TD 4

1 Complexity of LTL Model Checking

Exercise 1 (Model Checking a Path). We want to verify a model with a single run $w$, which is an ultimately periodic word $uv^\omega$ with $u$ in $\Sigma^*$ and $v$ in $\Sigma^+$. Give an algorithm for checking whether $w, 0 \models \varphi$ holds, where $\varphi$ is a LTL$(X, U)$ formula, in time bounded by $O(|u| \cdot |\varphi|)$.

2 CTL

Exercise 2 (Equivalences). Are the following formulæ equivalent?

1. $AXAG\varphi$ et $AXG\varphi$
2. $EXEG\varphi$ et $EXG\varphi$
3. $A(\varphi \land \psi)$ et $A\varphi \land A\psi$
4. $E(\varphi \land \psi)$ et $E\varphi \land E\psi$
5. $\neg A(\varphi \Rightarrow \psi)$ et $E(\varphi \land \neg \psi)$

Exercise 3 (Model Checking).

Check whether the above Kripke structure verifies the following CTL formula:

$$E(X(a \land \neg b) \land XA(b U (Ga)))$$
3 CTL and CTL$^+$

**Exercise 4 (CTL Equivalences).**

1. Are the two formulæ $\varphi = \mathsf{AG}(\mathsf{EF}p)$ and $\psi = \mathsf{EF}p$ equivalent? Does one imply the other?

2. Same questions for $\varphi = \mathsf{EG}q \lor (\mathsf{EG}p \land \mathsf{EF}q)$ and $\psi = \mathsf{E}(p \lor q)$.

**Exercise 5 (CTL$^+$).** CTL$^+$ extends CTL by allowing boolean connectives on path formulæ, according to the following abstract syntax:

\[
\begin{align*}
f &::= \top \mid a \mid f \land g \mid \neg f \mid \mathsf{E} \varphi \mid \mathsf{A} \varphi \\
\varphi &::= \varphi \land \psi \mid \neg \varphi \mid \mathsf{X} f \mid f \mathsf{U} g
\end{align*}
\]

where $a$ is an atomic proposition. The associated semantics is that of CTL$^*$.

We want to prove that, for any CTL$^+$ formula, there exists an equivalent CTL formula.

1. Give an equivalent CTL formula for $\mathsf{E}( ( a_1 \mathsf{U} b_1 ) \land ( a_2 \mathsf{U} b_2 ) )$.

2. Generalize your translation for any formula of form $\mathsf{E}( ( \bigwedge_{i=1}^n \psi_i \psi_i') \land \mathsf{G} \varphi )$.

What is the complexity of your translation?

3. Give an equivalent CTL formula for the following CTL$^+$ formula:

$$\mathsf{E}(\mathsf{X} a \land ( b \mathsf{U} c ) ).$$

4. Using subformulæ of form (1) and $\mathsf{EX}$ modalities, give an equivalent CTL formula to

$$\mathsf{E}(\mathsf{X} \varphi \land \bigwedge_{i=1}^n ( \psi_i \psi_i') \land \mathsf{G} \varphi' ).$$

What is the complexity of your translation?

5. We only have to transform any CTL$^+$ formula into (nested) disjuncts of form (2). Detail this translation for the following formula:

$$\mathsf{A}( ( \mathsf{F} a \lor \mathsf{X} a \lor \mathsf{X} \neg b \lor \mathsf{F} \neg d ) \land ( d \mathsf{U} \neg c ) ).$$
Exercise 6 (Fair CTL). We consider strong fairness constraints, which are conjunctions of formulæ of form

\[ \text{GF} \psi_1 \Rightarrow \text{GF} \psi_2. \]

We want to check whether the following Kripke structure fairly verifies

\[ \varphi = \text{AGAF} a \]

under the fairness requirement \( e \) defined by

\[ \begin{align*}
\psi_1 &= b \land \neg a \\
\psi_2 &= E(b U (a \land \neg b)) \\
e &= \text{GF} \psi_1 \Rightarrow \text{GF} \psi_2.
\end{align*} \]

1. Compute \([\psi_1]_e\) and \([\psi_2]_e\).
2. Compute \([\text{EGT}]_e\).
3. Compute \([\varphi]_e\).