Home Assignment 2: Stuttering and Bisimulation (Solutions)

Here are partial solutions and hints for the exercises that caused some difficulties.

Exercise 1 (Mutual Exclusion)**.**

1. Note that one needs for instance $(\langle \mathrm{n}_1, \mathrm{w}_2, f = 1\rangle, n)$ in the relation, due to the transition from q_1 to n.
2. The expected answer could contain a CTL* formula separating the two structures (e.g. $\mathrm{EXX}(\neg \mathrm{c}_1 \wedge \neg \mathrm{c}_2)$); alternatively, as bisimulation entails trace equivalence, a separating trace was also easy to find (e.g. $\emptyset^\omega$). Direct arguments are possible (and generally well done), but are harder to read.

Exercise 2 (Coarsest Stutter Bisimulation)**.** Usually well done. Be careful not to forget that (s_1', s_2) should not belong to R when applying condition 2(b) of the stutter simulation definition.

Exercise 3 (Quotients)**.**

2. Set $R = \{(s, [s]) \mid s \in S\}$; do not forget to prove that both R and R^{-1} are stutter simulations.

Exercise 5 (Divergence-sensitive Relations)**.** Transitivity of $\overset{d}{\approx}$ proved to be quite tricky. We consider three Kripke structures M_r, M_s, and M_t.

Let us suppose $r \overset{d}{\approx} s$ and $s \overset{d}{\approx} t$ for some states $r \in S_r$, $s \in S_s$, and $t \in S_t$, which implies that there exist $R_1 \cup R_1^{-1}$ and $R_2 \cup R_2^{-1}$ two divergent-sensitive stutter bisimulations with $(r, s) \in R_1$ and $(s, t) \in R_2$. We want to prove that $r \overset{d}{\approx} t$; to this end, we define

$$R = R_1 \cup R_1^{-1} \cup R_2 \cup R_2^{-1}$$

and prove that R^+ is a divergence-sensitive stutter bisimulation. Note that, if $R_1 \cup R_1^{-1}$ and $R_2 \cup R_2^{-1}$ were equivalence relations, then R^+ would be the finest equivalence relation such that $R_1 \cup R_1^{-1} \subseteq R^+$ and $R_2 \cup R_2^{-1} \subseteq R^+$—this shows that the definition of R^+ is not random...

Claim 5.1. The relation R^+ is symmetric.

Proof. Assume $(r, t) \in R^+$: there exist $n > 0$, $u_0, \ldots, u_n$ with $u_0 = r$, $u_n = t$, and $(u_{i-1}, u_i) \in R$ for all $0 < i \leq n$. Then $(u_i, u_{i-1}) \in R$ since the latter relation is trivially symmetric, hence $(t, r) \in R^+$. □

Claim 5.2. The relation R^+ is a stutter bisimulation.

Proof. By Claim 5.1, we only need to prove that R^+ is a stutter simulation. Let us check the conditions for this to hold:

1. For any initial state r in I_r, there exists s in I_s such that $(r,s) \in R_1$, and thus t in I_t such that $(s,t) \in R_2$, which verifies $(r,t) \in R^+$.

2. For all $(r,t) \in R^+$ there exist $n > 0$, $u_0, \ldots, u_n$ with $u_0 = r$, $u_n = t$, and $(u_{i-1}, u_i) \in R$ for all $0 < i \leq n$.

   (a) Define for all $0 \leq i \leq n$

   $$\ell(u_i) = \begin{cases} \ell_r(u_i) & \text{if } u_i \in S_r \\ \ell_s(u_i) & \text{if } u_i \in S_s \\ \ell_t(u_i) & \text{if } u_i \in S_t \text{ ;} \end{cases}$$

   then for all $0 < i \leq n$, $\ell(u_{i-1}) = \ell(u_i)$ since each of R_1, R_1^{-1}, R_2, and R_2^{-1} is a stutter simulation. Therefore $\ell(r) = \ell(t)$, i.e. $\ell_r(r) = \ell_t(t)$.

   (b) Let us assume wlog. that $u_{i-1} \in S_r$ and $(u_{i-1}, u_i) \in R_1$. If there exist $n_{i-1} \geq 0$ and a path $v_{0,i-1} \cdots v_{n_{i-1},i-1}$ with $v_{0,i-1} = u_{i-1}$ and $(v_{j-1,i-1}, v_{j,i-1}) \in T_r$ for all $0 < j \leq n_{i-1}$ starting from u_{i-1}, then there exist $n_i \geq 0$ and a path $v_{0,i} \cdots v_{n_i,i}$ with $v_{0,i} = u_i$ and $(v_{j-1,i}, v_{j,i}) \in T_s$ for all $0 < j \leq n_i$ starting from u_i, such that for each $0 \leq j < n_i$, there exists $0 \leq j' < n_{i-1}$ with $(v_{j',i-1}, v_{j,i}) \in R_1$, and such that $(v_{n_{i-1},i-1}, v_{n_i,i}) \in R_1$.

   Indeed, by induction on n_{i-1}, if $n_{i-1} = 0$ then $n_i = 0$ fits. Then, for $n_{i-1}+1$ and using the induction hypothesis, either $(v_{n_{i-1}+1,i-1}, v_{n_i,i}) \in R_1$, and then we can keep the same n_i and path, or $(v_{n_{i-1}+1,i-1}, v_{n_i,i}) \notin R_1$, and we find $m \geq 0$ and $m+2$ states $v_{n_i+0,i}, \ldots, v_{n_i+m+1,i}$ in S_s such that $v_{n_i+0,i} = v_{n_i,i}$, $(v_{n_{i-1}+1,i-1}, v_{n_i+m+1,i}) \in R_1$, and for each $0 \leq k \leq m$, $(v_{n_{i-1},i-1}, v_{n_i+k,i}) \in R_1$ and $(v_{n_i+k,i}, v_{n_i+k+1,i}) \in T_s$, which allows to conclude with $n_i + m + 1$ as the new path length.

