# Alternating Vector Addition Systems with States

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# Outline

alternating VASS and vector addition games

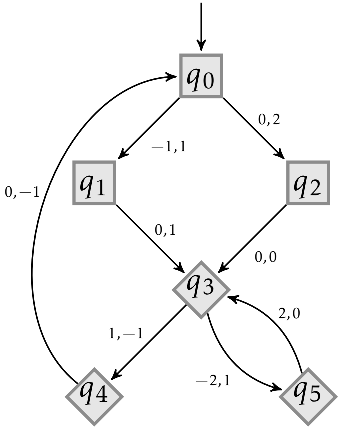
applications

- regular simulations
- energy games

upper bounds & lower bounds

# Vector Addition Games

- two Players $\Diamond$ and $\Box$: partitioned state space $Q_\Diamond \uplus Q_\Box$
- dimension $d \in \mathbb{N}$: transitions labeled with vectors in $\mathbb{Z}^d$
- defines an infinite arena in $Q \times \mathbb{N}^d$
- VASS semantics: a transition is blocked if it makes a value negative



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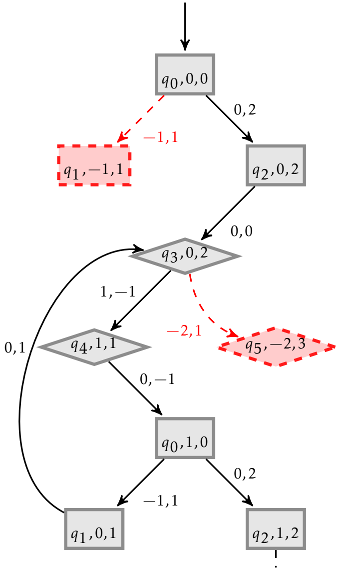

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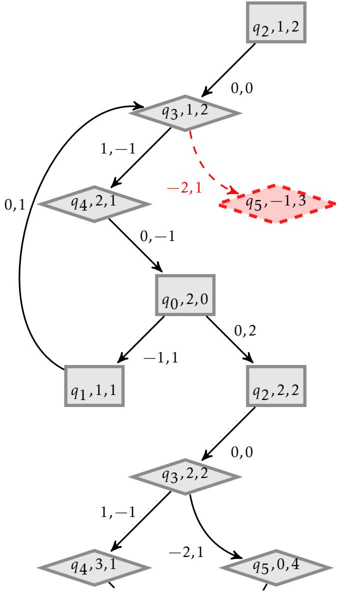


# Asymmetric VASS Games

aka. vector games (Kanovich, 1995), B-games (Raskin et al., 2005), single-sided games (Abdulla et al., 2013)

Asymmetric VASS (AVASS) game:

- $Q = Q_\diamond \uplus Q_\Box$, resp. Controller and Environment
- $T_\diamond \subseteq Q_\diamond \times \mathbb{Z}^d \times Q$:

$$q_\diamond \xrightarrow{\mathbf{u}} q'$$

- $T_\Box \subseteq Q_\Box \times \{\mathbf{0}\} \times Q$:

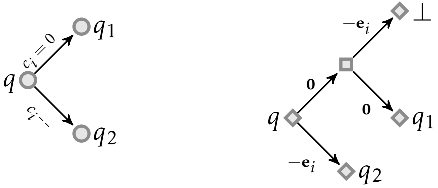
$$\begin{array}{ccc} & & q_1 \\ & \nearrow & \\ q_\Box & & \\ & \searrow & \\ & & q_2 \end{array}$$

# The Importance of Asymmetry

(Raskin et al., 2005)

- VASS game: $T \subseteq Q \times \mathbb{Z}^d \times Q$
- coverability objective: fix $q_\ell$, target $\{q_\ell\} \times \mathbb{N}^d$

Minsky machine Symmetric VASS Game




Player □ can simulate zero-tests!

