



The Power of Priority Channel Systems

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LaBRI, March 13th, 2014



OUTLINE

priority channel systems a model of
computation

priority embedding a well quasi ordering

Contents

Channel Systems with Priorities

Priority Embedding

Computational Power



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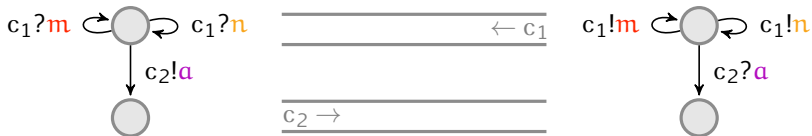
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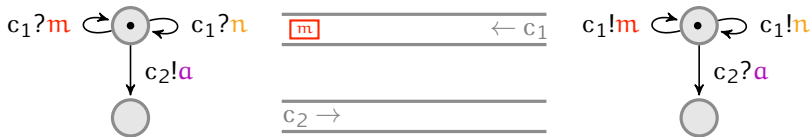
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- ▶ modeling network communications
- ▶ Turing-powerful: Σ_1^0 -complete



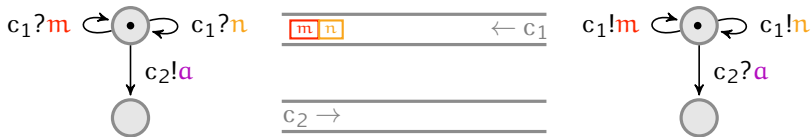
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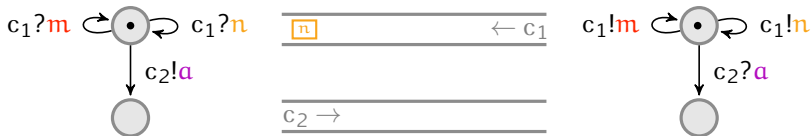
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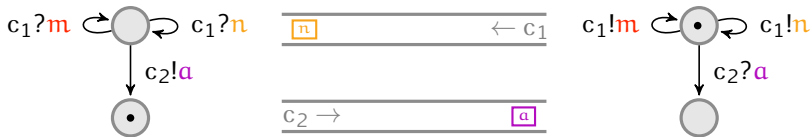
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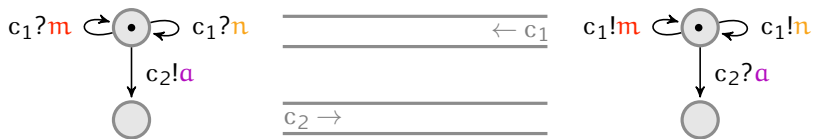


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LOSSY CHANNEL SYSTEMS (LCSs)

(ABDULLA AND JONSSON, 1996; CÉCÉ et al., 1996)



- ▶ apply **losing** rewriting rules to channel contents:

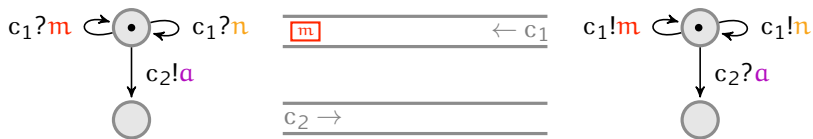
$$m \rightarrow_* \varepsilon \quad m \in M$$

- ▶ modeling imperfect communications, e.g. packet dropping policies against congestions
- ▶ decidable: $\mathbf{F}_{\omega}^{\omega}$ -complete (Chambart and Schnoebelen, 2008)



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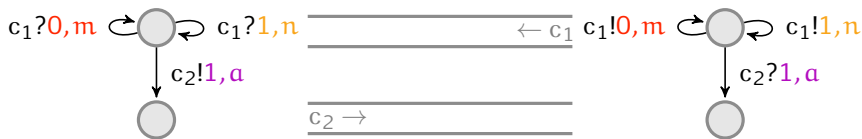


