Reachability in Vector Addition Systems is Primitive-Recursive in Fixed Dimension

Jérôme Leroux & Sylvain Schmitz

LICS 2019

Outline

vector addition systems (VAS)

- central model of computation

reachability problem

- hard conceptually and computationally
- decision via decomposition algorithm

this talk

- new complexity upper bounds

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Vector Addition Systems

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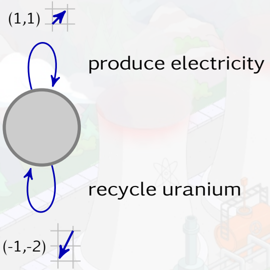
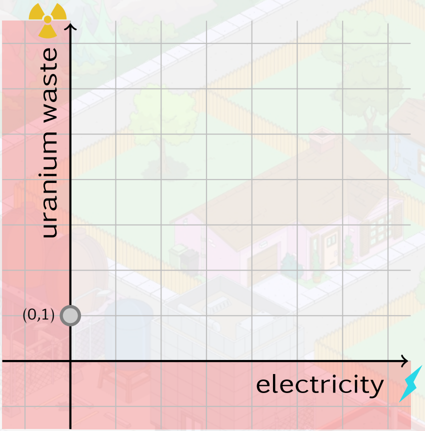
Vector Addition Systems

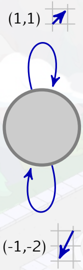
Springfield Power Plant

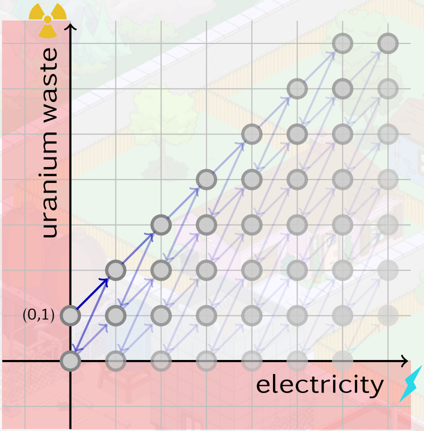




Vector Addition Systems

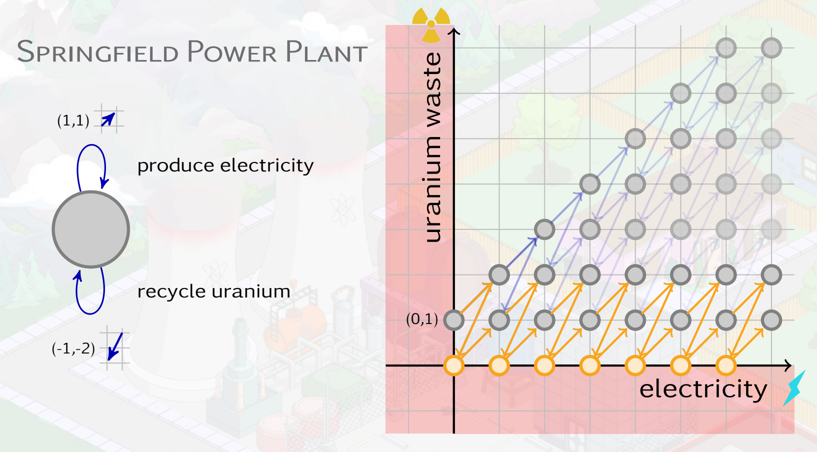
Springfield Power Plant





Can we produce unbounded electricity with no left-over uranium waste?

Vector Addition Systems



Can we produce unbounded electricity with no left-over uranium waste? Yes, $(\infty,0)$ is reachable

Importance of the Problem

Reachability Problem

input: *a vector addition system and two configurations* **source** *and* **target**

question: **source** $\rightarrow^*$ **target**?

- modelling of discrete resources (items, money, molecules, active threads, active data domain, …)
- many decision problems interreducible with reachability

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Importance of the Problem

1962
1976
1981
1982
1992
2011
2015
2017
2019


R. J. Lipton: EXPSPACE lower bound


E. W. Mayr: decidability by decomposition


S. R. Kosaraju: decidability by decomposition


J.-L. Lambert: decidability by decomposition


J. Leroux: decidability by Presburger inductive invariants

J. Leroux & S.: cubic Ackermann upper bound ($\mathbf{F}_{\omega^3}$)

S.: quadratic Ackermann upper bound ($\mathbf{F}_{\omega^2}$)






