Algorithmic Theory of WQOs

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Outline

well quasi orderings (wqo)

generic tools for termination arguments

this talk

beyond termination: complexity upper bounds

contents

A Bit of History...

- First bounds for Dickson's Lemma: McAloon (1984); Clote (1986), using Ramsey-type arguments.
- Application: finite containment for Petri nets, shown by Mayr and Meyer (1981) (see also Jančar, 2001) to be Ackermann-hard
- Result: Ackermannian upper bounds

Well Quasi Orderings

Definition (wqo)

A wqo is a quasi-order $(A, \leqslant)$ s.t.

$$\forall \mathbf{x} = x_0, x_1, x_2, \ldots \in A^\omega, \exists i_1 < i_2, x_{i_1} \leqslant x_{i_2} .$$

Example (Basic WQO's)

- $(\mathbb{N}, \leqslant)$,
- $(\{0, 1, \ldots, k\}, \leqslant)$ for any $k \in \mathbb{N}$,
- $(\Gamma_p, =)$ for any finite set Γ_p with p elements.

Algebra of WQO's

Finite Sequences

Lemma (Higman's Lemma)

If $(A, \leqslant)$ is a wqo, then $(A^*, \leqslant_*)$ is a wqo where $\leqslant_*$ is the subword embedding ordering:

$$a_1 \cdots a_m \leqslant_* b_1 \cdots b_n \overset{\text{def}}{\Leftrightarrow} \begin{cases} \exists 1 \leqslant i_1 < \cdots < i_m \leqslant n, \\ \bigwedge_{j=1}^{m} a_j \leqslant_A b_{i_j} \,. \end{cases}$$

Example

$$aba \leqslant_* baaacabbab$$

Algebra of WQO's

Disjoint Sums

Lemma

If $(A_1, \leqslant_{A_1})$ *and* $(A_2, \leqslant_{A_2})$ *are two wqo's, then* $(A_1 + A_2, \leqslant_+)$ *is a wqo,*

where $A_1 + A_2 \stackrel{\text{def}}{=} \{\langle i, a\rangle \mid i \in \{1,2\} \wedge a \in A_i\}$ *and* $\leqslant_+$ *is the sum ordering:*

$$\langle i, a\rangle \leqslant_+ \langle j, b\rangle \stackrel{\text{def}}{\Leftrightarrow} i = j \wedge a \leqslant_{A_i} b\,.$$

Algebra of WQO's

Cartesian Products

Lemma (Dickson's Lemma)

If $(A_1, \leqslant_{A_1})$ and $(A_2, \leqslant_{A_2})$ are two wqo's, then $(A_1 \times A_2, \leqslant_\times)$ is a wqo,
where $\leqslant_\times$ is the product ordering:

$$\langle a_1, a_2 \rangle \leqslant_\times \langle b_1, b_2 \rangle \overset{\text{def}}{\Leftrightarrow} a_1 \leqslant_{A_1} b_1 \wedge a_2 \leqslant_{A_2} b_2 \,.$$

WQO's for Termination

Bad Sequences

- $\mathbf{x} = x_0, x_1, \ldots$ in A^∞ is a good sequence if $\exists i_1 < i_2, x_{i_1} \leqslant x_{i_2}$,
- a bad sequence otherwise,
- if $(A, \leqslant)$ is a wqo: every bad sequence is finite

An Example

$$
\begin{array}{l}
\textsc{simple}\ (a,b) \\
c \longleftarrow 1 \\
\textbf{while}\ a>0 \wedge b>0 \\
\qquad \langle a,b,c\rangle \longleftarrow \langle a-1,b,2c\rangle \\
\quad \textbf{or} \\
\qquad \langle a,b,c\rangle \longleftarrow \langle 2c,b-1,1\rangle \\
\textbf{end}
\end{array}
$$

- in any run, $\langle a_0,b_0\rangle,\ldots,\langle a_n,b_n\rangle$ is a bad sequence over $(\mathbb{N}^2,\leqslant_\times)$,
- $(\mathbb{N}^2,\leqslant_\times)$ is a wqo: all the runs are finite
- How long can SIMPLE run?

A Computation of simple$(2,3)$

SIMPLE (a,b)
$c \longleftarrow 1$
while $a > 0 \wedge b > 0$
  $\langle a,b,c\rangle \longleftarrow \langle a-1,b,2c\rangle$
 or
  $\langle a,b,c\rangle \longleftarrow \langle 2c,b-1,1\rangle$
end

$\langle a,b,c\rangle$	loop iterations
$\langle 2,3,2^0\rangle$	0

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$\langle 2,3,2^0\rangle$	0
$\langle 1,3,2^1\rangle$	1
$\langle 2^2,2,2^0\rangle$	2

A COMPUTATION OF SIMPLE$(2,3)$

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end

$\langle a,b,c\rangle$	loop iterations
$\vdots$	$\vdots$
$\langle 2^2,2,2^0\rangle$	2
$\vdots$	$\vdots$
$\langle 1,2,2^{2^2-1}\rangle$	$2+2^2-1$