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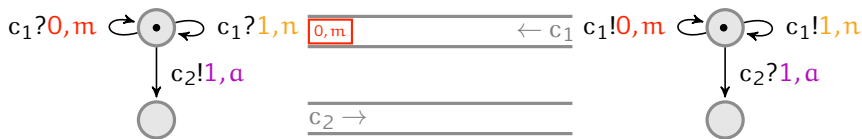
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$$(a, m)(b, n) \rightarrow_{\#} (b, n) \quad a \leq b \in \mathbb{N}, m, n \in M$$

- ▶ modeling communications with QoS, e.g. differentiated services (RFC2475)
- ▶ decidable: $\mathbf{F}_{\varepsilon_0}$ -complete



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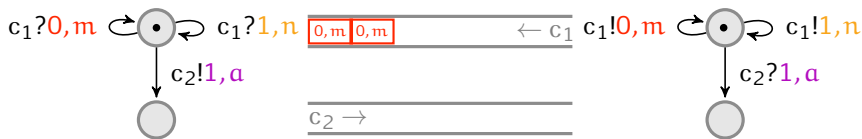
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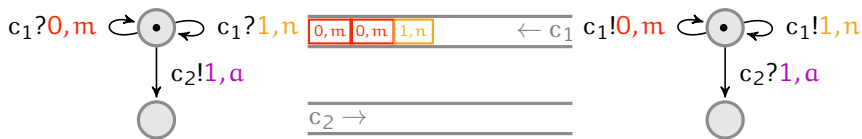


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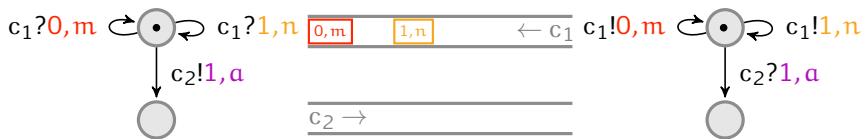
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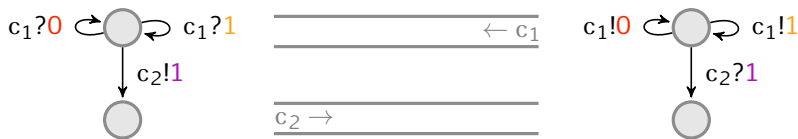
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REMARK: ALTERNATIVE MODELS

strict superseding Turing-powerful
ordered channels with rules

$$a b \rightarrow_s b \quad a < b \in \mathbb{N}$$

overtaking Turing-powerful
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priority queues decidable (VASS w. ordered 0-tests)
unordered channels, maximal priority
messages read first



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LOSING AS AN EMBEDDING

- ▶ losing rules define a quasi-ordering \sqsubseteq_*^* over M^*
- ▶ can be restated as **substring embedding**:

$$x \sqsubseteq_* y \stackrel{\text{def}}{\iff} x = m_1 \cdots m_\ell, y = z_1 m_1 z_2 \cdots z_\ell m_\ell z_{\ell+1}, \forall i. z_i \in M$$

- ▶ examples:
 - ▶ 201 \sqsubseteq_* 22011
 - ▶ 120 \sqsubseteq_* 10210
 - ▶ $\forall y \in M^*. \varepsilon \sqsubseteq_* y$



SUPERSEDING AS AN EMBEDDING

- ▶ if $d \in \mathbb{N}$, write $\Sigma_d \stackrel{\text{def}}{=} \{0, \dots, d\}$
- ▶ superseding rules define a quasi-ordering $\leftarrow^*_{\#}$ over Σ_d^*
- ▶ can be restated as **priority embedding**:

$$x \sqsubseteq_p y \stackrel{\text{def}}{\iff} x = a_1 \cdots a_l, y = z_1 a_1 z_2 \cdots z_l a_l, \forall i. z_i \in \Sigma_{a_i}^*$$

- ▶ examples:
 - ▶ $201 \sqsubseteq_p 22011$
 - ▶ $120 \not\sqsubseteq_p 10210$
 - ▶ $\varepsilon \sqsubseteq_p y$ iff $y = \varepsilon$

PRIORITY EMBEDDING IS WELL

C.F. RELATED ORDERINGS OF SCHÜTTE AND SIMPSON (1985)

Definition (wqo)

A quasi-order (A, \leq_A) is **well** $\stackrel{\text{def}}{\iff}$ in any infinite sequence x_0, x_1, \dots over A , there exist $i < j$ s.t. $x_i \leq_A x_j$.

