Well-quasi-orders in Logic

Sylvain Schmitz

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well-quasi-orders (wqo):
- robust notion

- selection of applications:
  - algorithm termination
  - relevance logic
  - preservation theorems
  - certain answers

research projects
Outline

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  ▶ robust notion
  ▶ selection of applications:
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OUTLINE

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research projects
A One-Player Game

- over $\mathbb{Q}_{\geq 0} \times \mathbb{Q}_{\geq 0}$
- given initially $(x_0, y_0)$
- Eloise plays $(x_j, y_j)$ s.t.
  $\forall 0 \leq i < j, x_i > x_j$ or $y_i > y_j$

- Can Eloise win, i.e. play indefinitely?
A One-Player Game

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- Can Eloise win, i.e. play indefinitely?
If \((x_0, y_0) \neq (0, 0)\), then choosing \((x_j, y_j) = \left( \frac{x_0}{2^j}, \frac{y_0}{2^j} \right)\) wins.
A One-Player Game

- over $\mathbb{N} \times \mathbb{N}$
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Assume there exists an infinite sequence \((x_j, y_j)_j\) of moves over \(\mathbb{N}^2\).
Assume there exists an infinite sequence \((x_j, y_j)_j\) of moves over \(\mathbb{N}^2\). Consider the pairs of indices \(i < j\): color \((i,j)\)

- **purple** if \(x_i > x_j\) but \(y_i \leq y_j\),
- **red** if \(x_i > x_j\) and \(y_i > y_j\),
- **orange** if \(y_i > y_j\) but \(x_i \leq x_j\).

\((3,4)\) \(\rightarrow\) \((5,2)\) \(\rightarrow\) \((2,3)\) \(\cdots\)
Assume there exists an infinite sequence \((x_j, y_j)_j\) of moves over \(\mathbb{N}^2\). Consider the pairs of indices \(i < j\): color \((i, j)\)

- **purple** if \(x_i > x_j\) but \(y_i \leq y_j\),
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- **orange** if \(y_i > y_j\) but \(x_i \leq x_j\).

\[(3,4) \quad (5,2) \quad (2,3) \quad \ldots\]

By the **infinite Ramsey Theorem**, there exists an infinite monochromatic subset of indices.
Assume there exists an infinite sequence \((x_j, y_j)_j\) of moves over \(\mathbb{N}^2\). Consider the pairs of indices \(i < j\): color \((i, j)\)

- purple if \(x_i > x_j\) but \(y_i \leq y_j\),
- red if \(x_i > x_j\) and \(y_i > y_j\),
- orange if \(y_i > y_j\) but \(x_i \leq x_j\).

\((3,4) \rightarrow (5,2) \rightarrow (2,3) \rightarrow \ldots\)

By the infinite Ramsey Theorem, there exists an infinite monochromatic subset of indices. In all cases, it implies the existence of an infinite decreasing sequence in \(\mathbb{N}\), a contradiction.
Well-Quasi-Orders

- multiple equivalent definitions
- algebraic constructions
Well-Quasi-Orders

multiple equivalent definitions: \((X, \leq)\) wqo iff

- **bad sequences** are finite: \(x_0, x_1, \ldots\) is bad if \(\forall i < j, x_i \not\leq x_j\)

- \(\leq\) is well-founded and has no infinite antichains

- finite basis property: \(\emptyset \subset U \subseteq X\) has at least one and finitely many minimal elements

- ascending chain condition: any chain \(U_0 \subset U_1 \subset \cdots\) of upwards-closed sets is finite

- etc.

- algebraic constructions
Well-Quasi-Orders

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  - ascending chain condition: any chain $U_0 \subset U_1 \subset \cdots$ of upwards-closed sets is finite
- etc.