W. Czerwinski, S. Lasota, R. Lazić, J. Leroux, F. Mazowiecki: Tower lower bound ($\mathbf{F}_3$)

New Upper Bounds

$$
\begin{aligned}
F_0(x) &= x+1 \\
F_1(x) &= \overbrace{F_0 \circ \cdots \circ F_0}^{x+1 \text{ times}}(x) &= 2x+1 \\
F_2(x) &= \overbrace{F_1 \circ \cdots \circ F_1}^{x+1 \text{ times}}(x) &\approx 2^x \\
F_3(x) &= \overbrace{F_2 \circ \cdots \circ F_2}^{x+1 \text{ times}}(x) &\approx \mathrm{tower}(x) \\
&\vdots \\
F_\omega(x) &= F_{x+1}(x) &\approx \mathrm{ackermann}(x)
\end{aligned}
$$

Upper Bound Theorem

VAS Reachability is in $\mathbf{F}_\omega$, and in $\mathbf{F}_{d+4}$ in fixed dimension d

New Upper Bounds

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\begin{aligned}
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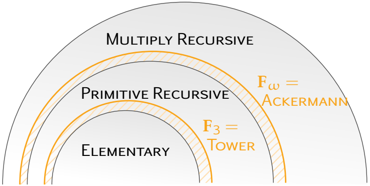

Upper Bound Theorem

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New Upper Bounds

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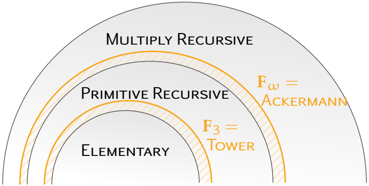

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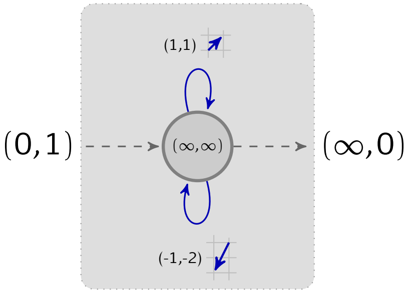
"Simple Runs" (Θ Condition)

[Mayr'81, Kosaraju'82, Lambert'92]



"Simple Runs" (Θ Condition)

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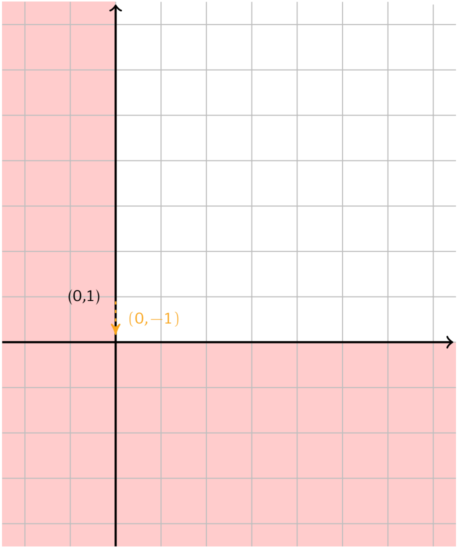

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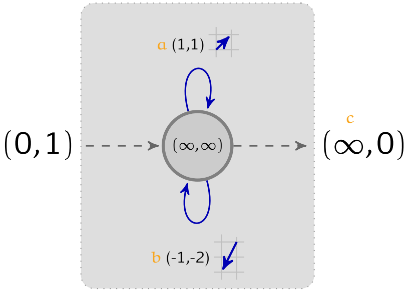
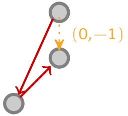


Characteristic System

$$0 + 1 \cdot a - 1 \cdot b = c$$
$$1 + 1 \cdot a - 2 \cdot b = 0$$

Solution Path

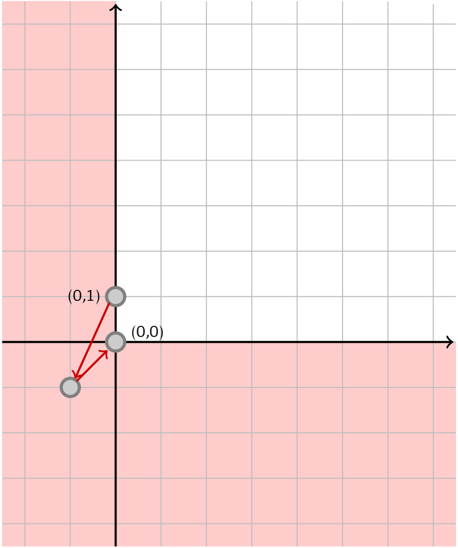


"Simple Runs" (Θ Condition)