   If $(r, r') \in T_r$ with $(r', t) \notin R^+$, then in particular $(r', u_1) \notin R$, and the above argument applied repeatedly produces first a path $v_{0,1} v_{1,1} \cdots v_{n_1,1}$ with $u_1 = v_{0,1}$, $(r, v_{j,1}) \in R$ for all $0 \leq j < n_1$ and $(r', v_{n_1,1}) \in R$, and eventually a path $v_{0,n} \cdots v_{n_n,n}$ with $t = u_n = v_{0,n}$, $(r, v_{j,n}) \in R^+$ for $0 \leq j < n_n$ and $(r', v_{n_n,n}) \in R^+$ by transitivity of R^+.

□

Claim 5.3. The relation R^+ is divergence-sensitive.

Proof. By Claim 5.1, we only need to prove that, for $(r,t) = (r_0, t_0) \in R^+$, if $\pi = r_0 r_1 \cdots$ is an infinite path such that $(t_0, r_i) \in R^+$ for all i, then there is an infinite path $\pi' =$

$t_0t_1\cdots$ with $(r_0,t_i)\in R^+$ for all i. Since $(r_0,t_0)\in R^+$, there exist $n>0$, $v_{0,0},\dots,v_{0,n}$ with $v_{0,0}=r_0$, $v_{0,n}=t_0$, and $(v_{0,i-1},v_{0,i})\in R$ for all $0<i\leq n$.

Let us assume wlog. that $v_{0,i-1}\in S_r$ and $(v_{0,i-1},v_{0,i})\in R_1$. If there exists an infinite path $v_{0,i-1}v_{1,i-1}\cdots$ starting in $v_{0,i-1}$ with $(v_{j,i-1},v_{0,i})\in R^+$ for all $j\geq 0$, then $(v_{j-1,i-1},v_{j,i-1})\in T_r$ for all $j>0$, i.e. this path remains in M_r. We show that there exists an infinite path $v_{0,i}v_{1,i}\cdots$ starting in $v_{0,i}$ with $(v_{0,i-1},v_{j,i})\in R^+$ for all $j\geq 0$.

Indeed, either

- $(v_{j,i-1},v_{0,i})\in R_1$ for all $j\geq 0$, which shows that $v_{0,i-1}$ is $R_1^{-1}(v_{0,i})$-divergent, and thus $v_{0,i}$ is $R_1(v_{0,i-1})$-divergent, and we find an appropriate infinite path starting from $v_{0,i}$, or
- we consider the smallest j such that $(v_{j,i-1},v_{0,i})\notin R_1$, thus with $(v_{j-1,i-1},v_{0,i})\in R_1$, and apply condition 2(b) on R_1 to add a finite path $v_{0,i}\cdots v_{m+1,i}$ with $(v_{j-1,i-1},v_{k,i})\in R_1$ for all $0\leq k\leq m$ and $(v_{j,i-1},v_{m+1,i})\in R_1$. By transitivity and symmetry of R^+, $(v_{j-1,i-1},v_{0,i})\in R^+$ and $(v_{j,i-1},v_{0,i})\in R^+$ imply $(v_{j-1,i-1},v_{j,i-1})\in R^+$ for all $j>0$, hence

  $$\begin{aligned}&v_{0,i-1}\ R^+\ v_{j-1,i-1}\ R_1\ v_{k,i} && \forall k\leq m\\ &v_{0,i-1}\ R^+\ v_{j,i-1}\ R_1\ v_{m+1,i}\ ,\end{aligned}$$

  which yields $(v_{0,i-1},v_{k,i})\in R^+$ for all $0\leq k\leq m+1$, thus this path fragment fits our requisites.

  Similarly $(v_{j+j',i-1},v_{m+1,i})\in R^+$ for all $j'\geq 0$, which allows us to repeat the process starting from $v_{j,i-1}$ and $v_{m+1,i}$ instead of $v_{0,i-1}$ and $v_{0,i}$: there is an infinite path $v_{j,i-1}v_{j+1,i-1}\cdots$ starting from $v_{j,i-1}$, with $(v_{j,i-1},v_{m+1,i})\in R_1$ and $(v_{j+j',i-1},v_{m+1,i})\in R^+$ for all $j'\geq 0$.

In order to apply this process for all $0<i\leq n$, we only need to note that, if $n>1$, $(v_{0,1},v_{0,n})\in R^+$ and $(v_{0,n},v_{j,0})\in R^+$ for all $j\geq 0$ yield $(v_{j,0},v_{0,1})\in R^+$ for all $j\geq 0$. We obtain an infinite path $v_{0,n}v_{1,n}\cdots$ with $(v_{0,0},v_{j,n})\in R^+$ for all $j\geq 0$, proving that t_0 is $R^+(r_0)$-divergent. □

Exercise 6 (Logical Characterization).

3. I am at loss here: why did so many of you put $\psi\in\mathrm{AP}$ as a base case of the structural induction for path formulæ? The abstract syntax only gave $\psi=\varphi$ as base case... While this is not wrong per se, it shows that something is not clear about inductions on the structure of terms. Another hint in this regard is that some of you defined “sizes” on formulæ that made sure that something would decrease—correctly, but a recurrence on the size is needlessly complex.