Exam: Non-Associative Lambek Calculus

Duration: 3 hours.
Written documents are allowed.
The numbers in front of questions are indicative of hardness or duration. The exercises are not independent, but you should not hesitate to skip a question.

This exam is centered on the *non-associative Lambek calculus*.

Recall the definition of product-free *syntactic types* over a set Γ of atomic types:

$$C ::= p \mid (C \setminus C) \mid (C \mathbin{/} C) \;,$$

where p ranges over Γ. The *size* $|C|$ of a syntactic type C is its number of connectives in $\{\setminus, /\}$.

A structural rule usually left implicit in presentations of sequent calculi is the *associativity* rule: using sets, multisets, or sequences for hypotheses of sequents indeed implicitly assumes associativity. In order to introduce a non-associative Lambek calculus, we first define the set of *sequent terms* by

$$T ::= C \mid (T \circ T)$$

where C is a syntactic type; thus sequent terms are binary trees with syntactic types for leaves. We note $C(\Gamma)$ and $T(\Gamma)$ for syntactic types and sequent terms over Γ. We employ the usual context notations for sequent terms: $X[Y]$ is a context $X[]$ containing a subterm Y. Given a sequent term X, its *yield* $y(X) = C_1 \cdots C_n$ is the sequence of its leaves in $(C(\Gamma))^+$ read in left-to-right order.

The rules of the (product-free) non-associative Lambek calculus follow, where A, B, C range over syntactic types and X, Y over sequent terms or contexts:

$$\frac{}{C \vdash C}\;(\mathsf{Id}) \qquad\qquad \frac{Y \vdash B \quad X[B] \vdash A}{X[Y] \vdash A}\;(\mathsf{Cut})$$

$$\frac{(B \circ X) \vdash A}{X \vdash (B \setminus A)}\;(\setminus \mathsf{R}) \qquad\qquad \frac{Y \vdash B \quad X[A] \vdash C}{X[(Y \circ (B \setminus A))] \vdash C}\;(\setminus \mathsf{L})$$

$$\frac{(X \circ B) \vdash A}{X \vdash (A \mathbin{/} B)}\;(/ \mathsf{R}) \qquad\qquad \frac{X[A] \vdash C \quad Y \vdash B}{X[((A \mathbin{/} B) \circ Y)] \vdash C}\;(/ \mathsf{L})$$

We call $(B \setminus A)$ (resp. $(A \mathbin{/} B)$) the *active formula* in rules $(\setminus \mathsf{R})$ and $(\setminus \mathsf{L})$ (resp. $(/ \mathsf{R})$ and $(/ \mathsf{L})$).

The calculus enjoys cut elimination.

1 Warming Up

Exercise 1 (Natural Deduction)**.**

[1] 1. Propose a *natural deduction* version of the calculus, i.e. provide two elimination rules ($\backslash\mathsf{E}$) and ($/\mathsf{E}$).

[1] 2. Show that your rule ($\backslash\mathsf{E}$) holds in the non associative Lambek calculus (the case of ($/\mathsf{E}$) being symmetric).

Exercise 2 (NL Categorial Grammars)**.** A *NL categorial grammar* is a tuple $\mathcal{C} = \langle \Sigma, \Gamma, S, \ell \rangle$ with Σ a finite alphabet, Γ a finite set of atomic types, S a distinguished syntactic type in $C(\Gamma)$, ℓ a finite lexical relation in $\Sigma \times C(\Gamma)$. The *language* of $\mathcal{C}$ is

$$L(\mathcal{C}) = \{a_1 \cdots a_n \in \Sigma^+ \mid \exists X \in T(\Gamma), \exists C_1 \in \ell(a_1), \ldots, \exists C_n \in \ell(a_n), X \vdash S \text{ and } y(X) = C_1 \cdots C_n\} \ .$$

Consider the grammar with $\Sigma = \{\text{John}, \text{Mary}, \text{loves}, \text{smiles}, \text{who}\}$, $\Gamma = \{NP, S\}$, and the lexical relation defined by

$$\begin{aligned} \text{John} &: NP \\ \text{Mary} &: NP \\ \text{loves} &: (NP \setminus S) \ / \ NP \\ \text{smiles} &: NP \setminus S \\ \text{who} &: (NP \setminus NP) \ / \ (NP \setminus S) \end{aligned}$$

[2] Show that “John who loves Mary smiles” is a sentence of this grammar.