[Mayr'81, Kosaraju'82, Lambert'92]

solution path

$\times 1$



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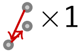

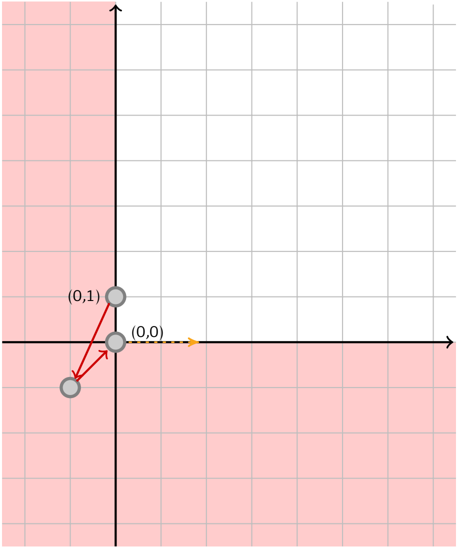

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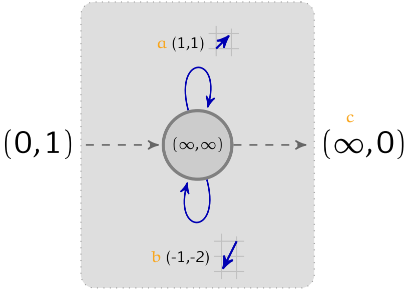
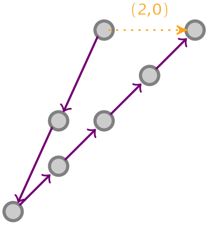


Homogeneous System

$$1 \cdot a - 1 \cdot b = c$$
$$1 \cdot a - 2 \cdot b = 0$$
$$a, b, c > 0$$

Unbounded Path



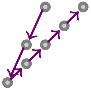
"Simple Runs" (Θ Condition)

[Mayr'81, Kosaraju'82, Lambert'92]

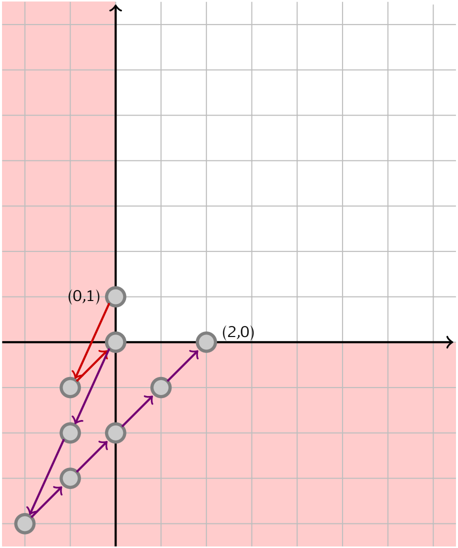
solution path

$\times 1$

unbounded path

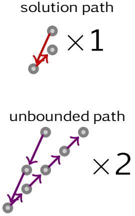


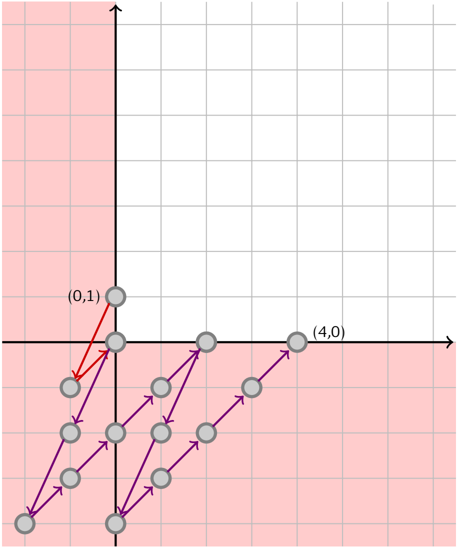
$\times 1$



“Simple Runs” (Θ Condition)

[Mayr’81, Kosaraju’82, Lambert’92]



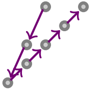


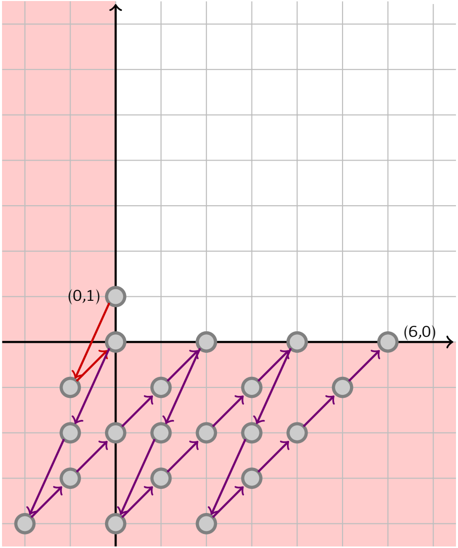
"Simple Runs" (Θ Condition)

