Games with Discrete Resources

Sylvain Schmitz
with Th. Colcombet, J.-B. Courtois, M. Jurdziński, and R. Lazić

LSV, ENS Paris-Saclay & CNRS

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Outline

multi-dimensional energy parity games

complexity through perfect half-space games
  (Colcombet et al., LICS ’17)

related problems:

- multi-dimensional mean-payoff parity games
  (Chatterjee et al., Concur ’12)
- VASS games (too many references!)
- regular VASS simulations (Courtois and S., MFCS ’14)
- $(!,\oplus)$-Horn linear logic (Kanovich, APAL ’95)
- $\mu$-calculus on VASS (Abdulla et al., Concur ’13)
- resource-bounded agent temporal logic $\text{RB}\pm\text{ATL}^*$
  (Alechina et al., RP ’16)
WHERE TO TRECK IN ICELAND?
WHERE TO TRECK IN ICELAND?

- Reykjavik
- Hrútafjördur
- Landmannalaugar
- Thórmörk
- Vatnajökull
- Mývatn
WHERE TO TRECK IN ICELAND?

MAXIMAL DRY TEMPERATURE
WHERE TO TRECK IN ICELAND?

MAXIMAL DRY TEMPERATURE
as a parity objective

Multi-Energy Parity Games

Perfect Half-Space Games
WHERE TO TRECK IN ICELAND?

MAXIMAL DRY TEMPERATURE
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WHERE TO TRECK IN ICELAND?

UNCONTROLLED EVENTS
WHERE TO TRECK IN ICELAND?

MAXIMAL DRY TEMPERATURE as a parity objective
UNCONTROLLED EVENTS as a two-players game

Maximal dry temperature as a parity objective
Uncontrolled events as a two-players game
WHERE TO TRECK IN ICELAND?

Maximal dry temperature as a parity objective
Uncontrolled events as a two-players game

Multi-Energy Parity Games

Perfect Half-Space Games
WHERE TO TRECK IN ICELAND?

DIScrete resources
WHERE TO TRECK IN ICELAND?

**Maximal dry temperature**
as a parity objective

**Uncontrolled events**
as a two-players game

**Discrete resources**
as a multi-energy objective

---

Diagram showing the transitions and states for the game with labels and positions of points indicating transitions and values.
WHERE TO TRECK IN ICELAND?

**Maximal dry temperature**
as a parity objective

**Uncontrolled events**
as a two-players game

**Discrete resources**
as a multi-energy objective

---

Maps of Iceland.

- **Reykjavik**
- **Landmannalaugar**
- **Thòrsmörk**
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Maps of Iceland.

**Maximal dry temperature** as a parity objective

**Uncontrolled events** as a two-players game

**Discrete resources** as a multi-energy objective
Multi-Dimensional Energy Parity Games

Player 1 wins a play if both

- **energy** objective: no component goes negative
- **parity** objective: the maximal priority is odd

**Example**

\[
R(0,0) \xrightarrow{(1,0)} R(1,0) \xrightarrow{(1,0)} R(2,0) \xrightarrow{(-1,0)} H(1,0) \xrightarrow{(0,0)} R(1,0) \rightarrow \cdots
\]

Decision problems: Does Player 1 have a winning strategy

- **given** initial credit as part of the input
- **existential**: for some initial credit
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Multi-Dimensional Energy Games

Complexity

lower bound     upper bound

w. initial credit

∃ initial credit
Multi-Dimensional Energy Games

Complexity

lower bound

upper bound

w. initial credit

EXPSPACE

(Lasota, IPL ’09)

∃ initial credit
Multi-Dimensional Energy Games

**Complexity**

<table>
<thead>
<tr>
<th>w. initial credit</th>
<th>lower bound</th>
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<tbody>
<tr>
<td></td>
<td><strong>EXPSPACE</strong></td>
<td><strong>TOWER</strong></td>
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<th>∃ initial credit</th>
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**Multi-Dimensional Energy Games**

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(Courtois and S., MFCS ’14)

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### Multi-Dimensional Energy Parity Games

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## Multi-Dimensional Energy Parity Games

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<td>EXP for $d \geq 2$</td>
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<td>pseudoP</td>
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| $\exists$ initial credit | pseudoP | (Colcombet et al., LICS ’17) |
Complexity of Multi-Energy Parity Games

Theorem (Colcombet et al., LICS ’17)

1. The given initial credit problem for multi-dimensional energy parity games is in 2-EXP.

2. With fixed dimension and number of priorities, it is in pseudo polynomial time.

- series of reductions using notably perfect half-space games

- fine understanding of Player 2’s strategies: 
  Player 2 can win by announcing in which perfect half space he will escape
Complexity of Multi-Energy Parity Games