2 Context-Freeness

Exercise 3 (Interpolation)**.** The purpose of the exercise is to establish an *interpolation* result: if $X[Y] \vdash A$ is a provable sequent, then there exists a syntactic type B such that $Y \vdash B$, $X[B] \vdash A$, and there exists a syntactic type occurring in $X[Y] \vdash A$ with at least as many connectives (in $\{\backslash, /\}$) as B.

The proof proceeds by induction over cut-free sequent derivations of $X[Y] \vdash A$.

[1] 1. Show that the result holds for a derivation consisting of a single (Id) rule.

This covers the base case. For the induction step, we assume that the premises of a rule R with $X[Y] \vdash A$ as conclusion verify the result, and need to prove that it then holds for $X[Y] \vdash A$.

[3] 2. Assume Y contains the active formula of R. Show that the result holds.

[2] 3. Assume Y occurs in one of the premises of R (and is thus not affected by R). Show that the result holds.

[1] 4. Conclude.

Exercise 4 (Bounded Calculus)**.** We consider the (m,Γ)-*bounded* non-associative Lambek calculus with rules

$$\frac{}{B \vdash A}\,(\mathsf{Ax1}) \qquad \frac{}{(B \circ C) \vdash A}\,(\mathsf{Ax2}) \qquad \frac{Y \vdash B \quad X[B] \vdash A}{X[Y] \vdash A}\,(\mathsf{Cut})$$

where every $B \vdash A$ in $(\mathsf{Ax1})$ and $(B \circ C) \vdash A$ in $(\mathsf{Ax2})$ is provable in the non-associative Lambek calculus with $|A| \leq m$, $|B| \leq m$, and $|C| \leq m$ (thus for fixed m and Γ there are finitely many possible instances of $(\mathsf{Ax1})$ and $(\mathsf{Ax2})$).

Say that a sequent term X is m-*bounded* if all its leaves C are of size $|C| \leq m$. Define

$$C_m(\Gamma) = \{C \in C(\Gamma) \mid |C| \leq m\} \qquad T_m(\Gamma) = \{X \in T(\Gamma) \mid X \text{ is } m\text{-bounded}\}\ .$$

[2] Let $X \vdash A$ be provable in the non-associative Lambek calculus with (X, A) in $T_m(\Gamma) \times C_m(\Gamma)$ for some m and Γ. Show by induction on X (i.e. on its number of $\circ$ connectives) that $X \vdash A$ is provable in the (m,Γ)-bounded non-associative Lambek calculus.

Exercise 5 (Context-Freeness)**.** We are now in position to prove that the languages of categorial grammars based on the non-associative Lambek calculus are context-free.
[4] Show using the previous exercise that for every NL categorial grammar, there exists an equivalent context-free grammar.

3 Montague Semantics

Exercise 6. Consider again the non-associative Lambek grammar of Exercise 2, together with the following semantics interpretation with $[\![S]\!] = o$ and $[\![NP]\!] = (\iota \to o) \to o$:

$$\begin{aligned}
[\![\text{John}]\!] &= \lambda k.\, k\, \mathbf{j} \\
[\![\text{Mary}]\!] &= \lambda k.\, k\, \mathbf{m} \\
[\![\text{loves}]\!] &= \lambda o\, s.\, s\, (\lambda x.\, o\, (\lambda y.\, \mathbf{love}\, x\, y)) \\
[\![\text{smiles}]\!] &= \lambda s.\, s\, (\lambda x.\, \mathbf{smile}\, x) \\
[\![\text{who}]\!] &= \ldots
\end{aligned}$$

where

$$\begin{aligned}
\mathbf{j} &: \iota \\
\mathbf{m} &: \iota \\
\mathbf{love} &: \iota \to (\iota \to o) \\
\mathbf{smile} &: \iota \to o
\end{aligned}$$

[2] Give a semantic interpretation to the relative pronoun "who" such that:

$$[\![\text{smiles}\,(\text{who}\,(\lambda\text{x}.\,\text{loves}\,\text{Mary}\,\text{x})\,\text{John})]\!] = (\mathbf{love}\,\mathbf{j}\,\mathbf{m}) \wedge (\mathbf{smile}\,\mathbf{j})$$