$ is the quotient of $M$ by $\simeq$, then $M/\simeq \preceq M$ and $M \preceq M/\simeq$.

4. Call a Kripke structure $M = \langle S, T, I, AP, l \rangle$ *AP-deterministic* if
Let us consider two (not necessarily different) Kripke structures $M$ and $\mathcal{M}$ such that there exists some state $s \in S$ such that $l(s) = a$. Let

$$\exists \psi \in \mathcal{M} \exists s \in S \lnot l(s) = a.$$ 

(b) for each state $s$, if there exist two transitions $(s, s_1)$ and $(s, s_2)$ in $T$ with $l(s_1) = l(s_2)$, then $s_1 = s_2$.

Show that, if two Kripke structures $M_1$ and $M_2$ are AP-deterministic, then they are bisimilar iff they are simulation equivalent.

**Exercise 5 (Logical Characterization).** Let us define existential $\text{CTL}^*$ as the fragment of $\text{CTL}^*$ defined by the following abstract syntax, where $p$ ranges over the set of atomic propositions $AP$:

$$\varphi ::= T \mid \bot \mid p \mid \neg \varphi \mid \varphi \land \varphi \mid \varphi \lor \varphi \mid \text{E}\psi \mid \text{EX}\varphi \mid \text{E}(\varphi \land \psi) \mid \text{E}(\varphi \lor \psi) \mid \text{F}\varphi \mid \text{F}(\varphi \land \psi) \mid \text{F}(\varphi \lor \psi) \mid \text{U}\psi \mid \text{U}\varphi \mid \text{U}(\varphi \land \psi) \mid \text{U}(\varphi \lor \psi) \mid \text{R}\psi \mid \text{R}\varphi \mid \text{R}(\varphi \land \psi) \mid \text{R}(\varphi \lor \psi).$$

Existential $\text{CTL}^*$ includes both LTL and existential $\text{CTL}$ (hereafter noted $\text{ECTL}$), which is defined by the following abstract syntax:

$$\varphi ::= T \mid \bot \mid p \mid \neg \varphi \mid \varphi \land \varphi \mid \varphi \lor \varphi \mid \text{E}\varphi \mid \text{EX}\varphi \mid \text{E}(\varphi \land \psi) \mid \text{E}(\varphi \lor \psi) \mid \text{F}\varphi \mid \text{F}(\varphi \land \psi) \mid \text{F}(\varphi \lor \psi) \mid \text{U}\psi \mid \text{U}\varphi \mid \text{U}(\varphi \land \psi) \mid \text{U}(\varphi \lor \psi) \mid \text{R}\psi \mid \text{R}\varphi \mid \text{R}(\varphi \land \psi) \mid \text{R}(\varphi \lor \psi).$$

Let us consider two (not necessarily different) Kripke structures $M_1 = \langle S_1, T_1, I_1, AP, l_1 \rangle$ and $M_2 = \langle S_2, T_2, I_2, AP, l_2 \rangle$. We assume these structures to be total, where for any state $s$ there exists some state $s'$ such that $(s, s')$ is a transition.

1. Prove the following two statements, for any two states $s_1$ and $s_2$, and any two infinite paths $\pi_1$ and $\pi_2$ in $M_1$ and $M_2$, resp.:

   (a) if $s_1 \leq s_2$, then for any existential $\text{CTL}^*$ state formula $\varphi$, $s_1 \models \varphi$ implies $s_2 \models \varphi$,

   (b) if $\pi_1 = s_{0,1}s_{1,1}\cdots$ and $\pi_2 = s_{0,2}s_{1,2}\cdots$ with $s_{i,1} \leq s_{i,2}$ for all $i$ in $\mathbb{N}$, then for any existential $\text{CTL}^*$ path formula $\psi$, $\pi_1 \models \psi$ implies $\pi_2 \models \psi$.

2. Let us consider the following relation on $S_1 \times S_2$:

   $$\mathcal{F} = \{(s_1, s_2) \in S_1 \times S_2 \mid \forall \varphi \in \text{ECTL}, s_1 \models \varphi \Rightarrow s_2 \models \varphi \}.$$ 

   Assuming that for all initial states $s$ in $I_1$, $\mathcal{F}(s) \cap I_2$ is not empty, show that $\mathcal{F}$ is a simulation between $M_1$ and $M_2$.

3. Conclude by proving the following theorem:

**Theorem 1 (Logical Characterization of Simulation).** Let $M_1 = \langle S_1, T_1, I_1, AP, l_1 \rangle$ and $M_2 = \langle S_2, T_2, I_2, AP, l_2 \rangle$ be two total Kripke structures and $s_1$ and $s_2$ be two states of $S_1$ and $S_2$ resp. The following three statements are equivalent:

1. $s_1 \leq s_2$,

2. for all existential $\text{CTL}^*$ formulae $\varphi$: $s_1 \models \varphi$ implies $s_2 \models \varphi$,

3. for all existential $\text{CTL}$ formulae $\varphi$: $s_1 \models \varphi$ implies $s_2 \models \varphi$. 

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