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- Non-elementary complexity
- Derive (matching) upper bounds for termination arguments based on $(\mathbb{N}^2,\leqslant_\times)$ being a wqo

Controlled Sequences

- bound the length of bad sequences over $(A, \leqslant)$

Controlled Sequences

- bound the length of bad sequences over $(A,\leqslant)$
- but: choose any N, and consider the bad sequence $N, N-1, \ldots, 0$ over $\mathbb{N}$
- similarly: $\langle 3,3\rangle, \langle 3,2\rangle, \langle 3,1\rangle, \langle 3,0\rangle, \langle 2,N\rangle, \langle 2,N-1\rangle, \ldots$

Controlled Sequences

- bound the length of bad sequences over $(A, \leqslant; |.|_A)$
- associate a norm function $|.|_A : A \to \mathbb{N}$ to each wqo $(A, \leqslant)$
- assume $|.|_A$ is proper $\overset{\text{def}}{\Leftrightarrow}$ for all n

$$A_{<n} \overset{\text{def}}{=} \{x \in A \mid |x|_A < n\} \text{ is finite}$$

Definition (Normed WQO's)

$$|k|_{\mathbb{N}} \overset{\text{def}}{=} k \qquad |a_i|_{\Gamma_p} \overset{\text{def}}{=} 0 \qquad |\langle a,b\rangle|_{A\times B} \overset{\text{def}}{=} \max(|a|_A, |b|_B)$$

$$|\langle i,a\rangle|_{A_1+A_2} \overset{\text{def}}{=} |a|_{A_i} \quad |a_1\cdots a_m|_{A^*} \overset{\text{def}}{=} \max(m, |a_1|_A, \ldots, |a_m|_A)$$

Controlled Sequences

- bound the length of controlled bad sequences over $(A, \leqslant; |.|_A)$
- fix a control function $g : \mathbb{N} \to \mathbb{N}$ (monotone with $g(x+1) \geqslant g(x) + 1 \geqslant x + 2$)
- $\mathbf{x} = x_0, x_1, \ldots$ over A is (g, n)-controlled iff

$$\forall i,\ |x_i|_A < g^i(n)$$

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Example (SIMPLE$(2,3)$)

$A = \mathbb{N}^2, n = 4, g(x) = 2x$

Controlled Sequences

- bound the length of controlled bad sequences over $(A, \leqslant; |.|_A)$
- for fixed A, g, n, there are finitely many bad (g, n)-controlled sequences over A
- maximal length function

$$L_{A,g}(n)$$

Technical Overview

1. residuals: inductive definition for $L_{A,g}$
2. reflections: approximations to obtain inequalities for $L_{A,g}$ in terms of "simpler" wqo's
3. ordinal notations: associate ordinal terms to wqo's in order to work with subrecursive hierarchies

Residuals

Definition (Residual)

The residual of a wqo $(A, \leqslant)$ by $a \in A$ is the wqo

$$A/a \stackrel{\text{def}}{=} \{b \in A \mid a \not\leqslant b\}\,.$$

Proposition (Descent Equation)

$$L_{A,g}(n) = \max_{a \in A_{<n}} \left\{1 + L_{A/a,g}(g(n))\right\}$$

Example

$$\Gamma_{p+1}/a \equiv \Gamma_p \qquad \mathbb{N}/k = \{0,\ldots,k-1\}$$

$$L_{\Gamma_p,g}(n) = p \qquad L_{\mathbb{N},g}(n) = n\,.$$

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REFLECTIONS

Definition (Normed Reflection)

A normed reflection is a mapping $h : A \to B$ between two normed wqo's satisfying

$$\forall a, b \in A, h(a) \leqslant_B h(b) \text{ implies } a \leqslant_A b$$
$$\forall a \in A, |h(a)|_B \leqslant |a|_A$$

Notation

$A \hookrightarrow B$: there exists a normed reflection $h : A \to B$

Example

$$\{0, \ldots, k-1\} \hookrightarrow \Gamma_k \qquad \Gamma_p \hookrightarrow \Gamma_{p+1}$$

Reflections

Proposition (Monotony of Length Function)

$$A \hookrightarrow B \text{ implies } \forall n, L_{A,g}(n) \leqslant L_{B,g}(n)\ .$$

Proposition (Reflections for Residuals)

$$\begin{aligned}
(A+B)/\langle 1,a\rangle &= (A/a)+B\\
(A+B)/\langle 2,b\rangle &= A+(B/b)\\
(A\times B)/\langle a,b\rangle &\hookrightarrow [(A/a)\times B]+[A\times(B/b)]\\
\Gamma^*_{p+1}/a_1\cdots a_m &\hookrightarrow \Gamma_m\times(\Gamma^*_p)^m
\end{aligned}$$

$\Gamma_2^*/aba \hookrightarrow \Gamma_3 \times (\Gamma_1^*)^3$

"Jullien's Decomposition"