# Alternating VASS

aka. "and-branching" (Lincoln et al., 1992; Urquhart, 1999)

$Q$ finite set of states

$q_r$ initial state in $Q$

$T_u$ finite set of unary transitions $\subseteq Q \times \mathbb{Z}^d \times Q$:

$$q \xrightarrow{\mathbf{u}} q'$$

$T_f$ set of fork transitions $\subseteq Q^3$:

$$\begin{tikzcd}[row sep=small] & q_1 \\ q \arrow[ur] \arrow[dr] & \\ & q_2 \end{tikzcd}$$

# Tree Semantics

run in $T(Q \times \mathbb{N}^d)$:

Initial

$$q_r, \mathbf{0}$$

unary rule in $T_u$

$$\begin{tikzcd} q, \mathbf{v} \arrow[d, "\mathbf{u}"] \\ q', \mathbf{v}+\mathbf{u} \end{tikzcd}$$

blocking transition: $\mathbf{v}+\mathbf{u} \geqslant \mathbf{0}$

fork rule in $T_f$

$$\begin{tikzcd} & q, \mathbf{v} \arrow[dl] \arrow[dr] & \\ q_1, \mathbf{v} & & q_2, \mathbf{v} \end{tikzcd}$$

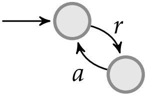
different possible acceptance conditions (on branches)

# Some Applications of AVASS

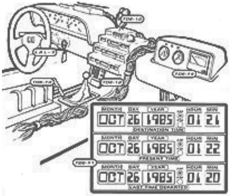
- propositional linear logic (Lincoln et al., 1992; Kanovich, 1995)
- relevance logic (Urquhart, 1999)
- multidimensional energy games (Brázdil et al., 2010; Chatterjee et al., 2012)
- multidimensional mean-payoff games (Chatterjee et al., 2010)
- one-sided μ-calculus (Abdulla et al., 2013)
- regular simulation games (Jančar and Moller, 1995; Lasota, 2009; Abdulla et al., 2013, 2014)

# Finite-state specifications

Required behaviours | Implementation | Safe behaviours




$\preceq$

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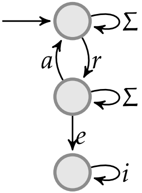


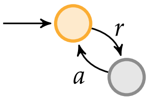

$\models \varphi \in \mathsf{ECTL}^*$

$\models \psi \in \mathsf{ACTL}^*$

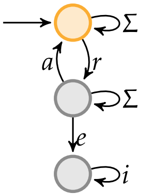
# Simulation Game

- two labeled transition systems $S_1$ and $S_2$
- two players Spoiler and Duplicator
- at each turn
  1. Spoiler chooses a successor state in $S_1$
  2. Duplicator must choose a successor state in $S_2$ with the same action label
- any blocked player loses; Duplicator wins if the play is infinite

$S_1$




$S_2$



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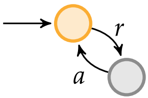


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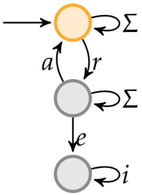

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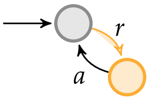


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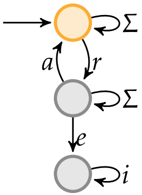

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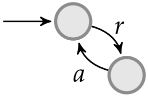
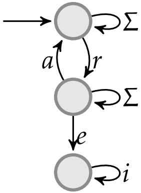



$S_2$



# VASS Regular Simulations

Simulation relations between

- a labeled VASS $\mathcal{V}$ (i.e. an AVASS with $Q_\Box = \emptyset$)
- a finite-state system $\mathcal{F}$

**Theorem** (Jančar and Moller, 1995)
*$\mathcal{V} \preceq \mathcal{F}$ and $\mathcal{F} \preceq \mathcal{V}$ are decidable.*

**Theorem** (Lasota, 2009)
*$\mathcal{V} \preceq \mathcal{F}$ and $\mathcal{F} \preceq \mathcal{V}$ are* ExpSpace*-hard, already if $\mathcal{V}$ is a BPP.*

