Complexity in substructural logics and counter systems

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The Leverhulme Trust

Leverhulme Lecture based on joint work with R. Lazić
Queen Mary University, April 29th, 2015

Outline

- VASS vector addition systems with states
  and alternating branching extensions (c.f. Kopylov, 2001)
- linear logic
  and its affine and contractive variants
- inter-reductions
  between counter systems and substructural logics
- algorithmic complexity
  on counter systems

Alternating Branching VASS$_0$

Syntax

- $\mathcal{A} = \langle Q, d, T_u, T_f, T_s, T_z \rangle$
- Q finite set of states
- $d \in \mathbb{N}^d$ dimension
- $q \xrightarrow{\mathbf{u}} q' \in T_u \subseteq_{\text{fin}} Q \times \mathbb{Z}^d \times Q$
- $q \to q_1 \wedge q_2 \in T_f \subseteq Q^3$
- $q \to q_1 + q_2 \in T_s \subseteq Q^3$
- $q \xrightarrow{\stackrel{?}{=}\mathbf{0}} q' \in T_z \subseteq Q^2$

Semantics

configurations

$q, \mathbf{v} \in Q \times \mathbb{N}^d$

$$\frac{q, \mathbf{v}}{q', \mathbf{v} + \mathbf{u}} \text{ (unary)}$$

$$\frac{q, \mathbf{v}}{q_1, \mathbf{v} \quad q_2, \mathbf{v}} \text{ (fork)}$$

$$\frac{q, \mathbf{v}_1 + \mathbf{v}_2}{q_1, \mathbf{v}_1 \qquad q_2, \mathbf{v}_2} \text{ (split)}$$

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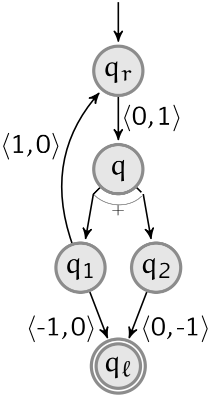
BVASS Example



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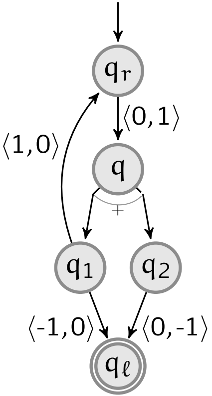
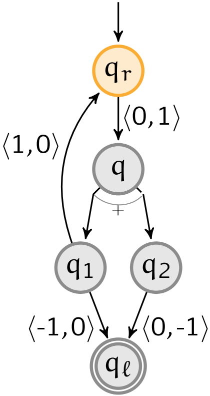


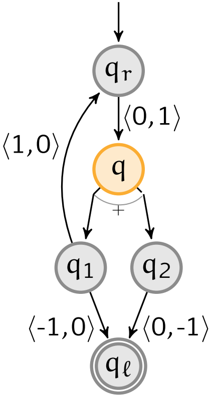
$q_r, 0, 0$

BVASS Example



$$\frac{q_r, 0, 0}{q, 0, 1}$$

BVASS Example



$$\dfrac{\dfrac{q_r, 0, 0}{q, 0, 1}}{q_1, 0, 0 \qquad\qquad q_2, 0, 1}$$

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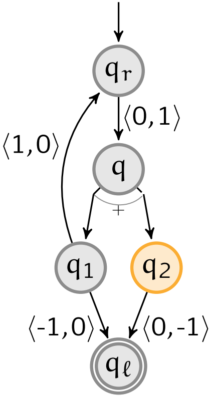

$$\dfrac{\dfrac{q_r,0,0}{q,0,1}}{q_1,0,0 \qquad \dfrac{q_2,0,1}{q_\ell,0,0}}$$

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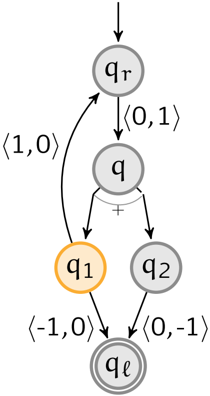

$$\cfrac{\cfrac{q_r,0,0}{q,0,1}}{\cfrac{q_1,0,0}{q_r,1,0} \qquad \cfrac{q_2,0,1}{q_\ell,0,0}}$$