Example

$$aba, \overbrace{bbba, bbb, aabb, baa, abb, bb}^{\in \Gamma_2^*/aba}$$

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"Jullien's Decomposition"

Example

$$aba,\ \overbrace{\begin{bmatrix} & bbb, & & & bb \\ bbba, & & & baa, & \\ & & aabb, & & abb, \end{bmatrix}}^{\in \Gamma_3 \times \Gamma_2^*}$$

$\Gamma_2^*/aba \hookrightarrow \Gamma_3 \times (\Gamma_1^*)^3$

"Jullien's Decomposition"

Example

$$aba, \overbrace{\left[\langle bbb,\varepsilon,\varepsilon\rangle, \langle bbb,\varepsilon,\varepsilon\rangle, \langle \varepsilon,a,b\rangle, \langle b,a,\varepsilon\rangle, \langle \varepsilon,\varepsilon,b\rangle, \langle bb,\varepsilon,\varepsilon\rangle \right]}^{\in \Gamma_3 \times (\Gamma_1^*)^3}$$

Ordinal Terms

- maximal order type
  $o : WQO \to \mathrm{CNF}(\omega^{\omega^{\omega}})$ (de Jongh and Parikh, 1977; Hasegawa, 1994)
- well-founded relations ∂_n over $\mathrm{CNF}(\omega^{\omega^{\omega}})$ implement reflection of residuals of norm $< n$

Example

$$\begin{tikzcd}
\Gamma_2^* \arrow[rr, "\bigcup_{|x|<4}[\cdot/x \hookrightarrow \cdot]"] \arrow[d, "o"'] & & \Gamma_3 \times (\Gamma_1^*)^3 \arrow[d, "o"] \\
\omega^{\omega} \arrow[rr, "\partial_4"'] & & \omega^3 \cdot 3
\end{tikzcd}$$

Main Inequality

$$L_{o^{-1}(\alpha),g}(n) \leqslant \max_{\alpha' \in \partial_n \alpha} \left\{ 1 + L_{o^{-1}(\alpha'),g}(g(n)) \right\} .$$

A Bounding Function

$$M_{\alpha,g}(n) \stackrel{\text{def}}{=} \max_{\alpha' \in \partial_n \alpha} \{1 + M_{\alpha',g}(g(n))\}\,.$$

- Then for all α and n

$$L_{A,g}(n) \leqslant M_{o(A),g}(n)$$

- find the functional complexity of M

Subrecursive Hierarchies

Hierarchies of functions (and function classes) indexed by ordinal terms.

Fundamental Sequences

Subrecursive hierarchies are defined through an assignment of fundamental sequences $(\lambda_x)_{x<\omega}$ for limit ordinal terms λ, s.t. $\lambda_x < \lambda$ and $\lambda = \sup_x \lambda_x$: e.g.

$$(\gamma + \omega^{\beta+1})_x \stackrel{\text{def}}{=} \gamma + \omega^{\beta} \cdot (x+1)$$
$$(\gamma + \omega^{\lambda})_x \stackrel{\text{def}}{=} \gamma + \omega^{\lambda_x} ,$$

Example

$$\omega_x = x + 1 \qquad (\omega^{\omega^{p+1}})_x = \omega^{\omega^{p} \cdot (x+1)}$$

Fast Growing Hierarchy: $(F_\alpha)_\alpha$

(Löb and Wainer, 1970)

$$F_0(x) \stackrel{\text{def}}{=} x+1,\ \ F_{\alpha+1}(x) \stackrel{\text{def}}{=} F_\alpha^{x+1}(x),\ \ F_\lambda \stackrel{\text{def}}{=} F_{\lambda_x}(x)\,.$$

Example

$$F_1(x) = 2x+1$$

$$F_2(x) = (x+1)\cdot 2^{x+1} - 1$$

F_3 is non elementary

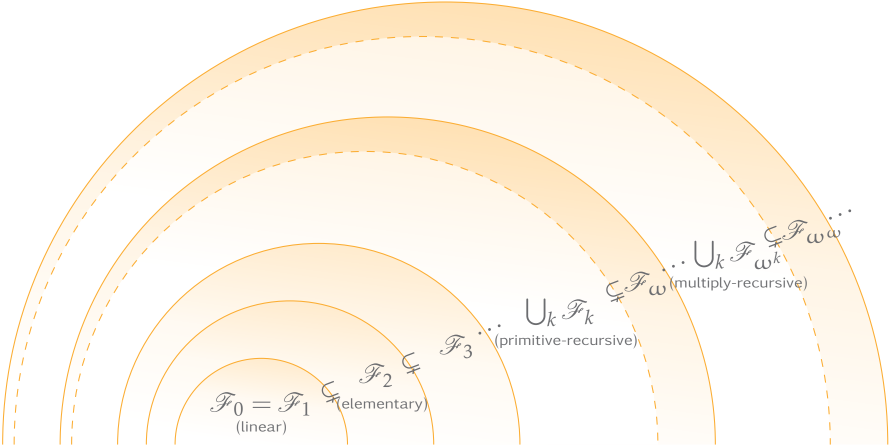
F_ω is non primitive-recursive

F_{ω^ω} is non multiply-recursive

Extended Grzegorczyk Hierarchy: $(\mathscr{F}_\alpha)_\alpha$

(Löb and Wainer, 1970)

Elementary-recursive closure of the $(F_\alpha)_\alpha$



Comparison with $(\mathscr{F}_\alpha)_\alpha$

Theorem (Higman's Lemma)

Let $p \geqslant 2$ and g primitive-recursive. Then $L_{\Gamma_p^*,g}$ is bounded by a function in $\mathscr{F}_{\omega^{p-1}}$.