[Mayr'81, Kosaraju'82, Lambert'92]

solution path

unbounded path

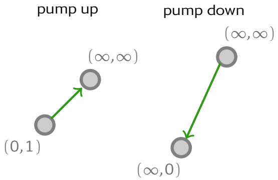




"Simple Runs" (Θ Condition)

[Mayr'81, Kosaraju'82, Lambert'92]

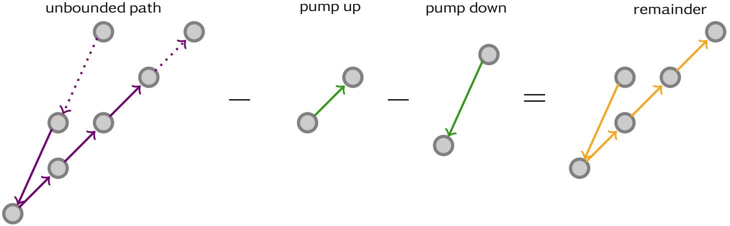
Pumpable Paths



"Simple Runs" (Θ Condition)

[Mayr'81, Kosaraju'82, Lambert'92]

Pumpable Paths

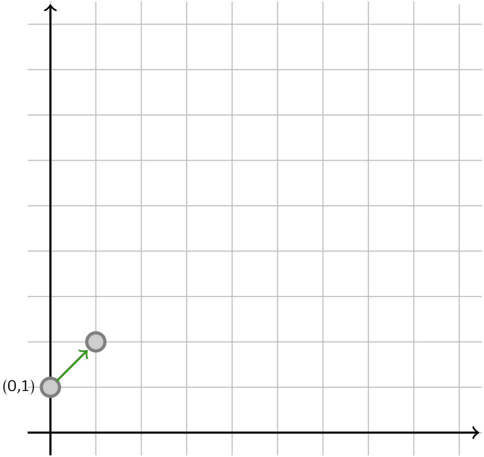


"Simple Runs" (Θ Condition)

[Mayr'81, Kosaraju'82, Lambert'92]

pump up

$\times 1$



"Simple Runs" (Θ Condition)

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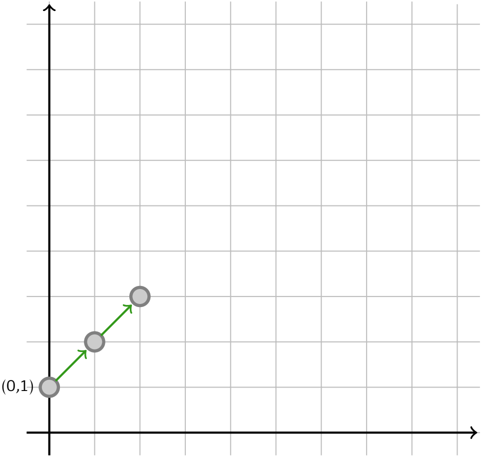
pump up



"Simple Runs" (Θ Condition)

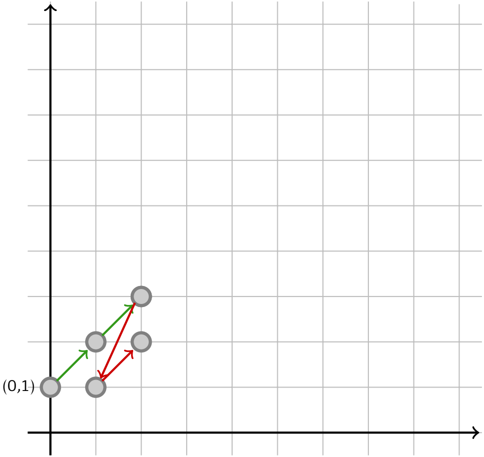
[Mayr'81, Kosaraju'82, Lambert'92]

pump up

$\times 2$

solution path

$\times 1$



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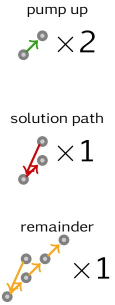
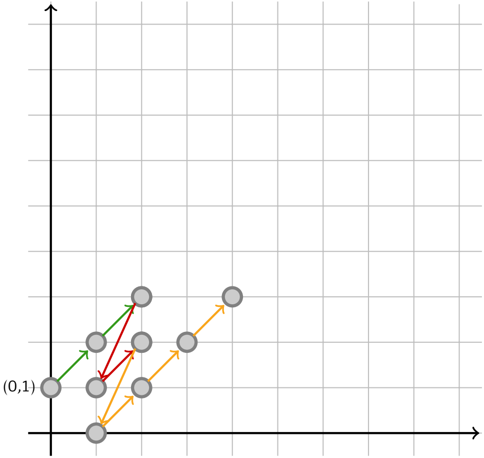