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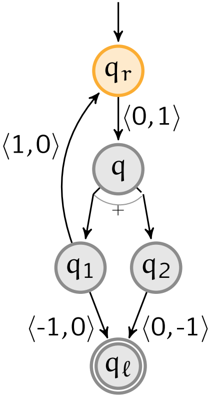

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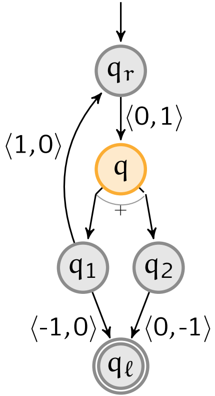

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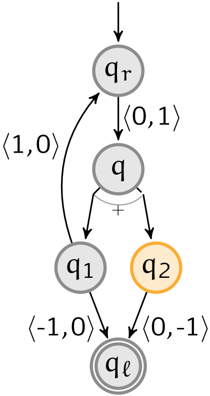

$$\dfrac{q_r, 0, 0}{q, 0, 1}$$

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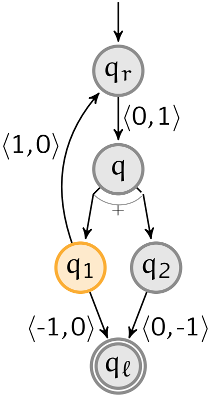

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Some Application Domains of AVASS

substructural logics

- undecidability of propositional linear logic (Lincoln et al., 1992)
- $(\oplus, !)$-Horn linear logic (Kanovich, 1995)
- Ackermann-completeness of $R_{\rightarrow,\wedge}$ (Urquhart, 1999)

energy games (Brázdil et al., 2010; Chatterjee et al., 2010)

regular VASS simulations (Abdulla et al., 2013; Courtois and S., 2014)

Some Application Domains of BVASS

computational linguistics (survey in S., 2010)

- dominance links (Rambow, 1994)
- abstract categorial grammars (de Groote, 2001)
- minimal grammars (Salvati, 2011)

linear logic inter-reductions with MELL (de Groote et al., 2004; Lazić and S., 2015)

protocol verification Horn deduction modulo AC (Verma and Goubault-Larrecq, 2005)

data logics for XML $\mathrm{FO}^2(<,+1,\sim)$ (Bojańczyk et al., 2009; Dimino et al., 2013)

parallel programming (Bouajjani and Emmi, 2013)

ABVASS$_0$ Reachability

Given $\mathcal{A} = \langle Q, d, T_u, T_f, T_s, T_z \rangle$, $q_r \in Q$, $Q_\ell \subseteq Q$
Does there exist a deduction tree rooted by $q_r, \mathbf{0}$ and with every leaf in $Q_\ell \times \{\mathbf{0}\}$?

Theorem (Lincoln et al., 1992)
AVASS Reachability is undecidable.

ABVASS_0 Reachability

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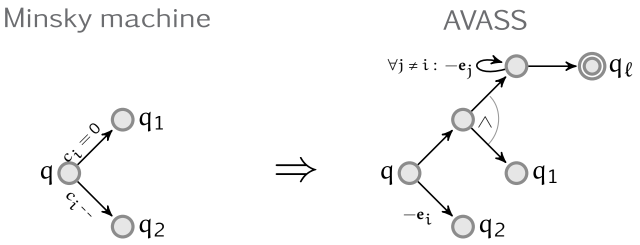

Eliminating Full-Zero Tests

Proposition

$ABVASS_0$ Reachability $<^{T}_{\text{PTime}}$ ABVASS Reachability

Proof Idea.

Let $\mathcal{A} = \langle Q, d, T_u, T_f, T_s, T_z \rangle$, $q_r \in Q$, $Q_\ell \subseteq Q$. Define for all $X \subseteq Q$

$$\text{Root}_{\mathcal{A}}(X) \stackrel{\text{def}}{=} \{q \in Q \mid \exists \text{ derivation s.t. root} = q, \mathbf{0} \wedge \text{leaves} \in Q_\ell \times \{\mathbf{0}\}\}$$

Then $ABVASS_0$ Reachability is equivalent to $q_r \in \text{Root}_{\mathcal{A}}(Q_\ell)$.