Theorem

$(\Sigma_d^*, \sqsubseteq_p)$ is a wqo.

- ▶ proof by induction over d
- ▶ nested applications of Higman's Lemma

PCSs ARE WELL-STRUCTURED

(ABDULLA et al., 2000; FINKEL AND SCHNOEBELEN, 2001)

For a PCS with state set Q and m channels:
transition system $(Q \times (\Sigma_d^*)^m, \rightarrow)$ with
superseding steps or perfect steps

wqo $(Q \times (\Sigma_d^*)^m, \sqsubseteq_p)$ by Dickson's Lemma

monotonicity $\forall (p, \bar{x}), (q, \bar{x}'), (p, \bar{y}) \in Q \times (\Sigma_d^*)^m$,
if $(p, \bar{x}) \rightarrow (q, \bar{x}')$ and $\bar{x} \sqsubseteq_p \bar{y}$,
then $\exists \bar{y}' \in (\Sigma_d^*)^m$, $\bar{x}' \sqsubseteq_p \bar{y}'$ and
 $(p, \bar{y}) \rightarrow (q, \bar{y}')$.

Generic Algorithms

for Reachability, Inevitability, Simulation w. a

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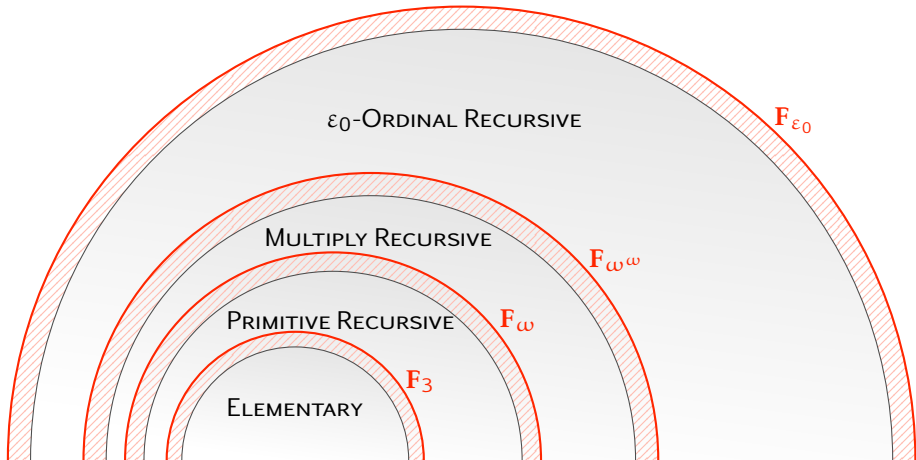
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FAST-GROWING COMPLEXITY CLASSES

(SCHMITZ, 2013)

Ordinal-indexed complexity hierarchy inside R:





COMPLEXITY OF PCS PROBLEMS

Theorem

Reachability and Termination in PCSs are F_{ε_0} -complete.

upper bound using **length function theorems**
for applications of Higman's Lemma
(Schmitz and Schnoebelen, 2011)

lower bound reduction from acceptance of a
Turing machine working in $H^{\varepsilon_0}(n)$
space



LOWER BOUND: HARDY FUNCTIONS

fundamental sequences $(\lambda(\alpha))_\alpha$

for limit ordinals λ in $\varepsilon_0 + 1$: $\lambda(\alpha) < \lambda$

with $\lim_{\alpha \rightarrow \omega} \lambda(\alpha) = \lambda$

Example

$$\omega(\alpha) = \alpha + 1,$$

$$\omega^{\omega \cdot 2}(\alpha) = \omega^{\omega + \alpha + 1},$$

$$(\varepsilon_0)(\alpha) = \Omega_{\alpha+1} \stackrel{\text{def}}{=} \omega^{\omega^{\dots^{\omega}}} \}_{\alpha+1} \text{ stacked } \omega\text{'s}$$