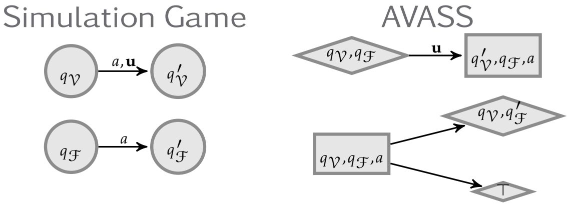
# VASS $\preceq$ FS

## Coverability

| | |
|---|---|
| input | AVASS $\mathcal{A}$ and state $q_\ell$ |
| question | can Controller win for the reachability objective $\{q_\ell\} \times \mathbb{N}^d$? |

## Proposition

*$\mathcal{V} \not\preceq \mathcal{F}$ and AVASS Coverability are* LogSpace*-equivalent (already holds for BPP).*



# FS $\preceq$ VASS

## Non-termination

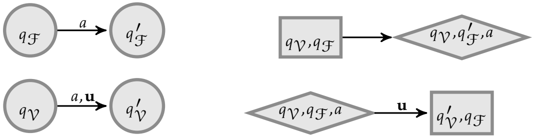
input AVASS $\mathcal{A}$

question can Controller force an infinite play?

## Proposition

*$\mathcal{F} \preceq \mathcal{V}$ and AVASS Non-termination are* LogSpace*-equivalent (already holds for BPP).*

Simulation Game AVASS

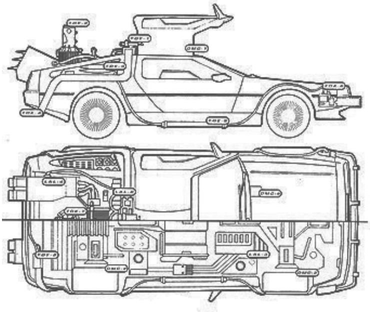


# Controller Synthesis

Property

$$F^{-1}F\,p$$

Environment



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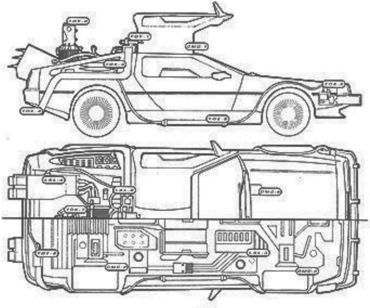
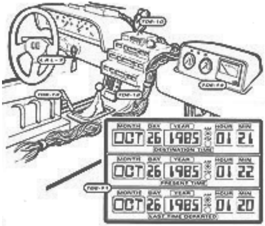


$$\Leftarrow$$

Controller



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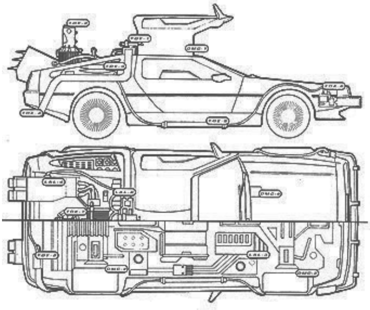

$\Leftarrow$

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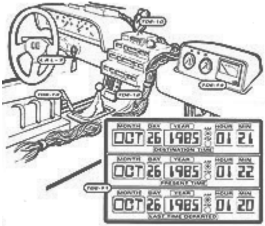
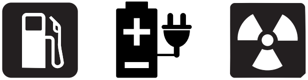



Resources

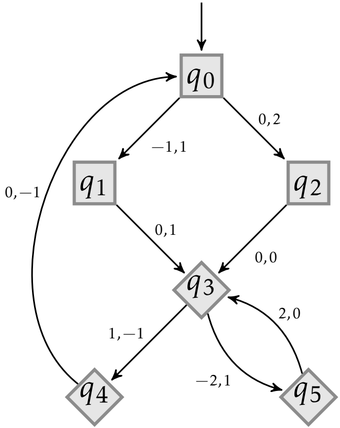
$\Leftarrow$



must remain non-negative

# Multidimensional Energy Games (1/2)

- defines an infinite arena in $Q \times \mathbb{Z}^d$
- energy semantics: transitions are non-blocking
- non-termination + energy objective: Controller must keep the values non-negative along an infinite play