Let $\mathcal{A}' \stackrel{\text{def}}{=} \langle Q, d, T_u, T_f, T_s, \emptyset \rangle$: $\text{Root}_{\mathcal{A}'}(X)$ can be computed using $|X|$ calls to an oracle for ABVASS Reachability. Then we can compute in polynomial time relative to the oracle:

$$\text{Root}_{\mathcal{A}}(Q_\ell) = \mu X.\text{Root}_{\mathcal{A}'}(Q_\ell) \cup \text{Root}_{\mathcal{A}'}(X \cup T_z^{-1}(X)) \quad \square$$

Lossy Reachability

New deduction rule: $\forall q \in Q, \mathbf{v} \in \mathbb{N}^d, 1 \leqslant i \leqslant d$

$$\frac{q, \mathbf{v} + \mathbf{e}_i}{q, \mathbf{v}} \text{ (loss)}$$

Given $\mathcal{A} = \langle Q, d, T_u, T_f, T_s, T_z \rangle$, $q_r \in Q$, $Q_\ell \subseteq Q$

Lossy Reachability

Does there exist a lossy deduction tree rooted by $q_r, \mathbf{0}$ and with every leaf in $Q_\ell \times \{\mathbf{0}\}$?

Leaf Coverability

Does there exist a deduction tree rooted by $q_r, \mathbf{0}$ and with every leaf in $Q_\ell \times \mathbb{N}^d$?

Lemma

Lossy reachability and leaf coverability coincide in ABVASS (without full zero-tests).

Lossy Reachability

Lemma

Lossy reachability and leaf coverability coincide in ABVASS (without full zero-tests).

Proof of Lossy reachability $\Longrightarrow$ Leaf coverability.

Rewrite derivations to perform losses at the leaves:

$$\dfrac{\dfrac{q, \mathbf{v}+\mathbf{e}_i}{q, \mathbf{v}}\ (\text{loss})}{q', \mathbf{v}+\mathbf{u}}\ (\text{unary}) \quad\Downarrow\quad \dfrac{\dfrac{q, \mathbf{v}+\mathbf{e}_i}{q', \mathbf{v}+\mathbf{u}+\mathbf{e}_i}\ (\text{unary})}{q', \mathbf{v}+\mathbf{u}}\ (\text{loss})$$

$$\dfrac{\dfrac{q, \mathbf{v}+\mathbf{e}_i}{q, \mathbf{v}}\ (\text{loss})}{q_1, \mathbf{v} \qquad q_2, \mathbf{v}}\ (\text{fork}) \quad\Downarrow\quad \dfrac{q, \mathbf{v}+\mathbf{e}_i}{\dfrac{q_1, \mathbf{v}+\mathbf{e}_i}{q_1, \mathbf{v}}\ (\text{loss}) \qquad \dfrac{q_2, \mathbf{v}+\mathbf{e}_i}{q_2, \mathbf{v}}\ (\text{loss})}\ (\text{fork})$$

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Lossy reachability and leaf coverability coincide in ABVASS (without full zero-tests).

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□

Gainy / Expansive Reachability

New deduction rules: $\forall q \in Q, \mathbf{v} \in \mathbb{N}^d, 1 \leqslant i \leqslant d$

$$\frac{q, \mathbf{v}}{q, \mathbf{v} + \mathbf{e}_i}\ (\text{gain}) \qquad \frac{q, \mathbf{v} + \mathbf{e}_i}{q, \mathbf{v} + 2\mathbf{e}_i}\ (\text{expansion})$$

Given $\mathcal{A} = \langle Q, d, T_u, T_f, T_s, T_z \rangle$, $q_r \in Q$, $Q_\ell \subseteq Q$

Gainy Reachability

Does there exist a gainy deduction tree rooted by $q_r, \mathbf{0}$ and with every leaf in $Q_\ell \times \{\mathbf{0}\}$?

Expansive Reachability

Does there exist an expansive deduction tree rooted by $q_r, \mathbf{0}$ and with every leaf in $Q_\ell \times \{\mathbf{0}\}$?