Theorem (Dickson's Lemma)

Let $k,\gamma \geqslant 1$ and g in $\mathscr{F}_\gamma$. Then $L_{\mathbb{N}^k,g}$ is bounded by a function in $\mathscr{F}_{\gamma+k}$.

Concluding Remarks

- practical applications of wqo's yield upper bounds!
- out-of-the-box upper bounds
- "essentially" matching lower bounds

F_ω-Complete Problems

Decision of problems on

- well-structured counter systems (Finkel and Schnoebelen, 2001), e.g.
  - finite VASS containment (Mayr and Meyer, 1981; Jančar, 2001)
  - lossy systems (Schnoebelen, 2010),
- transition invariants in $\mathbb{N}$ (Podelski and Rybalchenko, 2004),
- relevance logics (Urquhart, 1999),
- data logics (Demri and Lazić, 2009; Figueira and Segoufin, 2009), ...

F_{ω^ω}-Complete Problems

Decision of problems on

- lossy channel systems (Chambart and Schnoebelen, 2008),
- Post embedding problem $\mathrm{PEP}^{\mathrm{reg}}$ (Chambart and Schnoebelen, 2007),
- 1-clock alternating timed automata (Lasota and Walukiewicz, 2008),
- Metric temporal logic (Ouaknine and Worrell, 2007),
- finite concurrent programs under weak (TSO/PSO) memory models (Atig et al., 2010)
- alternating register automata over ordered domains (Figueira et al., 2010),...

Future Work

- full algebra based on $+, \times, ^*,$
- more algebraic operations: set, multisets,...
- applications
  - lower bounds for timed Petri nets,
  - upper bounds for Petri net reachability

References

Upper Bounds for Dickson's Lemma

McAloon, K., 1984. Petri nets and large finite sets. *Theor. Comput. Sci.*, 32(1–2): 173–183. doi:10.1016/0304-3975(84)90029-X.

Clote, P., 1986. On the finite containment problem for Petri nets. *Theor. Comput. Sci.*, 43:99–105. doi:10.1016/0304-3975(86)90169-6.

Figueira, D., Figueira, S., Schmitz, S., and Schnoebelen, Ph., 2011. Ackermannian and primitive-recursive bounds with Dickson's Lemma. In *LICS 2011*. IEEE Press. arXiv:1007.2989[cs.LO].

Upper Bounds for Higman's Lemma

Cichoń, E.A. and Tahhan Bittar, E., 1998. Ordinal recursive bounds for Higman's Theorem. *Theor. Comput. Sci.*, 201(1–2):63–84. doi:10.1016/S0304-3975(97)00009-1.

Schmitz, S. and Schnoebelen, Ph., 2011. Multiply recursive upper bounds with Higman's Lemma. In *ICALP 2011* volume 6756 of *LNCS*, pages 441–452. Springer. arXiv:1103/4399[cs.LO].

Upper Bounds for Kruskal's Theorem

Weiermann, A., 1994. Complexity bounds for some finite forms of Kruskal's Theorem. *J. Symb. Comput.*, 18(5):463–488. doi:10.1006/jsco.1994.1059.

Fast Growing Hierarchy

Löb, M. and Wainer, S., 1970. Hierarchies of number theoretic functions, I. *Arch. Math. Logic*, 13:39–51. doi:10.1007/BF01967649.

WSTS

Abdulla, P.A., Čerāns, K., Jonsson, B., and Tsay, Y.K., 2000. Algorithmic analysis of programs with well quasi-ordered domains. *Inform. and Comput.*, 160(1–2): 109–127. doi:10.1006/inco.1999.2843.

Finkel, A. and Schnoebelen, Ph., 2001. Well-structured transition systems everywhere! *Theor. Comput. Sci.*, 256(1–2):63–92. doi:10.1016/S0304-3975(00)00102-X.

Lower Bounds

Schnoebelen, Ph., 2010. Revisiting Ackermann-hardness for lossy counter machines and reset Petri nets. In *MFCS 2010*, volume 6281 of *LNCS*, pages 616–628. Springer. doi:10.1007/978-3-642-15155-2_54.

Chambart, P. and Schnoebelen, Ph., 2008. The ordinal recursive complexity of lossy channel systems. In *LICS 2008*, pages 205–216. IEEE. doi:10.1109/LICS.2008.47.