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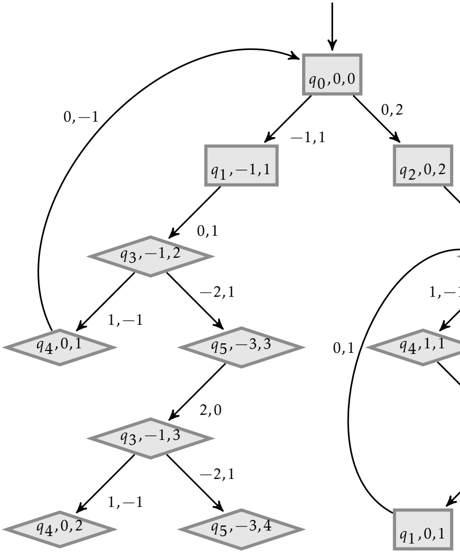


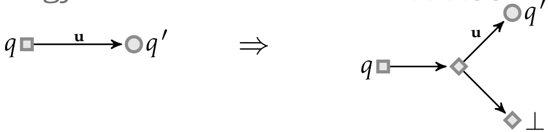
# Multidimensional Energy Games (2/2)

**Theorem** (Abdulla et al., 2013)

*AVASS Non-termination and multidimensional energy games are* LogSpace*-equivalent.*

Energy Games

AVASS



# Complexity Bounds

## Theorem (Brázdil et al., 2010)

*AVASS Non-termination is in* TOWER *and* EXPSPACE*-hard.*

## Proposition (Lower Bounds)

*AVASS Coverability and Non-termination are 2*EXPTIME*-hard, and* EXPTIME*-hard in fixed dimension $d \geqslant 4$.*

## Proposition (Upper Bound)

*AVASS Coverability is in 2*EXPTIME*, and in* EXPTIME *in fixed dimension (more precisely pseudo-polynomial).*

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# Complexity Bounds

upper bound Rackoff (1978)'s technique: small witness property

lower bounds Lipton (1976)'s technique: reduction from alternating Minsky machines

# Proof Plan for Coverability

- if coverable, then there exists a small witness of double exponential height
  - alternating TM can check the existence of a witness in AExpSpace = 2ExpTime
- induction on dimension: $i$-witness for $(q, \mathbf{v})$
  - root label $q, \mathbf{v}$
  - enforces coverability: every leaf labeled by $q_\ell$
  - allows negative values on coordinates $i < j \leqslant d$
- $H_i$: bound on $\sup_{q,\mathbf{v}}$ of the heights of minimal $i$-witnesses for $(q, \mathbf{v})$

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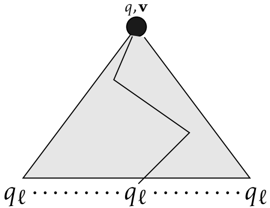
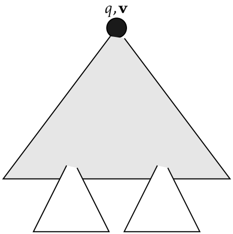
# Small Witnesses: Base Case




No state can appear twice along a branch of a minimal 0-witness:

$$H_0 = |Q|$$

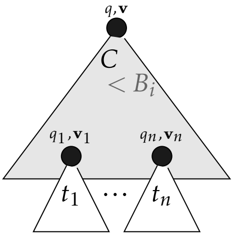
# Small Witnesses: Induction Step




an $(i+1)$-witness $t$

# Small Witnesses: Induction Step

$$B_i \stackrel{\text{def}}{=} \|T_u\|_\infty \cdot H_i$$




an $(i+1)$-witness $t = C[t_1, \ldots, t_n]$

$\forall 1 \leqslant j \leqslant n . \exists 1 \leqslant k \leqslant d . \mathbf{v}_j(k) \geqslant B_i$