Well-Structured Systems

Lemma (Dickson's Lemma)

$Q \times \mathbb{N}^d$ *is well quasi ordered: in any infinite sequence* $(q_0, \mathbf{v}_0), (q_1, \mathbf{v}_1), \ldots, \exists j < k$ *s.t.* $q_j = q_k$ *and* $\mathbf{v}_j \leqslant \mathbf{v}_k$ *componentwise*

Theorem (using e.g. Finkel and Schnoebelen, 2001)

Lossy (resp. gainy, resp. expansive) $ABVASS_0$ Reachability is decidable.

Theorem (using e.g. Figueira et al., 2011)

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Propositional Linear Logic

Single-Sided Sequent calculus

A, B, C formulæ; Γ, Δ multisets of formulæ

$$\frac{}{\vdash A, A^\perp}\ \text{init}$$

multiplicatives

$$\frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \parr B}\ \parr \qquad \frac{\vdash \Gamma, A \quad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \otimes B}\ \otimes \qquad \frac{\vdash \Gamma}{\vdash \Gamma, \perp}\ \perp \qquad \frac{}{\vdash \mathbf{1}}\ \mathbf{1}$$

additives

$$\frac{\vdash \Gamma, A \quad \vdash \Gamma, B}{\vdash \Gamma, A \,\&\, B}\ \& \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, A \oplus B}\ \frac{\vdash \Gamma, B}{\vdash \Gamma, A \oplus B}\ \oplus \qquad \frac{}{\vdash \Gamma, \top}\ \top$$

exponentials

$$\frac{\vdash \Gamma, A}{\vdash \Gamma, ?A}\ ?\text{D} \qquad \frac{\vdash \Gamma}{\vdash \Gamma, ?A}\ ?\text{W} \qquad \frac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A}\ ?\text{C} \qquad \frac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A}\ ?\text{P}$$

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$$\frac{}{\vdash A, A^\perp}\,\text{init}$$

multiplicatives

$$\frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \parr B}\,\parr \qquad \frac{\vdash \Gamma, A \quad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \otimes B}\,\otimes \qquad \frac{\vdash \Gamma}{\vdash \Gamma, \perp}\,\perp \qquad \frac{}{\vdash \mathbf{1}}\,\mathbf{1}$$

additives

$$\frac{\vdash \Gamma, A \quad \vdash \Gamma, B}{\vdash \Gamma, A \,\&\, B}\,\& \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, A \oplus B}\;\frac{\vdash \Gamma, B}{\vdash \Gamma, A \oplus B}\,\oplus \qquad \frac{}{\vdash \Gamma, \top}\,\top$$

exponentials

$$\frac{\vdash \Gamma, A}{\vdash \Gamma, ?A}\,?\text{D} \qquad \frac{\vdash \Gamma}{\vdash \Gamma, ?A}\,?\text{W} \qquad \frac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A}\,?\text{C} \qquad \frac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A}\,?\text{P}$$

Propositional Linear Logic

Single-Sided Sequent calculus

A, B, C formulæ; Γ, Δ multisets of formulæ

$$\frac{}{\vdash A, A^\perp}\ \text{init}$$

multiplicatives

$$\frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \parr B}\ \parr \qquad \frac{\vdash \Gamma, A \quad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \otimes B}\ \otimes \qquad \frac{\vdash \Gamma}{\vdash \Gamma, \perp}\ \perp \qquad \frac{}{\vdash \mathbf{1}}\ \mathbf{1}$$

additives

$$\frac{\vdash \Gamma, A \quad \vdash \Gamma, B}{\vdash \Gamma, A \,\&\, B}\ \& \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, A \oplus B}\ \frac{\vdash \Gamma, B}{\vdash \Gamma, A \oplus B}\ \oplus \qquad \frac{}{\vdash \Gamma, \top}\ \top$$

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$$\frac{\vdash \Gamma, A}{\vdash \Gamma, ?A}\ ?\text{D} \qquad \frac{\vdash \Gamma}{\vdash \Gamma, ?A}\ ?\text{W} \qquad \frac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A}\ ?\text{C} \qquad \frac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A}\ ?\text{P}$$

Provability

Given a formula F, is $\vdash F$ provable?