“Simple Runs” (Θ Condition)

[Mayr’81, Kosaraju’82, Lambert’92]

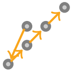
pump up

$\times 2$

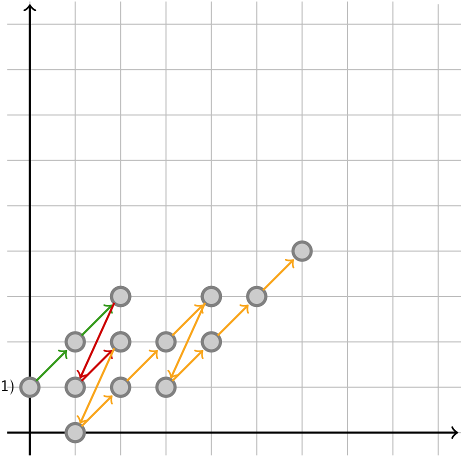
solution path

$\times 1$

remainder



$\times 2$



"Simple Runs" (Θ Condition)

[Mayr'81, Kosaraju'82, Lambert'92]

pump up

$\times 2$

solution path

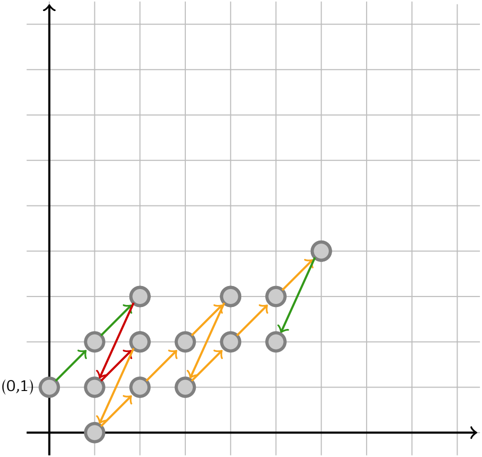
$\times 1$

remainder

$\times 2$

pump down

$\times 1$



"Simple Runs" (Θ Condition)

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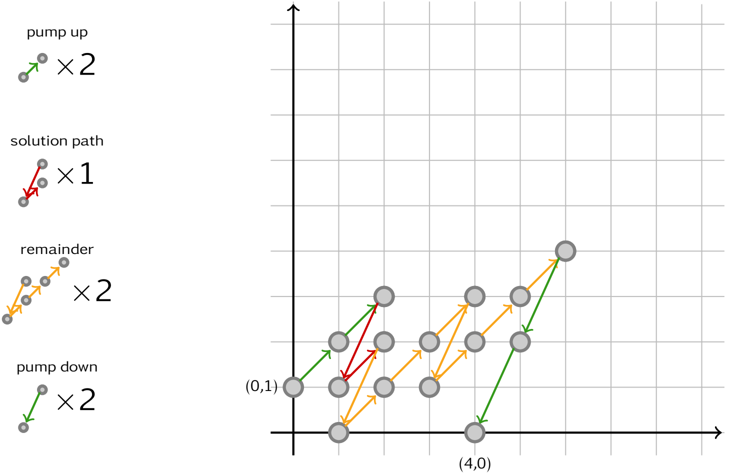

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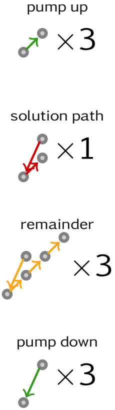
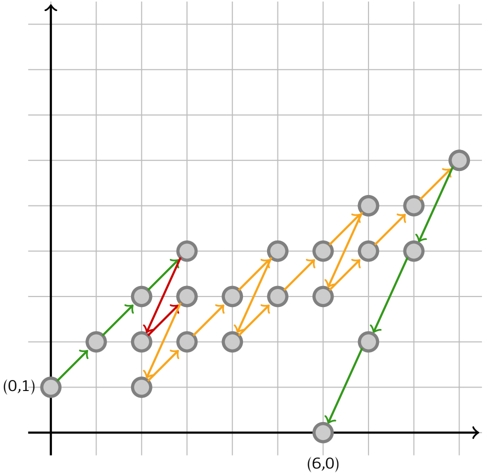
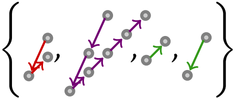




Decomposition Algorithm

[Mayr'81, Kosaraju'82, Lambert'92]

can we build a "simple run"?