- algebraic constructions
**Well-Quasi-Orders**

- multiple equivalent definitions
- algebraic constructions
  - Cartesian products (Dickson’s Lemma),
  - finite sequences (Higman’s Lemma),
  - disjoint sums,
- finite sets with Hoare’s quasi-ordering,
- finite trees (Kruskal’s Tree Theorem),
- graphs with minors (Robertson and Seymour’s Graph Minor Theorem),
- etc.
Example: Ordinals

ordinal: well-founded linear order

bad sequences are descending sequences:

\[ \alpha \not\leq \beta \text{ iff } \alpha > \beta \]
**Example: Dickson’s Lemma**

**Lemma (Dickson 1913)**

If \((X, \leq_X)\) and \((Y, \leq_Y)\) are two wqos, then \((X \times Y, \leq_X \times)\) is a wqo, where \(\leq_X\) is the **product ordering**:

\[
\langle x, y \rangle \leq_X \langle x', y' \rangle \iff x \leq_X x' \land y \leq_Y y'.
\]

**Example**

- \((\mathbb{N}^d, \leq_X)\) using the product ordering
- \((\mathbb{M}(X), \leq_m)\) for finite multiset embedding over finite \(X\)
**Example: Higman's Lemma**

**Lemma (Higman 1952)**

If \((X, \leq)\) is a wqo, then \((X^*, \leq^*)\) is a wqo where \(\leq^*\) is the subword embedding ordering:

\[
a_1 \cdots a_m \leq^* b_1 \cdots b_n \iff \exists 1 \leq i_1 < \ldots < i_m \leq n, \land_{j=1}^{m} a_j \leq_A b_{i_j}.
\]

**Example**

\[aba \leq^* baaacabbbab\]
Example: Bounded Tree-Depth

**Lemma (Ding 1992)**
For all $k$, $(\text{Graphs} \setminus \uparrow P_k, \subseteq)$ is wqo.

Non-Examples
APPLICATION: ALGORITHM TERMINATION

**SIMPLE** \((a, b)\)

\[
\begin{align*}
c & \leftarrow 1 \\
\text{while } a > 0 \land b > 0 & \\
\langle a, b, c \rangle & \leftarrow \langle a - 1, b, 2c \rangle \\
\text{or} & \\
\langle a, b, c \rangle & \leftarrow \langle 2c, b - 1, 1 \rangle \\
\text{end}
\end{align*}
\]

- in any execution, \(\langle a_0, b_0 \rangle, \ldots, \langle a_n, b_n \rangle\) is a bad sequence over \((\mathbb{N}^2, \leq_x)\),
- \((\mathbb{N}^2, \leq_x)\) is a wqo: all the runs are finite
- c.f. Podelski & Rybalchenko’s transition invariants
**APPLICATION: ALGORITHM TERMINATION**

```plaintext
SIMPLE (a, b)  
c ←− 1  
while a > 0 ∧ b > 0  
    ⟨a, b, c⟩ ←− ⟨a − 1, b, 2c⟩  
    or  
    ⟨a, b, c⟩ ←− ⟨2c, b − 1, 1⟩  
end
```

- in any execution, ⟨a₀, b₀⟩, ..., ⟨aₙ, bₙ⟩ is a bad sequence over (\(\mathbb{N}^2, \leq_x\)),
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end

⟨a_0, b_0, c_0⟩
⟨a_1, b_1, c_1⟩
  ...
⟨a_i, b_i, c_i⟩
  ...
⟨a_j, b_j, c_j⟩
```

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**Application: Algorithm Termination**

```plaintext
SIMPLE (a, b)
c ← 1
while a > 0 ∧ b > 0
    ⟨a, b, c⟩ ← ⟨a − 1, b, 2c⟩
or
    ⟨a, b, c⟩ ← ⟨2c, b − 1, 1⟩
end

in any execution, ⟨a₀, b₀⟩, ..., ⟨aₙ, bₙ⟩ is a bad sequence over (\(\mathbb{N}^2, \preceq_x\)),