Theorem (Lincoln et al., 1992)

Provability in propositional linear logic is undecidable.

Proof.

By a reduction from AVASS Reachability! □

Affine / Contractive Linear Logic

Affine Linear Logic (LLW)

$$\frac{\vdash \Gamma}{\vdash \Gamma, A}\ \mathsf{W}$$

Theorem (Kopylov, 2001; Lafont, 1997)

LLW Provability is decidable.

Contractive Linear Logic (LLC)

$$\frac{\vdash \Gamma, A, A}{\vdash \Gamma, A}\ \mathsf{C}$$

Theorem (Okada and Terui, 1999)

LLC Provability is decidable.

Inter-reductions

Theorem (LL to $ABVASS_0$; Lazić and S., 2015)

1. *LL Provability* $<^{m}_{\text{PSPACE}}$ *$ABVASS_0$ Reachability*
2. *MELL Provability* $<^{m}_{\text{PSPACE}}$ *$BVASS_0$ Reachability*

Inter-reductions

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Inter-reductions

Theorem (LL to $\text{ABVASS}_\mathbf{0}$; Lazić and S., 2015)

1. *LL(W / C) Provability* $<^{m}_{\text{PSPACE}}$ *$\text{ABVASS}_\mathbf{0}$ (Lossy / Expansive) Reachability*
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Theorem (ABVASS_0 to LL; Lazić and S., 2015 & S., 2015)

1. *ABVASS Reachability* $<^m_{\text{LogSpace}}$ *LL Provability*
2. *BVASS Reachability* $<^m_{\text{LogSpace}}$ *MELL Provability*
3. *ABVASS Expansive Reachability* $<^m_{\text{LogSpace}}$ *MALLC Provability*
4. *BVASS Expansive Reachability* $<^m_{\text{LogSpace}}$ *MLLC Provability*

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From LL to $\text{ABVASS}_\mathbf{0}$ (1/2)

- subformula property
- given a formula F, set

$$S_? \stackrel{\text{def}}{=} ?\text{-Subformulæ}(F)\,, \quad S \stackrel{\text{def}}{=} \text{Subformulæ}(F) \setminus S_?\,, \quad d \stackrel{\text{def}}{=} |S|\,.$$

- all the sequents in a proof of $\vdash F$ are of form $\vdash ?\Psi, \Delta$
  - $?\Psi$ is a multiset over $S_?$
  - Δ is a multiset over S
- define a state set $Q \supseteq 2^{S_?}$
- associate to $\vdash ?\Psi, \Delta$ the configuration $\sigma(?\Psi), \mathbf{v}_\Delta \in Q \times \mathbb{N}^d$
- $\text{ABVASS}_\mathbf{0}$ rules implement LL proof search

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From LL to ABVASS_0 (2/2)

$$\frac{}{\vdash A, A^\perp}\ \text{init}$$

$$\frac{\vdash \Gamma, A \qquad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \otimes B}\ \otimes$$

$$\frac{\vdash \Gamma, A \qquad \vdash \Gamma, B}{\vdash \Gamma, A \,\&\, B}\ \&$$

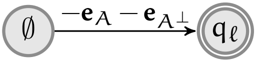
$$\frac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A}\ ?\mathrm{P}$$

$$\frac{\vdash \Gamma, A, A}{\vdash \Gamma, A}\ \mathrm{C}$$

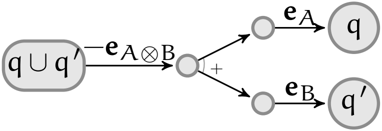
From LL to ABVASS$_0$ (2/2)

Assuming $A, B \in S$, $q, q' \in 2^{S_?}$

$$\frac{}{\vdash A, A^\perp}\ \text{init}$$



$$\frac{\vdash \Gamma, A \qquad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \otimes B}\ \otimes$$