Applications of Higman's Lemma

Chambart, P. and Schnoebelen, Ph., 2007. Post embedding problem is not primitive recursive, with applications to channel systems. In Arvind, V. and Prasad, S., editors, *FSTTCS 2007*, volume 4855 of *LNCS*, pages 265–276. Springer. doi:10.1007/978-3-540-77050-3_22.

Ouaknine, J.O. and Worrell, J.B., 2007. On the decidability and complexity of Metric Temporal Logic over finite words. *Logic. Meth. in Comput. Sci.*, 3(1):8. doi:10.2168/LMCS-3(1:8)2007.

Lasota, S. and Walukiewicz, I., 2008. Alternating timed automata. *ACM Trans. Comput. Logic*, 9(2):10. doi:10.1145/1342991.1342994.

Atig, M.F., Bouajjani, A., Burckhardt, S., and Musuvathi, M., 2010. On the verification problem for weak memory models. In *POPL 2010*, pages 7–18. ACM Press. doi:10.1145/1706299.1706303.

Figueira, D., Hofman, P., and Lasota, S., 2010. Relating timed and register automata. In Fröschle, S. and Valencia, F., editors, *EXPRESS 2010*, volume 41 of *EPTCS*, pages 61–75. doi:10.4204/EPTCS.41.5.

Applications of Dickson's Lemma

Demri, S., 2006. Linear-time temporal logics with Presburger constraints: An overview. *J. Appl. Non-Classical Log.*, 16(3–4):311–347. doi:10.3166/jancl.16.311-347.

Demri, S. and Lazić, R., 2009. LTL with the freeze quantifier and register automata. *ACM Trans. Comput. Logic*, 10(3). doi:10.1145/1507244.1507246.

Figueira, D. and Segoufin, L., 2009. Future-looking logics on data words and trees. In *MFCS 2009*, volume 5734 of *LNCS*, pages 331–343. Springer. doi:10.1007/978-3-642-03816-7_29.

Jančar, P., 2011. Nonprimitive recursive complexity and undecidability for Petri net equivalences. *Theor. Comput. Sci.*, 256(1–2):23–30. 10.1016/S0304-3975(00)00100-6.

Mayr, E.W. and Meyer, A.R., 1981. The complexity of the finite containment problem for Petri nets. *J. ACM*, 28(3):561–576. 10.1145/322261.322271.

Podelski, A. and Rybalchenko, A., 2004. Transition invariants. In *LICS 2004*, pages 32–41. IEEE. doi:10.1109/LICS.2004.1319598.

Revesz, P.Z., 1993. A closed-form evaluation for Datalog queries with integer (gap)-order constraints. *Theor. Comput. Sci.*, 116(1):117–149. doi:10.1016/0304-3975(93)90222-F.

Urquhart, A., 1999. The complexity of decision procedures in relevance logic II. *J. Symb. Log.*, 64(4):1774–1802. doi:10.2307/2586811.

Program Termination Proofs

(Podelski and Rybalchenko, 2004)

Monolithic Termination Argument

- prove that the program's transition relation R is well-founded
- ranking function ρ from program configurations $\mathbf{x} = x_0, x_1, \ldots$ into a wqo s.t. $R \subseteq \{(x_i, x_j) \mid \rho(x_i) \not\leq \rho(x_j)\}$
- for SIMPLE: $\rho(a, b, c) = \omega \cdot b + a$

Program Termination Proofs

(Podelski and Rybalchenko, 2004)

Disjunctive Termination Argument

- find well-founded relations $T_1, \ldots, T_k$ on program configurations
- prove $R^+ \subseteq T_1 \cup \cdots \cup T_k$
- for SIMPLE:

$$T_1 = \{(\langle a,b,c\rangle, \langle a',b',c'\rangle) \mid a > 0 \wedge a' < a\}$$
$$T_2 = \{(\langle a,b,c\rangle, \langle a',b',c'\rangle) \mid b > 0 \wedge b' < b\}$$

- at the heart of the Terminator tool

Termination by Dickson's Lemma

- each T_j shown well-founded thanks to a ranking function ρ_j into a wqo $(S_j, \leqslant_j)$
- map any sequence of program configurations

$$\mathbf{x} = x_0, x_1, \ldots$$

to a sequence of tuples

$$\mathbf{y} = \langle \rho_1(x_0), \ldots, \rho_k(x_0) \rangle, \langle \rho_1(x_1), \ldots, \rho_k(x_1) \rangle, \ldots$$

in $S_1 \times \cdots \times S_k$

- $\mathbf{y}$ is bad: if $i_1 < i_2$, there exists j s.t.

$$(x_{i_1}, x_{i_2}) \in R^+ \cap T_j \text{ but } \rho_j(x_{i_1}) \not\leq \rho_j(x_{i_2})$$