**Theorem** (Colcombet et al., LICS ’17)

1. The given initial credit problem for multi-dimensional energy parity games is in $2\text{-}\text{EXP}$.

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**REDUCTIONS AND STRATEGY TRANSFERS**

multi-dimensional energy parity games

(\text{Jančar, RP ’15})

\[ \downarrow \]

extended multi-dimensional energy games (Brázdil et al., ICALP ’10)

\[ \downarrow \]

bounding games (Jurdziński et al., ICALP ’15)

\[ \downarrow \]

perfect half space games (Colcombet et al., LICS ’17)

\[ \downarrow \]

lexicographic energy games (Colcombet and Niwiński)

\[ \downarrow \]

mean-payoff games (Comin and Rizzi, Algorithmica ’16)
**Extended Multi-Dimensional Energy Games**

Encode Priorities as Energy (Jančar, RP '15)

Two new dimensions: tolerance to humid low/high temperature
Bounding Games

Player 1’s Objective

existential energy

bounding
**Bounding Games**

**Player 1’s Objective**

- **Existential Energy**
- **Bounding**
Bounding Games

Player 1’s Objective

existential energy

bounding
Bounding Games

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bounding
**Bounding Games**

**Player 1’s Objective**

existential energy

bounding
Bounding Games

Encoding Extended Energy Games

Bin excess energy

Unbounded replenishing

(..., −1,...)

(..., ω,...)

(0, 1, 0)
**Perfect Half Space Games**

**Player 2’s Objective in a Bounding Game**

**Key Intuition**
Player 2 can escape in a perfect half space
**Perfect Half Space Games**

**Player 2’s Objective in a Bounding Game**

![Diagram](image)

**Key Intuition**

Player 2 can escape in a *perfect half space*
Perfect Half-Space Games

Perfect Half-Space

\[ \{ (x, y) : x + y < 0 \} \]
Perfect Half-Space Games

Perfect Half-Space

\[ \{(x, y) : x + y < 0\} \]

boundary: \[ \{(x, y) : x + y = 0\} \]
**Perfect Half Space Games**

**Perfect Half Space**

\[
\{(x, y) : x + y < 0\}\ 
\cup \{(x, y) : x + y = 0 \land x < 0\}
\]
**Perfect Half Space Games**

**Plays**

- pairs of vertices and perfect half spaces:
  
  \[(v_0, H_0) \xrightarrow{w_1} (v_1, H_1) \xrightarrow{w_2} (v_2, H_2) \cdots\]

- in his vertices, Player 2 chooses the current perfect half space
Perfect Half Space Games

- Player 2 wins if $\exists i$ s.t. $\sum_{j} w_j \geq 0$ diverges into $\cap_{j \geq i} H_j$

Example

$$H_L \cap H_R = \hfill$$
SOLVING PERFECT HALF SPACE GAMES

**Theorem**

Perfect half space games on multi-weighted game graphs \((V, E, d)\) are solvable in \((|V| \cdot \|E\|)^{O(d^3)}\).

**Proof Idea**

- reduce to a lexicographic energy game (Colcombet and Niwiński)
- \(\approx\) perfect half space game with a single fixed \(H\)
- itself reduced to a mean-payoff game
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Player 2 Strategies

Oblivious Strategy
Player 2 chooses the same $H_v$ every time it visits vertex $v$.

Theorem
If Player 2 has a winning strategy in a perfect half space game, then it has an oblivious one.

“Counterless” Strategy

Corollary (Brázdil et al., ICALP ’10)
If Player 2 has a winning strategy in an existential multi-dimensional energy parity game, then it has a positional one.
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VASS Games

multi-energy game configuration arena over $\mathbb{Q} \times \mathbb{Z}^d$

+ energy objective

VASS game configuration arena over $\mathbb{Q} \times \mathbb{N}^d$

Example

$R(0,0) \not\leftrightarrow H(-1,0)$
**Objectives**

**Monotone objectives:**

State reachability given $q_\ell \in Q$, Player 1 wins if any configuration in $\{q_\ell\} \times \mathbb{N}^d$ is visited.

Non-termination Player 1 wins if the play is infinite.

Parity given a colouring $c: Q \to \{1, \ldots, k\}$, Player 1 wins if the least colour seen infinitely often is odd.