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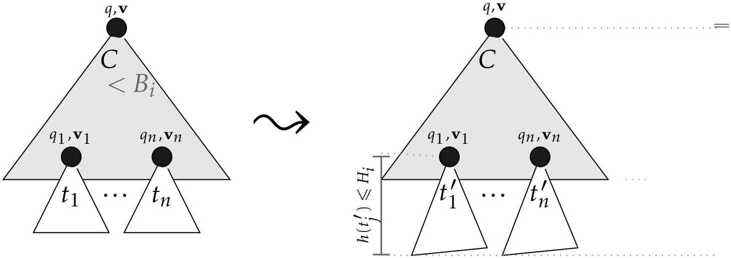


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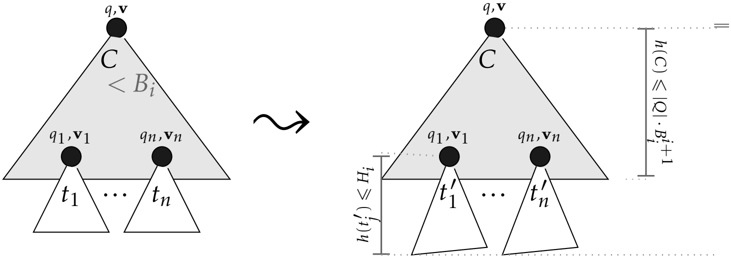


an $(i+1)$-witness $t = C[t_1, \dots, t_n]$ $\qquad t' = C[t'_1, \dots, t'_n]$

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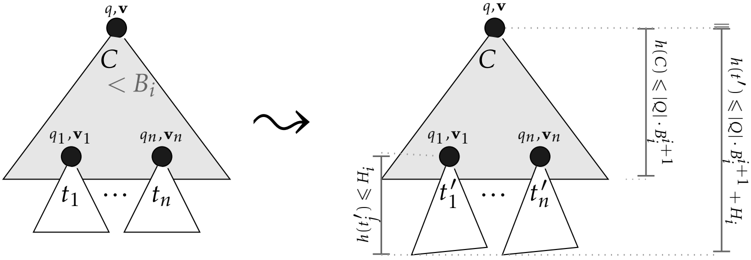


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$$H_{i+1} \leqslant |Q| \cdot B_i^{i+1} + H_i$$

# Concluding Remarks

- alternating VASS / asymmetric VASS games as a sensible model for counter games
- forgotten connections with substructural logics
- importance of Rackoff (1978)'s techniques
- open problem: gap 2ExpTime-hard/Tower-easy for AVASS Non-termination and $\mathcal{F} \preceq \mathcal{V}$

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# Substructural Logics

- Restrict the use of *structural* rules: e.g.

$$\frac{\Gamma, A, A \vdash B}{\Gamma, A \vdash B}(\mathsf{C}) \qquad \frac{\Gamma \vdash B}{\Gamma, A \vdash B}(\mathsf{W})$$

- track resource usage in logic
- example: *relevance logic*
  - in $A \to B$, $A$ should be relevant to the proof of $B$
  - forbids weakening (W) but allows contraction (C)
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# (Intuitionistic) Linear Logic

$$\frac{}{A \vdash A}(\mathsf{I}) \qquad \frac{\Gamma, !A, !A \vdash B}{\Gamma, !A \vdash B}(\mathsf{C!}) \qquad \frac{\Gamma, A \vdash B}{\Gamma, !A \vdash B}(\mathsf{L!})$$

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$$\frac{\Gamma, A \vdash C \quad \Gamma, B \vdash C}{\Gamma, A \oplus B \vdash C}(\mathsf{L}_{\oplus}) \qquad \frac{\Gamma \vdash A}{\Gamma \vdash A \oplus B} \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \oplus B}(\mathsf{R}_{\oplus})$$

$$\frac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C}(\mathsf{L}_{\otimes}) \qquad \frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B}(\mathsf{R}_{\otimes})$$