Termination by Dickson's Lemma

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- map any sequence of program configurations

$$\mathbf{x} = x_0, x_1, \ldots$$

  to a sequence of tuples

$$\mathbf{y} = \langle \rho_1(x_0), \ldots, \rho_k(x_0) \rangle, \langle \rho_1(x_1), \ldots, \rho_k(x_1) \rangle, \ldots$$

  in $S_1 \times \cdots \times S_k$
- $\mathbf{y}$ is bad: if $i_1 < i_2$, there exists j s.t.

$$(x_{i_1}, x_{i_2}) \in R^+ \cap T_j \text{ but } \rho_j(x_{i_1}) \not\leqslant \rho_j(x_{i_2})$$

Bounds on Program Complexity

Make some assumptions:

- complexity bound f on atomic program operations
  - for instance linear
- complexity bound ρ on ranking functions into $\mathbb{N}$
  - for instance linear
- $\mathbf{y}$ controlled by $f \circ \rho$ in some $\mathscr{F}_\gamma$
  - in this case a linear function in $\mathscr{F}_1$
- time complexity in $\mathscr{F}_{\gamma+k}$
  - in this case $\mathscr{F}_{k+1}$
- matches the lower bound (expand SIMPLE to dimension k instead of 2)

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$\mathbb{N}^2/\langle 2,2\rangle \hookrightarrow \mathbb{N}/2 \times \mathbb{N} + \mathbb{N}/2 \times \mathbb{N}$

Example

$$\mathbf{x} = \langle 2,2\rangle, \langle 1,5\rangle, \langle 4,0\rangle, \langle 1,1\rangle, \langle 0,100\rangle, \langle 0,99\rangle, \langle 3,0\rangle$$

$$\langle 2,2\rangle, \left[\begin{array}{llllll} \langle 1,5\rangle, & & \langle 1,1\rangle,\langle 0,100\rangle,\langle 0,99\rangle, & & (\{0,1\}\times\mathbb{N}) \\ & \langle 4,0\rangle, & & \langle 3,0\rangle & (\{0,1\}\times\mathbb{N}) \end{array}\right]$$

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Ordinal Terms for WQO's

Quick Reminder

Definition (Ordinal Terms)

$$\alpha ::= 0 \mid \omega^{\alpha} \mid \alpha + \alpha$$

Definition (Cantor Normal Form)

$$\alpha = \omega^{\beta_1} + \cdots + \omega^{\beta_m}$$

with $\alpha > \beta_1 \geqslant \cdots \geqslant \beta_m \geqslant 0$ with each β_i in CNF itself.

Ordinal Terms for WQO's

Operations on CNF

Definition (Natural Sum)

$$\sum_{i=1}^{m} \omega^{\beta_i} \oplus \sum_{j=1}^{n} \omega^{\beta'_j} \stackrel{\text{def}}{=} \sum_{k=1}^{m+n} \omega^{\gamma_k},$$

where $\gamma_1 \geqslant \cdots \geqslant \gamma_{m+n}$ is a rearrangement of $\beta_1, \ldots, \beta_m, \beta'_1, \ldots, \beta'_n$.

Definition (Natural Product)

$$\sum_{i=1}^{m} \omega^{\beta_i} \otimes \sum_{j=1}^{n} \omega^{\beta'_j} \stackrel{\text{def}}{=} \bigoplus_{i=1}^{m} \bigoplus_{j=1}^{n} \omega^{\beta_i \oplus \beta'_j}.$$

Ordinal Terms for WQO's

Maximal Order Type (de Jongh and Parikh, 1977; Hasegawa, 1994)

A bijection between normed WQO's with $+$, $\times$, and * over finite sets, and $\mathrm{CNF}(\omega^{\omega^\omega})$:

$$o(\Gamma_p) \stackrel{\text{def}}{=} p\,, \qquad o(\Gamma_0^*) \stackrel{\text{def}}{=} \omega^0\,, \qquad o(\Gamma_{p+1}^*) \stackrel{\text{def}}{=} \omega^{\omega^p}\,,$$

$$o(A+B) \stackrel{\text{def}}{=} o(A) \oplus o(B)\,, \quad o(A \times B) \stackrel{\text{def}}{=} o(A) \otimes o(B)\,.$$

Example

$$o((\Gamma_{p+2}^*)^{k+1} \times \Gamma_{q+1}) = \omega^{\omega^{p+1}\cdot(k+1)} \cdot (q+1)\,.$$

Ordinal Terms for WQO's

- translate reflections of residuals into derivative ordinal terms
- $\forall n$, we define a well-founded relation ∂_n over $\mathrm{CNF}(\omega^{\omega^\omega})$ s.t. if $a \in A_{<n}$, then $\exists \alpha' \in \partial_n o(A)$ s.t. $A/a \hookrightarrow o^{-1}(\alpha')$.