Decomposition Algorithm

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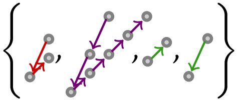
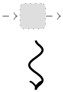
can we build a “simple run”? yes



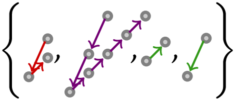
DECOMPOSITION ALGORITHM

[Mayr'81, Kosaraju'82, Lambert'92]

can we build a "simple run"? no



☹

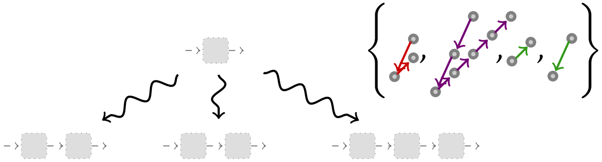


decompose

Decomposition Algorithm

[Mayr'81, Kosaraju'82, Lambert'92]

can we build a "simple run"? no



decompose

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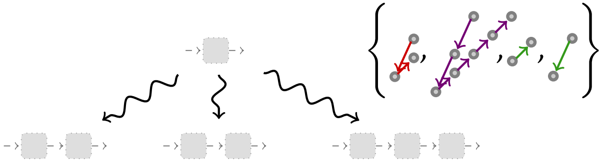
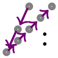
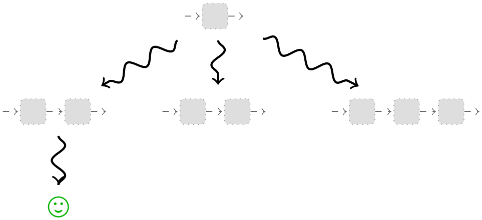


- no : no execution ⇝ empty decomposition
- no :
  - bounded ∞: saturate with bounded value
  - bounded transition use: unfold and track bounded transition count
- no  or no : unfold and track bounded counter value

Decomposition Algorithm

[Mayr'81, Kosaraju'82, Lambert'92]



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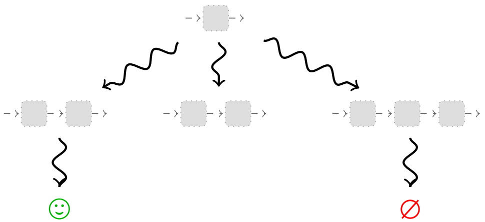

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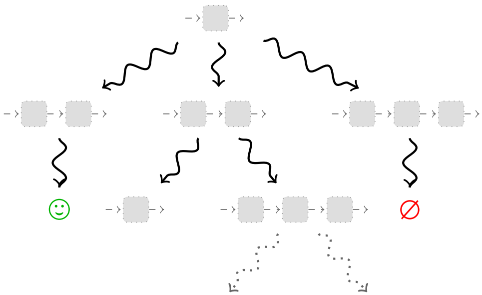


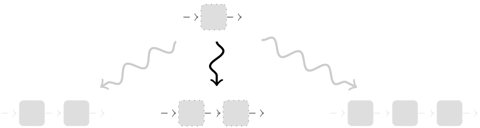
Termination

Ranking Function

α_0

Termination

Ranking Function



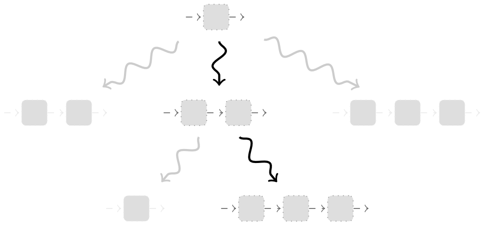
$$\alpha_0$$

$$\vee$$

$$\alpha_1$$

Termination

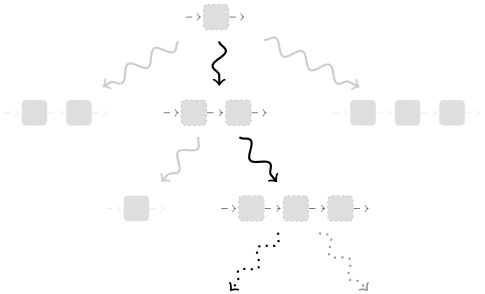
Ranking Function



$$\alpha_0 > \alpha_1 > \alpha_2$$

Termination

Ranking Function



$$\begin{array}{c} \alpha_0 \\ \vee \\ \alpha_1 \\ \vee \\ \alpha_2 \\ \vee \\ \vdots \end{array}$$