**Non-monotone objective:**

Configuration reachability given $q_\ell \in Q$, Player 1 wins if the configuration $(q_\ell, 0)$ is visited.
**Objectives**

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STATE REACHABILITY VASS GAMES

Player 2 can enforce zero-tests:

Minsky machine

Symmetric VASS Game

\[
\begin{array}{c}
q \\
\downarrow c_i = 0 \\
q_1 \\
\uparrow \\
\downarrow c_i \\
q_2 \\
\end{array}
\quad \Rightarrow 
\quad
\begin{array}{c}
q \\
\downarrow 0 \\
-\varepsilon_i \\
q_1 \\
\uparrow \\
\downarrow 0 \\
-\varepsilon_i \\
q_2 \\
\end{array}
\]

THEOREM (RASKIN ET AL., AVoCS ’04)

State reachability VASS games with given initial credit are undecidable.
State Reachability VASS Games

Player 2 can enforce zero-tests:

Minsky machine

\[
\begin{align*}
q & \quad c_i = 0 \\
q_1 & \\
q_2 & \\
\end{align*}
\]

Symmetric VASS Game

\[
\begin{align*}
q & \quad -e_i \\
\bot & \\
q_1 & \\
q_2 & \quad 0 \\
\end{align*}
\]

Theorem (Raskin et al., AVoCS ’04)

State reachability VASS games with given initial credit are undecidable.
ASYNMMETRIC VASS GAMES

Player 2 moves restricted to use the zero vector.

A FREQUENT ASSUMPTION

- and-branching VASS (Lincoln et al., APAL ’92)
- vector games (Kanovich, APAL ’95)
- B-games (Raskin et al., AVoCS ’04)
- single-sided games (Abdulla et al., Concur ’13)
- alternating VASS (Courtois and S., MFCS ’14)
**Monotone Objectives** (1/2)

**Lemma (Monotone Objectives)**

If Player 1 wins a monotone AVASS game from a configuration \(q, v\) and \(v' \geq v\), then she also wins from \(q, v'\).

**Corollary (by Dickson’s Lemma)**

- finite-memory strategies suffice for Player 1
- state reachability and non-termination objectives are decidable
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**Monotone Objectives**

Multi-Energy Games

\[ q \xrightarrow{u} q' \implies q \xrightarrow{u} \triangle \xrightarrow{u} q' \]

**Theorem (Abdulla et al., Concur '13)**

Monotone AVASS games and multi-dimensional energy games are LOGSPACE-equivalent.

**Corollary**

Monotone AVASS games with given initial credit are 2-EXP-complete.
**Configuration Reachability Objective** (1/2)

Player 2 can enforce zero-tests using the reachability objective \((q_\ell, 0)\):

- **Minsky machine**
  - \(q\) \(\xrightarrow{c_i = 0} q_1\)
  - \(q_1 \xrightarrow{c_i} q_2\)

- **AVASS**
  - \(q \xRightarrow{\forall j \neq i: -e_j} q_\ell\)
  - \(q \xRightarrow{-e_i} q_1\)
  - \(q_1 \xRightarrow{} q_2\)

**Theorem (Lincoln et al., APAL '92)**

Configuration reachability AVASS games with given initial credit are undecidable.
Configuration Reachability Objective

Existential initial credit \(\equiv\) gainy game
where \(\forall q \in Q. \forall 1 \leq i \leq d. q \overset{e_i}{\rightarrow} q\)

Theorem (Urquhart, JSL ’99; Lazić and S., ToCL ’15)

Configuration reachability AVASS games with existential initial credit are ACKERMANN-complete.
MODEL-CHECKING RESOURCE-AWARE LOGICS

VASS models fragment of the $\mu$-calculus on VASS executions

(Abdulla et al., Concur ’13)

resource-bounded concurrent game structures $\text{RB}\pm \text{ATL}^*$

(Alechina et al., RP ’16)

Both are 2-EXP-complete by reduction to multi-energy parity games / parity AVASS games.
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**Propositional (Intuitionistic) Linear Logic**

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<tr>
<th>Rule</th>
<th>Formula</th>
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<tbody>
<tr>
<td>(I)</td>
<td>$\frac{\Gamma \vdash A}{\Gamma, !A \vdash A}$</td>
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<tr>
<td>(C!)</td>
<td>$\frac{\Gamma, !A, !A \vdash B}{\Gamma, !A \vdash B}$</td>
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<td>$\frac{\Gamma, !A \vdash B}{\Gamma, !A \vdash B}$</td>
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<tr>
<td>($\La$)</td>
<td>$\frac{\Gamma \vdash A \quad \Delta, B \vdash C}{\Gamma, \Delta, A \rightarrow B \vdash C}$</td>
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<tr>
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**Propositional (Intuitionistic) Linear Logic**

\[
\frac{A \vdash A}{(I)} \quad \frac{\Gamma, !A, !A \vdash B}{(C!)} \quad \frac{\Gamma, A \vdash B}{(L!)}
\]