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# $(!,\oplus)$-Horn Programs (1/3)

connectives $\{\otimes, \multimap, \oplus, !\}$

simple products $W, X, Y, Z ::= p_1 \otimes p_2 \otimes \cdots \otimes p_m$
for atomic $p_i$'s

Horn implications $X \multimap Y$

$\oplus$-Horn implications $X \multimap (Y_1 \oplus \cdots \oplus Y_n)$

$(!,\oplus)$-Horn sequents $W, !\Gamma \vdash Z$ where $\Gamma$ contains
Horn and $\oplus$-Horn implications

# $(!, \oplus)$-Horn Programs (2/3)

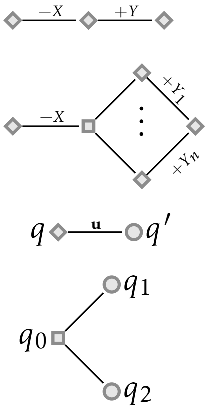
Horn programs | AVASS

$$X \multimap Y \quad \Rightarrow$$

$$X \multimap (Y_1 \oplus \cdots \oplus Y_n) \quad \Rightarrow$$

$$q \otimes \mathbf{u}^- \multimap q' \otimes \mathbf{u}^+ \quad \Leftarrow$$

$$q_0 \multimap (q_1 \oplus q_2) \quad \Leftarrow$$



# $(!,\oplus)$-Horn Programs (3/3)

## AVASS Reachability

input AVASS $\mathcal{A}$, configuration $q_\ell, \mathbf{v} \in Q \times \mathbb{N}^d$

question can Controller win for the reachability objective $\{(q_\ell, \mathbf{v})\}$?

**Theorem** (Lincoln et al., 1992; Kanovich, 1995; Raskin et al., 2005)

*AVASS Reachability is undecidable.*

**Corollary** (Lincoln et al., 1992)

*Provability in propositional linear logic is undecidable.*

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# Relevance Logic $R_{\&,\multimap}$ (1/2)

connectives $\{\oplus, \otimes, \&, \multimap\}$

rules with contraction (C)

$$\frac{\Gamma, A, A \vdash B}{\Gamma, A \vdash B}\text{(C)}$$

but *without weakening* (W)

# Relevance Logic $R_{\&,\multimap}$ (2/2)

add *increasing* transitions to account for (C):

$\forall q \in Q_\diamond, \forall i \leqslant d. q \xrightarrow{\mathbf{e}_i} q$

## AVASS "Bottom-up" Coverability

input increasing AVASS $\mathcal{A}$ and configuration $q, \mathbf{v} \in Q_\diamond \times \mathbb{N}^d$

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**Theorem** (Urquhart, 1999)
*Bottom-up Coverability is* Ackermann*-complete.*

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# Unknown Initial Credit

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input AVASS $\mathcal{A}$ with an objective (reachability, coverability, etc.)

question $\exists \mathbf{v} \in \mathbb{N}^d$ s.t. Controller wins when starting from $q_r, \mathbf{v}$?

**Theorem** (Chatterjee et al., 2012)
*AVASS Non-termination with unknown initial credit is coNP-complete.*

**Theorem** (Urquhart, 1999)
*AVASS Reachability with unknown initial credit is Ackermann-complete.*

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# Multidimensional Mean-Payoff Games

- integer vector game over $Q \times \mathbb{Z}^d$
- *payoff*: $\liminf_{n\to\infty} \frac{1}{n}\mathbf{v}_n$ if $\mathbf{v}_n$ is the $n$th vector of the play
- *threshold* vector $\mathbf{r} \in \mathbb{Q}^d$: a payoff $\geqslant r$ is seeked

## Theorem (Chatterjee et al., 2010)

*Finite-memory strategies for multidimensional mean-payoff games with unknown initial credit are co*NP*-complete.*