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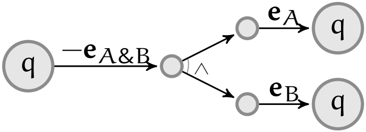
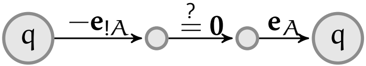


$$\frac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A}\ ?P$$



$$\frac{\vdash \Gamma, A, A}{\vdash \Gamma, A}\ C$$

$$\frac{q, \mathbf{v} + \mathbf{e}_i}{q, \mathbf{v} + 2\mathbf{e}_i}\ \text{(expansion)}$$

A Tower Upper Bound

Tower Complexity

$$\textsc{Tower} \stackrel{\text{def}}{=} \bigcup_{e \in \textsc{FElem}} \textsc{DTime}\Big(2^{\cdot^{\cdot^{\cdot^{2}}}}\Big\}e(n) \text{ times}\Big)$$

Theorem (Lazić and S., 2015)

ABVASS Leaf Coverability is in Tower*; it is in* d-ExpTime *for fixed dimension* $d > 0$*, and in* PTime *for fixed* $d = 0$.

- Rackoff (1978)'s technique: small derivation property
- matching lower bound already for BVASS

Corollary

LLW Provability is in Tower.

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Corollary

LLW Provability is in Tower.

Proof Plan for Upper Bound

- if coverable, then $\exists$ small derivation of height $\approx 2^{\cdot^{\cdot^{\cdot 2^n}}} \}$ d times
- induction on dimension: i-derivation for $(q, \mathbf{v})$
  - root label $q, \mathbf{v}$
  - enforces leaf coverability: every leaf labelled by some state $\in Q_\ell$
  - allows negative values on coordinates $i < j \leqslant d$
- H_i: bound on $\sup_{q,\mathbf{v}}$ of the heights of minimal i-derivations for $(q, \mathbf{v})$

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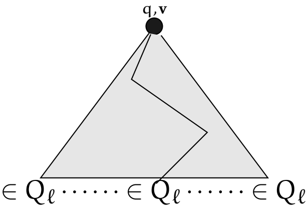
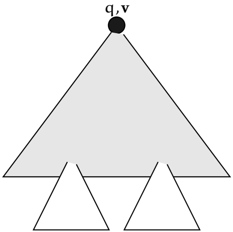
Small Derivations: Base Case



No state can appear twice along a branch of a minimal 0-derivation:

$$H_0 = |Q|$$

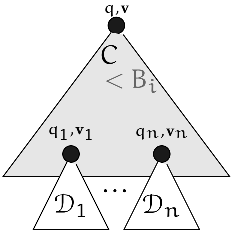
Small Derivations: Induction Step



an $(i+1)$-derivation $\mathcal{D}$

Small Derivations: Induction Step

$$B_i \stackrel{\text{def}}{=} \|T_u\|_\infty \cdot 2^{H_i}$$



an $(i+1)$-derivation $\mathcal{D} = C[\mathcal{D}_1, \ldots, \mathcal{D}_n]$

$\forall 1 \leqslant j \leqslant n. \exists 1 \leqslant k \leqslant d. \mathbf{v}_j(k) \geqslant B_i$

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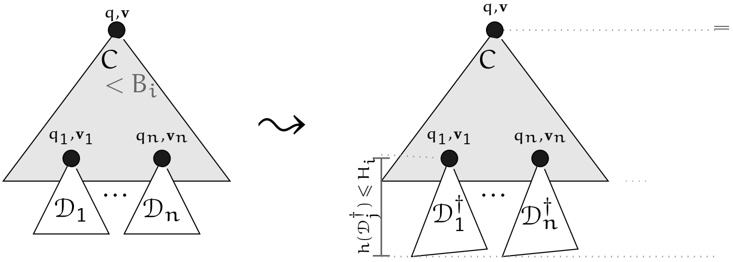

an $(i+1)$-derivation $\mathcal{D} = C[\mathcal{D}_1, \ldots, \mathcal{D}_n]$ $\quad$ $\mathcal{D}^\dagger = C[\mathcal{D}_1^\dagger, \ldots, \mathcal{D}_n^\dagger]$