\[
\frac{\Gamma \vdash A}{\Delta, B \vdash C} \quad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \Rightarrow B} \quad \frac{\Gamma \vdash A}{\Gamma, A \vdash B} \quad \frac{\Gamma, B \vdash C}{\Gamma, A \& B \vdash C} \quad \frac{\Gamma \vdash A \& B}{\Gamma \vdash A} \quad \frac{\Gamma \vdash B}{\Gamma \vdash A \& B} \quad \frac{\Gamma, A \vdash C}{\Gamma, B \vdash C} \quad \frac{\Gamma, A \& B \vdash C}{\Gamma, A \oplus B \vdash C} \quad \frac{\Gamma \vdash A \oplus B}{\Gamma \vdash A} \quad \frac{\Gamma \vdash A \oplus B}{\Gamma \vdash B} \quad \frac{\Gamma, A \vdash C}{\Gamma, B \vdash C} \quad \frac{\Gamma, A \& B \vdash C}{\Gamma, A \otimes B \vdash C} \quad \frac{\Gamma \vdash A}{\Gamma, A \otimes B \vdash C}\]

...
**Propositional (Intuitionistic) Linear Logic**

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\begin{align*}
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\frac{\Gamma \vdash A \quad \Delta, B \vdash C}{\Gamma, \Delta, A \rightarrow B \vdash C} & \quad \frac{\Gamma, A \vdash B}{(R \rightarrow)} \\
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\frac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C} & \quad \frac{\Gamma \vdash A \quad \Delta \vdash B}{(R \otimes)} \\
\frac{\Gamma, \Delta \vdash A \otimes B}{(R \otimes)} \\
\end{align*}
\]
(!, ⊕)-HORN PROGRAMS

connectives \{\otimes, \neg, \oplus, !\}

simple products \( W, X, Y, Z \coloneqq p_1 \otimes p_2 \otimes \cdots \otimes p_m \) for atomic \( p_i \)'s

Horn implications \( X \rightarrow Y \)

⊕-Horn implications \( X \rightarrow (Y_1 \oplus \cdots \oplus Y_n) \)

(!, ⊕)-Horn sequents \( W, !\Gamma \vdash Z \) where \( \Gamma \) contains Horn and ⊕-Horn implications
(!, ⊕)-HORN PROGRAMS

Horn programs

\[ X \rightarrow Y \]

\[ X \rightarrow (Y_1 \oplus \cdots \oplus Y_n) \]

AVASS

\[ \triangle \rightarrow \triangle \]

\[ \neg X \rightarrow +Y \]

\[ q \otimes u \rightarrow q' \otimes u \]

\[ q_0 \rightarrow (q_1 \oplus q_2) \]

\[ q_0 \rightarrow q_1 \rightarrow q_2 \]

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$(!, \oplus)$-HORN PROGRAMS

Horn programs

$X \rightarrow Y$

$X \rightarrow (Y_1 \oplus \cdots \oplus Y_n)$

$q \otimes u^- \rightarrow q' \otimes u^+$

$q_0 \rightarrow (q_1 \oplus q_2)$

AVASS

$\Rightarrow$

$\Rightarrow$

$\Leftarrow$

$\Leftarrow$

$q \quad u \rightarrow\quad q'$

$q_0 \quad q_1 \quad q_2$
### (!, ⊕)-Horn Programs (3/3)

#### Theorem (Kanovich, APAL ’95)

*Provability of (!, ⊕)-Horn sequents and configuration reachability AVASS games are PSPACE equivalent.*

#### Corollary (Lincoln et al., APAL ’92)

*Provability in propositional linear logic is undecidable.*

#### Corollary (Courtois and S., MFCS ’14; Lazić and S., ToCL ’15)

- *Provability of affine (!, ⊕)-Horn sequents is 2-EXP-complete.*
- *Provability of contractive (!, ⊕)-Horn sequents is ACKERMANN-complete.*
(!, ⊕)-HORN PROGRAMS  

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Concluding Remarks

- tight 2-EXP bounds for multi-energy parity games
- impacts numerous problems
  - affine ($\oplus,!$)-Horn linear logic
    (Kanovich, APAL ’95)
  - (weak) simulation of finite-state systems by Petri nets
    (Abdulla et al., Concur ’13)
  - model-checking Petri nets with a fragment of $\mu$-calculus
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  - resource-bounded agent temporal logic $\text{RB}^\pm\text{ATL}^*$
    (Alechina et al., RP ’16)

- fine understanding of Player 2’s strategies:
  
  Player 2 can win by announcing in which perfect half space he will escape


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