Example

$$\partial_n 0 = \emptyset, \qquad \partial_n 1 = \{0\}, \qquad \partial_n \omega = \{n-1\},$$

$$\partial_n(\omega^{\omega^{p+1}\cdot(k+1)} \cdot (q+1)) = \{\omega^{\omega^{p+1}\cdot(k+1)} \cdot q + \omega^{[\omega^{p+1}\cdot k + \omega^p \cdot (n-1)]} \cdot (k+1)(n-1)\}.$$

Fundamental Sequences

Subrecursive hierarchies are defined through an assignment of fundamental sequences $(\lambda_x)_{x<\omega}$ for limit ordinal terms λ, s.t. $\lambda_x < \lambda$ and $\lambda = \sup_x \lambda_x$: e.g.

$$(\gamma + \omega^{\beta+1})_x \stackrel{\text{def}}{=} \gamma + \omega^{\beta} \cdot (x+1)$$
$$(\gamma + \omega^{\lambda})_x \stackrel{\text{def}}{=} \gamma + \omega^{\lambda_x},$$

Example

$$\omega_x = x + 1 \qquad (\omega^{\omega^{p+1}})_x = \omega^{\omega^{p} \cdot (x+1)}$$

WELL-STRUCTURED TRANSITION SYSTEMS

- transition systems $(Q, \rightarrow, q_0)$ with a wqo $\leqslant$ on Q compatible with transitions:

$$\forall p, q, p' \in Q, (p \xrightarrow{a} q \wedge p \leqslant p') \Rightarrow \exists q', (q \leqslant q' \wedge p' \xrightarrow{a} q')$$

- a generic framework for decidability results: safety, termination, EF model checking, ...
- many classes of concrete systems are WSTS:
  - over $(\mathbb{N}^k, \leqslant_\times)$: vector addition systems, resets/transfer Petri nets, increasing counter

Example: (Non) Termination

- given $(Q, \to, q_0)$, decide whether there exists an infinite run $q_0 \to q_1 \to \cdots$
- holds iff there exists $q_i \leqslant q_j$ with $q_0 \to^* q_i \to^+ q_j$
- thanks to wqo, termination is both r.e. and co-r.e.
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Example: Lossy Channel Systems

- $\langle Q, M, C, \delta\rangle$
- Q finite set of q states, M size-m message alphabet, C set of c channels, $\delta \subseteq Q \times (\{\text{nop}\} \uplus (\{!,?\} \times M))^C \times Q$ set of transitions,
- configurations in $A = Q \times (M^*)^C$,
- WSTS for $(s, w_1, \ldots, w_{|C|}) \to (s', w'_1, \ldots, w'_{|C|})$ iff $\exists (s, e_1, \ldots, e_{|C|}, s') \in \delta$ s.t. $\forall i$

$$\begin{cases} w'_i \leqslant_* w_i & \text{if } e_i = \text{nop} \\ \exists w \in M^*, w \leqslant_* w_i \wedge w'_i \leqslant_* aw & \text{if } e_i = !a \\ \exists w \in M^*, wa \leqslant_* w_i \wedge w'_i \leqslant_* w & \text{if } e_i = ?a \end{cases}$$

Example: Lossy Channel Systems

- $o(A) = \omega^{\omega^{m-1} \cdot c} \cdot q$
- linear control by $g(x) = x + 1$ in $\mathscr{F}_1$
- non-terminating run from $s_{\text{init}} = (s_0, w_1, \ldots, w_{|C|})$ iff there exists a run of length $L_{A,g}(|s_{\text{init}}|)$
- non termination in $\mathscr{F}_{\omega^{m-1} \cdot c}$

Hardy Hierarchy: $(h^{\alpha})_{\alpha}$

Fix $h : \mathbb{N} \to \mathbb{N}$:

$$h^{0}(x) \stackrel{\text{def}}{=} x,\ \ h^{\alpha+1}(x) \stackrel{\text{def}}{=} h^{\alpha}(h(x)),\ \ h^{\lambda}(x) \stackrel{\text{def}}{=} h^{\lambda_x}(x)\,.$$

Example

For $h(x) = x+1$:

$$H^{\omega}(x) = H^{x+1}(x) = 2x+1 \quad H^{\omega\cdot 2}(x) = H^{\omega+x+1}(x) = 4x+3$$

Lemma

For all $r < \omega$, α, and x,

$$h^{\omega^{\alpha}\cdot r}(x) = f_{\alpha}^{r}(x)\,.$$

Length Hierarchy: $(h_\alpha)_\alpha$

Fix $h : \mathbb{N} \to \mathbb{N}$:

$$h_0(x) \stackrel{\text{def}}{=} 0,\; h_{\alpha+1}(x) \stackrel{\text{def}}{=} 1 + h_\alpha(h(x)),\; h_\lambda(x) \stackrel{\text{def}}{=} h_{\lambda_x}(x)\,.$$

Lemma

For all α, x

$$h_\alpha(x) \leqslant h^\alpha(x) - x$$

Lemma

Define the predecessor at x of $\alpha > 0$ as

$$P_x(\alpha+1) \stackrel{\text{def}}{=} \alpha, \qquad P_x(\lambda) \stackrel{\text{def}}{=} P_x(\lambda_x)$$