(\(\mathbb{N}^2, \preceq_x\)) is a wqo: all the runs are finite

c.f. Podelski & Rybalchenko’s transition invariants
```
APPLICATION: ALGORITHM TERMINATION

\[ \text{SIMPLE } (a, b) \]
\[ c \leftarrow 1 \]
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\[ \text{or} \]
\[ \langle a, b, c \rangle \leftarrow \langle 2c, b - 1, 1 \rangle \]
\[ \text{end} \]

- in any execution, \( \langle a_0, b_0 \rangle, \ldots, \langle a_n, b_n \rangle \) is a bad sequence over \((\mathbb{N}^2, \leq_x)\),
- \((\mathbb{N}^2, \leq_x)\) is a wqo: all the runs are finite
- c.f. Podelski & Rybalchenko’s transition invariants
APPLICATION: RELEVANCE LOGIC

Example ($A \rightarrow (B \rightarrow A)$)

"if it’s raining ($A$), then if your favorite color is green ($B$) then it’s raining ($A$)"

A theorem in classical logic, **not** in relevance logic.

Gentzen-style sequent calculus

$A, B, C$ formulæ; $\Gamma, \Delta$ multisets of formulæ; no weakening

\[
\begin{align*}
\Gamma \vdash A & \quad (\text{Id}) \\
\Gamma, \Delta, A \rightarrow B & \vdash C \quad (\rightarrow_\text{L})
\end{align*}
\]

\[
\begin{align*}
\Gamma, A, A \vdash B & \quad (\text{C}) \\
\Gamma, A \vdash B & \vdash C \quad (\rightarrow_\text{R})
\end{align*}
\]
APPLICATION: RELEVANCE LOGIC

Example \((A \rightarrow (B \rightarrow A))\)

“if it’s raining (A), then if your favorite color is green (B) then it’s raining (A)”

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\[
\frac{}{A \vdash A} \quad \text{(Id)}
\]

\[
\frac{\Gamma \vdash A \quad \Delta, B \vdash C}{\Gamma, \Delta, A \rightarrow B \vdash C} \quad \text{\((\rightarrow_L)\)}
\]

\[
\frac{\Gamma, A \vdash B}{\Gamma, A, A \vdash B} \quad \text{\((C)\)}
\]

\[
\frac{\Gamma \vdash A \rightarrow B}{\Gamma \vdash A} \quad \text{\((\rightarrow_R)\)}
\]
**APPLICATION: RELEVANCE LOGIC**

**Gentzen-style sequent calculus**

\( \Gamma \vdash A \), \( \Delta \vdash C \) formulæ; \( \Gamma, \Delta \) multisets of formulæ; no weakening

\[
\begin{align*}
\frac{A \vdash A}{(Id)} & \quad \frac{\Gamma, A, A \vdash B}{\Gamma, A \vdash B} (C) \\
\frac{\Gamma \vdash A \quad \Delta, B \vdash C}{\Gamma, \Delta, A \rightarrow B \vdash C} (\rightarrow L) & \quad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \rightarrow B} (\rightarrow R)
\end{align*}
\]

**Problem (provability)**

*Given a sequent \( \Gamma \vdash A \), is it provable?*

**Theorem (Kripke 1959)**

*Provability is decidable in implicational relevance logic.*
APPLICATION: RELEVANCE LOGIC

Gentzen-style sequent calculus

\( A, B, C \) formulæ; \( \Gamma, \Delta \) multisets of formulæ; no weakening

\[
\frac{A}{A} (\text{Id})
\]

\[
\frac{\Gamma \vdash A \quad \Delta, B \vdash C}{\Gamma, \Delta, A \rightarrow B \vdash C} (\rightarrow_L)
\]

\[
\frac{\Gamma, A 
\vdash B}{\Gamma 
\vdash A 
\rightarrow B} (\rightarrow_R)
\]

- subformula property

- irredundant proof searches

- \((C)\) and \((\rightarrow_R)\) commute: \((C)\)'s only below a \((\rightarrow_L)\)

- rewrite proofs to apply \((C)\) whenever possible

- irredundant proof branches are bad sequences for contraction

- …which is wqo over the subformulæ of \(\Gamma \vdash A\)
APPLICATION: RELEVANCE LOGIC

Gentzen-style sequent calculus
A, B, C formulæ; Γ, Δ multisets of formulæ; no weakening

\[
\begin{align*}
\Gamma \vdash A & \quad \text{(Id)} \\
\Gamma, A, A \vdash B & \quad \text{((C))} \\
\Gamma \vdash A & \quad \Delta, B \vdash C \\
\Gamma, \Delta, A \rightarrow B \vdash C & \quad \text{((→L))} \\
\Gamma, A \vdash B & \quad \text{((→R))} \\
\end{align*}
\]

- subformula property
- irredundant proof searches
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APPLICATION: RELEVANCE LOGIC

GENTZEN-STYLE SEQUENT CALCULUS