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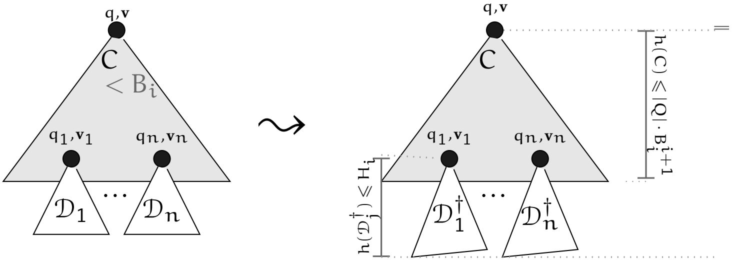

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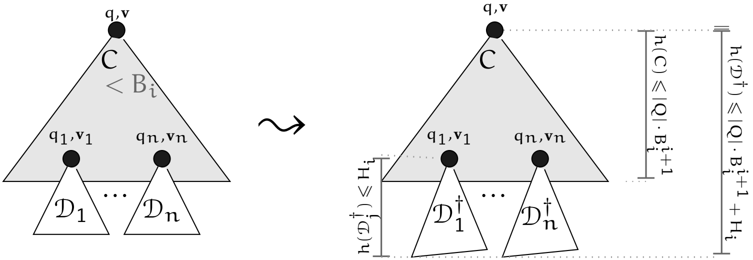

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$\forall 1 \leqslant j \leqslant n. \exists 1 \leqslant k \leqslant d. \mathbf{v}_j(k) \geqslant B_i$

$$H_{i+1} \leqslant |Q| \cdot B_i^{i+1} + H_i$$

Complexity of Reachability

	VASS	BVASS	$ABVASS_0$	AVASS
	EXPSPACE $\leqslant$? $\leqslant$ $\mathbf{F}_{\omega^3}$ (Lipton, 1976; Leroux and S., 2015)	TOWER $\leqslant$? $\leqslant$ Σ^0_1 (Lazić and S., 2015)	Σ^0_1 (Lincoln et al., 1992)	
Lossy	EXPSPACE (Lipton, 1976; Rackoff, 1978)	TOWER (Lazić and S., 2015)		2EXPTIME (Courtois and S., 2014)
Gainy		2EXPTIME (Demri et al., 2013)	ACKERMANN (Urquhart, 1999)	

Complexity of Provability

	MLL	MELL	LL	MALL
	NPTime (Kanovich, 1991)	Tower $\leqslant$? $\leqslant$ Σ_1^0 (Lazić and S., 2015)	Σ_1^0 (Lincoln et al., 1992)	PSpace (Lincoln et al., 1992)
+W	?	—— Tower —— (Lazić and S., 2015)		?
+C	—— 2ExpTime —— (S., 2015)		—— Ackermann —— (Urquhart, 1999; Lazić and S., 2015)	

Concluding Remarks

- long-known connections between

  counter systems and linear logic

  (Lincoln et al., 1992; Kanovich, 1995; Kopylov, 2001; de Groote et al., 2004)
  - also relevance logic (Urquhart, 1999)
  - also BBI & separation logic (Larchey-Wendling and Galmiche, 2013; Brotherston and Kanovich, 2014)
- extract complexity statements from inter-reductions
- counter systems provide an algorithmic backdrop
- to learn more: references in the next slides and
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Root Coverability

Meet BVASS

Replace fork semantics by *meet* semantics:

$$\mathbf{v}_1 \sqcap \mathbf{v}_2(i) \stackrel{\text{def}}{=} \min(\mathbf{v}_1(i), \mathbf{v}_2(i)) \qquad \frac{q, \mathbf{v}_1 \sqcap \mathbf{v}_2}{q_1, \mathbf{v}_1 \quad q_2, \mathbf{v}_2}\,(\text{meet})$$

Given $\mathcal{A} = \langle Q, d, T_u, T_f, T_s, T_z \rangle$, $q_r \in Q$, $Q_\ell \subseteq Q$

Root Coverability

Does there exist a deduction tree rooted by $q_r, \mathbf{v}$ for some $\mathbf{v} \in \mathbb{N}^d$ and with every leaf in $Q_\ell \times \{\mathbf{0}\}$?

Lemma

ABVASS gainy reachability coincides with meet BVASS root coverability (without full zero-tests).