Then $$h_\alpha(x) = 1 + h_{P_x(\alpha)}(h(x))\,.$$

Monotonicity Matters

Lemma

For h monotone with $h(x) \geqslant x$ and any α,

$$x < y \text{ implies } h^\alpha(x) \leqslant h^\alpha(y) .$$

But: for $x < n$,
$H^\omega(x) = 2x + 1 < x + n + 1 = H^{n+1}(x)$, i.e.

$$\alpha < \beta \text{ does not imply } h^\alpha(x) \leqslant h^\beta(x)$$

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Monotonicity Matters

Definition (Pointwise Ordering)

For all x, $\prec_x$ is the smallest transitive relation s.t.

$$\alpha \prec_x \alpha + 1 \qquad\qquad \lambda_x \prec_x \lambda\,.$$

Lemma

For h monotone with $h(x) \geqslant x$ and any x,

$$\alpha \prec_x \beta \textit{ implies } h^{\alpha}(x) \leqslant h^{\beta}(x)\,.$$

Comparison with $(h_\alpha)_\alpha$

Contrast $M_{\alpha,g}(n) \stackrel{\text{def}}{=} \max_{\alpha' \in \partial_n \alpha}\{1 + M_{\alpha',g}(g(n))\}$ with $h_\alpha(x) = 1 + h_{P_x(\alpha)}(h(x))$:

Proposition

For all α in $\mathrm{CNF}(\omega^{\omega^\omega})$, there is a constant k s.t. for all $n > 0$, $M_{\alpha,g}(n) \leqslant h_\alpha(kn)$ where $h(x) \stackrel{\text{def}}{=} x \cdot g(x)$.

Example (Higman's Lemma)

For bad (g,n)-controlled sequences in Γ_p^*:

$$L_{\Gamma_p^*,g}(n) \leqslant h_{\omega^{\omega^{p-1}}}((p-1)n) \quad \text{where } h(x) \stackrel{\text{def}}{=} x \cdot g(x)\,.$$

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Lower Bound

Specific sequence, bad for $(\mathbb{N}^k, \leqslant_{\mathrm{lex}})$, of length $\ell_{k,f}(t)$.

Example

$k = 2, t = 1, f(x) = x + 3$:

i	0	1	2	3	4	5	$\cdots$	10	11	12	13	$\cdots$	26	27	28	29	$\cdots$	58	59
$x_i[1]$	3	3	3	3	2	2	$\cdots$	2	2	1	1	$\cdots$	1	1	0	0	$\cdots$	0	0
$x_i[2]$	3	2	1	0	7	6	$\cdots$	1	0	15	14	$\cdots$	1	0	31	30	$\cdots$	1	0
$f(i+t)$	4	5	6	7	8	9	$\cdots$	14	15	16	17	$\cdots$	30	31	32	33	$\cdots$	62	63

$$\begin{aligned} 5 &= \quad 1+4 &&= 1+\ell_{1,f}(1) \\ 13 &= \quad 5+8 &&= 5+\ell_{1,f}(5) \\ 29 &= 16+13 &&= 13+\ell_{1,f}(13) \end{aligned}$$

Lower Bound

Specific sequence, bad for $(\mathbb{N}^k, \leqslant_{\mathrm{lex}})$, of length $\ell_{k,f}(t)$. In general, on the $k+1$th coordinate:

$$\underbrace{f(t)-1\, f(t)-1 \cdots f(t)-1}_{\ell_{k,f}(t)\text{ times}} \quad \underbrace{f(t)-2, f(t)-2, \cdots, f(t)-2}_{\ell_{k,f}(o_{k,f}(t))\text{ times}}$$

$$\cdots \quad \underbrace{0, 0, \cdots, 0}_{\ell_{k,f}\left(o_{k,f}^{f(t)-1}(t)\right)\text{ times}}$$

$$o_{k,f}(t) \stackrel{\text{def}}{=} t + \ell_{k,f}(t)$$

$$\ell_{k+1,f}(t) = \sum_{j=1}^{f(t)} \ell_{k,f}\left(o_{k,f}^{j-1}(t)\right)$$

Lower Bound

Specific sequence, bad for $(\mathbb{N}^k, \leqslant_{\text{lex}})$, of length $\ell_{k,f}(t)$. One can have $\ell_{k,f}(t) < L(\omega^k, t)$: let $f(x) = 2x$ and $t = 1$,

$$\langle 1,1\rangle, \langle 1,0\rangle, \langle 0,5\rangle, \langle 0,4\rangle, \langle 0,3\rangle, \langle 0,2\rangle, \langle 0,1\rangle, \langle 0,0\rangle$$

$$\langle 1,1\rangle, \langle 0,3\rangle, \langle 0,2\rangle, \langle 0,1\rangle, \langle 9,0\rangle, \langle 8,0\rangle, \langle 7,0\rangle, \langle 6,0\rangle, \langle 5,0\rangle, \ldots, \langle 0,0\rangle$$

$$\ell_{2,f}(1) = 8 \qquad\qquad L_{\omega^2,f}(1) \geqslant 14$$