$A, B, C$ formulæ; $\Gamma, \Delta$ multisets of formulæ; no weakening

\[
\frac{}{A \vdash A} \quad \text{(Id)}
\]

\[
\frac{\Gamma \vdash A, \Delta, B \vdash C}{\Gamma, \Delta, A \rightarrow B \vdash C} \quad \text{($\rightarrow_L$)}
\]

\[
\frac{\Gamma, A, A \vdash B}{\Gamma, A \vdash B} \quad \text{(C)}
\]

\[
\frac{\Gamma, A \vdash B}{\Gamma \vdash A \rightarrow B} \quad \text{($\rightarrow_R$)}
\]

▶ subformula property

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▶ ($C$) and ($\rightarrow_R$) commute: ($C$)’s only below a ($\rightarrow_L$)

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▶ ...which is wqo over the subformulæ of $\Gamma \vdash A$
APPLICATION: RELEVANCE LOGIC

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\Gamma, \Delta, A \rightarrow B \vdash C & \\
\Gamma \vdash A \rightarrow B & \quad \text{(-}\rightarrow\text{R})
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- subformula property
- irredundant proof searches
  - (C) and (\(\rightarrow_R\)) commute: (C)’s only below a (\(\rightarrow_L\))
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- irredundant proof branches are bad sequences for contraction
- \( \ldots \) which is wqo over the subformulæ of \( \Gamma \vdash A \)
**Application: Preservation Theorems**

<table>
<thead>
<tr>
<th>logic $\mathcal{L}$</th>
<th>example</th>
<th>hom$_{\mathcal{L}}$</th>
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<tbody>
<tr>
<td>$\exists FO$</td>
<td>$\exists z. x \xrightarrow{G} y \land \neg(y \xrightarrow{R} z)$</td>
<td>strong injective</td>
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**Fact**

If $\psi \in \mathcal{L}$, $h \in$ hom$_{\mathcal{L}}$, and $D \models \psi(x)$, then $h(D) \models \psi(h(x))$. 
## Application: Preservation Theorems

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### Definition

$D \preceq_{\mathcal{L}} D'$ if $\exists h \in \text{hom}_{\mathcal{L}}$ s.t. $D' = h(D)$. 
Application: Preservation Theorems

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Over arbitrary structures

**Theorem (Łoś, Lyndon, Tarski)**

If $\varphi$ is an FO-sentence s.t. $\llbracket \varphi \rrbracket$ is upwards-closed for $\leq_{\mathcal{L}}$, then there exists $\psi \in \mathcal{L}$ with $\llbracket \varphi \rrbracket = \llbracket \psi \rrbracket$. 
# Application: Preservation Theorems

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Over finite (relational) structures?
## Application: Preservation Theorems

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<td>injective</td>
</tr>
<tr>
<td>$\exists \text{FO}^+$</td>
<td>yes [Rossman 2008]</td>
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Over finite (relational) structures?
APPLICATION: PRESERVATION THEOREMS

Over finite (relational) structures?

finite $\sigma$-structures
Application: Preservation Theorems

Over finite (relational) structures?
APPLICATION: PRESERVATION THEOREMS

Over finite (relational) structures?

\[ \lbrack \varphi \rbrack \]
(upwards-closed inside \( K \))

finite \( \sigma \)-structures

class \( K \)
**APPLICATION: PRESERVATION THEOREMS**

Over finite (relational) structures?

\[ \llbracket \varphi \rrbracket \]

(upwards-closed inside \( \mathcal{K} \))

finite \( \sigma \)-structures

class \( \mathcal{K} \)

find finitely many structures \( \Lambda_1, \ldots, \Lambda_n \) s.t.

\[ \llbracket \varphi \rrbracket \cap \mathcal{K} \subseteq \uparrow \{ \Lambda_1, \ldots, \Lambda_n \} \]
**APPLICATION: PRESERVATION THEOREMS**

**Over finite (relational) structures?**

\[ \left[ \varphi \right] \quad \text{(upwards-closed inside } K) \]

\[ \left[ \varphi \right] \cap K \subseteq \uparrow \{ A_1, \ldots, A_n \} \]