Technical Contributions

1. new ranking function:

   order type ω^{d+1}

   ω^{ω^3} in [Leroux & S. '15]

   $\omega^{\omega} \cdot (d+1)$ in [S. '17]

2. refined analysis of pumpable paths:

   Rackoff-style analysis

   improves complexity from $\mathbf{F}_{2d+2}$ to $\mathbf{F}_{d+4}$

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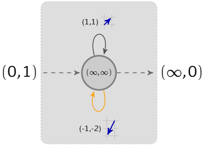
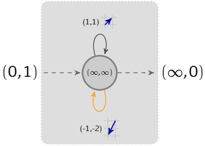
Rank of a Transition

For a transition t in 

$$\{\text{effects of cycles } C \mid t \in C\}$$

Rank of a Transition

For a transition t in 

$$\{m \cdot \nearrow + n \cdot \swarrow \mid m \geqslant 0, n > 0\}$$

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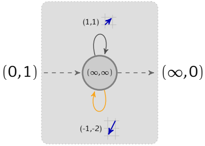
For a transition t in 

$$\mathrm{span}_{\mathbb{Q}}\Big(\big\{m \cdot \nearrow + n \cdot \swarrow \mid m \geqslant 0, n > 0\big\}\Big) = \mathbb{Q}^2$$

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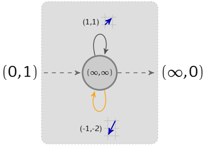
For a transition t in 

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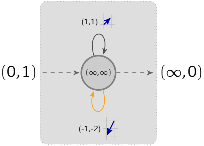
For a transition t in 

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here, $\quad \mathrm{rank}(t) = (1,0,0) \quad \in \mathbb{N}^{d+1}$

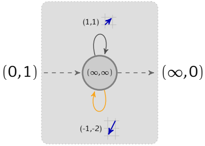
Definition

$$\mathrm{rank}(G) \stackrel{\text{def}}{=} \sum_{t \in G} \mathrm{rank}(t) \quad \in \mathbb{N}^{d+1}$$

ordered lexicographically

Rank of a VAS

For a transition t in



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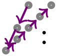
Decreasing Ranks

Recall:

- no : no execution $\rightsquigarrow$ empty decomposition
- no :
  - bounded ∞: saturate with bounded value
  - bounded transition use: unfold and track bounded transition count
- no or no : unfold and track bounded counter value

Decreasing Ranks

Recall:

- no : no execution ⇝ empty decomposition
- no :
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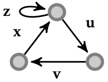
Decreasing Ranks

Proof Idea

Consider a strongly connected VAS G:



Claim: if $T' \subsetneq T$, then $\mathrm{rank}(G') < \mathrm{rank}(G)$

- let $\mathbf{V}$, resp. $\mathbf{V}'$ be the vector space associated to cycles of T, resp. T'
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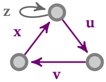
Proof Idea

Consider a strongly connected VAS G:

$T \setminus T'$: not in any hom. sol.



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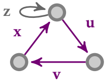

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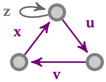

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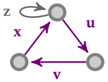

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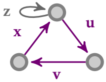

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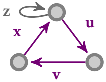

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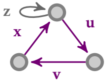

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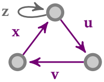

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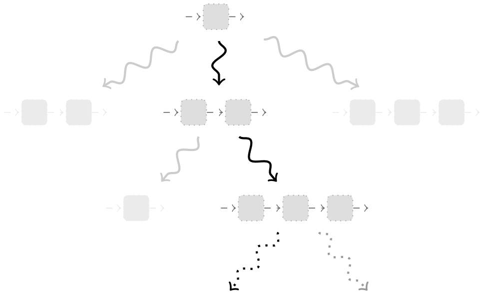
The Length of Decomposition Branches



$$\omega^{d+1} > \alpha_0 > \alpha_1 > \alpha_2 > \cdots$$

Consequence of (S. '14)

The decomposition tree is of size at most $F_{d+4}(e(n))$ *for some elementary function* e.