find finitely many structures \( A_1, \ldots, A_n \) s.t.
APPLICATION: PRESERVATION THEOREMS

OVER FINITE (RELATIONAL) STRUCTURES?

\[ \left[ \varphi \right] \] (upwards-closed inside \( \mathcal{K} \))

find finitely many structures \( \Lambda_1, \ldots, \Lambda_n \) s.t.

\[ \left[ \varphi \right] \cap \mathcal{K} \subseteq \uparrow \{ \Lambda_1, \ldots, \Lambda_n \} \]
Application: Preservation Theorems

Over finite (relational) structures?

\[
\begin{align*}
\langle \varphi \rangle & \subseteq \uparrow\{A_1, \ldots, A_n\} \\
\langle \varphi \rangle & \text{inside } \mathcal{K} \\
\mathcal{K} & \text{downwards-closed} \\
\text{class } \mathcal{K} & \text{finitely many structures } A_1, \ldots, A_n \in \mathcal{K} \text{ s.t.}
\end{align*}
\]
APPLICATION: PRESERVATION THEOREMS

OVER finite (relational) structures?

- by finite basis property: if \((\mathcal{K}, \leq_{\mathcal{L}})\) wqo and downwards-closed then such \(A_1, \ldots, A_n\) exist
- associate \(\psi_i \in \mathcal{L}\) to each \(A_i\) s.t. \(\llbracket \psi_i \rrbracket = \uparrow A_i\)

Find finitely many structures \(A_1, \ldots, A_n \in \mathcal{K}\) s.t.
\[\llbracket \varphi \rrbracket \cap \mathcal{K} \subseteq \uparrow \{A_1, \ldots, A_n\}\]
APPLICATION: CERTAIN ANSWERS

incomplete database I

query

possible completions \(\mathcal{G}(I)\)
APPLICATION: CERTAIN ANSWERS

incomplete database $I$

query $\varphi$

certain answers: $true$ in all completions

possible completions $\Phi(I)$
Application: Certain Answers

incomplete database $I$

query $\varphi$

certain answers: true in all completions

certain$_I(\varphi) = \bigcap_{D \in \mathcal{D}(I)} \{x \mid D \models \varphi(x)\}$
CHASE of $x \xrightarrow{G} z \implies \exists y. x \xrightarrow{R} y \land y \xrightarrow{B} z$ for $\varphi \in \exists FO^+(\neq)$.
CHASE of $x \xrightarrow{G} z \implies \exists y. x \xrightarrow{R} y \land y \xrightarrow{B} z$ for $\varphi \in \exists \text{FO}^+(\neq)$
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\[
\uparrow S_0 \subseteq \uparrow S_1
\]
CHASE of $x \xrightarrow{G} z \implies \exists y. x \xrightarrow{R} y \land y \xrightarrow{B} z$ for $\varphi \in \exists \mathbf{FO}^+(\neq)$.
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- Over a wqo: by ascending chain condition, \( \uparrow S_0 \subseteq \uparrow S_1 \subseteq \cdots \) always stabilises to \( \uparrow S_\ast \)

- \( \text{certain}_I(\varphi) = (\text{dom } I)^* \cap \bigwedge_{B \in S_\ast} \{ x : B \models \varphi(x) \} \)
\[ \text{CHASE of } x \xrightarrow{G} z \implies \exists y. x \xrightarrow{R} y \land y \xrightarrow{B} z \text{ for } \varphi \in \exists \text{FO}^+(\neq) \]

- over a wqo: by ascending chain condition, \[ \uparrow S_0 \subseteq \uparrow S_1 \subseteq \cdots \] always stabilises to \[ \uparrow S_* \]

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Research Projects

- Perspectives of a doctoral thesis
- Job applications
- Funding applications
- For yourselves
RESEARCH PROJECTS

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RESEARCH DRIVES

PROBLEMS

FOUNDATIONS
RESEARCH DRIVES

PROBLEMS

- attainable?
- specialised?

FOUNDATIONS
**RESEARCH DRIVES**

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CURIOSITY
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PROBLEMS
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FOUNDATIONS
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CURIOSITY
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Will you collaborate on your project?

▷ locally at your institution?

▷ internationally?

▷ by supervising students?

▷ by animating a local seminar?

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