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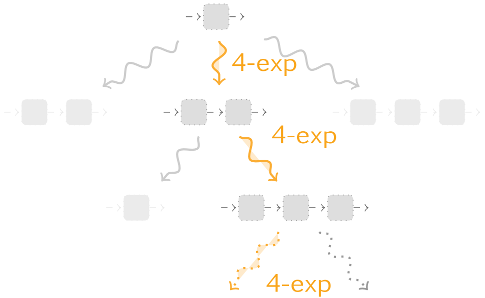

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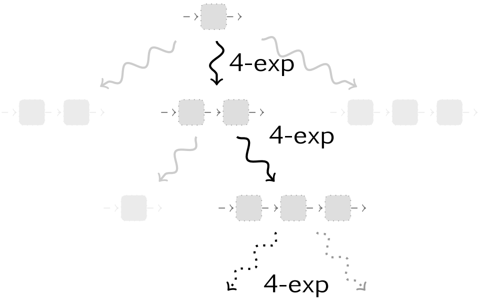

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New Upper Bounds

$$
\begin{aligned}
F_0(x) &= x + 1 \\
F_1(x) &= \overbrace{F_0 \circ \cdots \circ F_0}^{x+1 \text{ times}}(x) = 2x + 1 \\
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F_3(x) &= \overbrace{F_2 \circ \cdots \circ F_2}^{x+1 \text{ times}}(x) \approx \mathrm{tower}(x) \\
&\vdots \\
F_\omega(x) &= F_{x+1}(x) \approx \mathrm{ackermann}(x)
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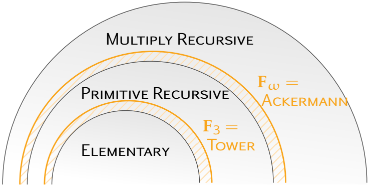

Upper Bound Theorem

VAS Reachability is in $\mathbf{F}_\omega$, and in $\mathbf{F}_{d+4}$ in fixed dimension d

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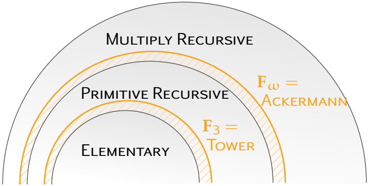

Upper Bound Theorem

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Theorem

VAS Reachability reduces to bounded VAS Reachability

A Related Problem

labelled VAS transitions carry labels from some alphabet

$L(\mathcal{V}, \mathbf{source}, \mathbf{target})$ the language of labels in runs from **source** to **target**

$\downarrow L$ the set of scattered subwords of the words in the language L

Example

$$aba \leqslant_* baaacabbab$$

A Related Problem

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$\downarrow L$ the set of scattered subwords of the words in the language L

Downwards Language Inclusion Problem

input: *two labelled VAS $\mathcal{V}$ and $\mathcal{V}'$ and configurations* $\mathbf{source}, \mathbf{target}, \mathbf{source}', \mathbf{target}'$

question: $\downarrow L(\mathcal{V}, \mathbf{source}, \mathbf{target}) \subseteq \downarrow L(\mathcal{V}', \mathbf{source}', \mathbf{target}')$?

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Theorem (Habermehl, Meyer & Wimmel'10)

Given a labelled VAS $\mathcal{V}$ and configurations $\mathbf{source}$ *and* $\mathbf{target}$ *and its decomposition, one can construct a finite automaton for* $\downarrow L(\mathcal{V}, \mathbf{source}, \mathbf{target})$ *in polynomial time*.

Corollary

The Downwards Language Inclusion is in Ackermann.

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Corollary

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Theorem (Zetzsche'16)

The Downwards Language Inclusion is Ackermann*-hard.*

Perspectives

1. complexity gap for VAS reachability
   - Tower-hard [Czerwinski et al.'19]
   - decomposition algorithm: requires $\mathbf{F}_\omega$ = Ackermann time, because downward language inclusion is $\mathbf{F}_\omega$-hard [Zetzsche'16]
2. reachability in VAS extensions?
   - decidable in VAS with hierarchical zero tests [Reinhardt'08]
   - what about
     - branching VAS
     - unordered data Petri nets
     - pushdown VAS

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   - what about
     - branching VAS
     - unordered data Petri nets
     - pushdown VAS

Perspectives

1. complexity gap for VAS reachability
   - Tower-hard [Czerwinski et al.'19]
   - decomposition algorithm: requires $\mathbf{F}_\omega = \text{Ackermann}$ time, because downward language inclusion is $\mathbf{F}_\omega$-hard [Zetzsche'16]
2. reachability in VAS extensions?
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   - what about
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