LOWER BOUND: HARDY FUNCTIONS

Hardy functions $(H^\alpha)_{\alpha \leq \varepsilon_0}$

$$H^0(x) \stackrel{\text{def}}{=} x, \quad H^{\alpha+1}(x) \stackrel{\text{def}}{=} H^\alpha(x+1), \quad H^\lambda(x) \stackrel{\text{def}}{=} H^{\lambda(x)}(x).$$

Example

$$H^n(x) = x + n,$$

$$H^\omega(x) = 2x + 1,$$

$$H^{\omega^2}(x) = 2^{x+1}(x+1) - 1,$$

H^{ω^3} non elementary,

H^{ω^ω} Ackermannian,

H^{ε_0} not provably total in Peano arithmetic



LOWER BOUND: HARDY COMPUTATIONS

rewrite system over $(\varepsilon_0 + 1) \times \omega$:

$$\alpha + 1, x \xrightarrow{H} \alpha, x + 1$$

$$\lambda, x \xrightarrow{H} \lambda(x), x$$

computations $\alpha_0, x_0 \xrightarrow{H} \alpha_1, x_1 \xrightarrow{H} \cdots \xrightarrow{H} \alpha_n, x_n$

- ▶ preserve $H^{\alpha_i}(x_i)$
- ▶ in particular if $\alpha_n = 0$ then $x_n = H^{\alpha_0}(x_0)$



LOWER BOUND: ENCODING ORDINALS

$$\alpha \in \Omega_{d+1}$$

3

$$\omega^2 + 1$$

$$t(\alpha) \in T_{d+1}$$



$$s_d(\alpha) \in \Sigma_d^*$$

222

1122

Proposition (Robustness)

If $s_d(\alpha) \sqsubseteq_p s_d(\beta)$, then $\forall x, H^\alpha(x) \leq H^\beta(x)$.

LOWER BOUND: WEAK HARDY COMPUTATIONS

Implement Hardy steps $\alpha, n \rightarrow \beta, m$ as a PCS:

- ▶ work on string encodings: $s_d(\alpha), n \xrightarrow{H}_{\#} s_d(\beta'), m'$
- ▶ **weak**: $s_d(\beta') \sqsubseteq_p s_d(\beta)$ and $m' \leq m$, but the perfect behaviour is possible
- ▶ also for **inverse** steps $s_d(\beta), m \xrightarrow{H^{-1}}_{\#} s_d(\alpha'), n'$ with $s_d(\alpha') \sqsubseteq_p s_d(\alpha)$ and $n' \leq n$

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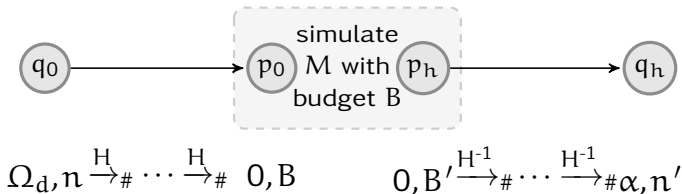
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LOWER BOUND: WRAPPING UP

reduction from a Turing machine M working in space $H^{\varepsilon_0}(n) = H^{\Omega_d}(n)$ for $d = n + 1$.

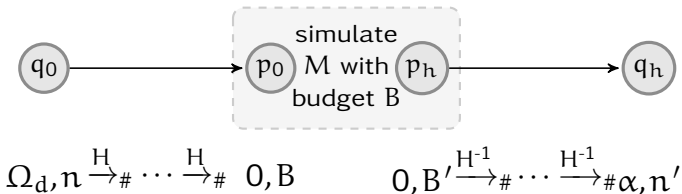


- ▶ robustness: $H^{\Omega_d}(n) \geq B \geq B' \geq H^\alpha(n')$
- ▶ coverability: $\alpha = \Omega_d \wedge n = n'$
- ▶ implies perfect simulation



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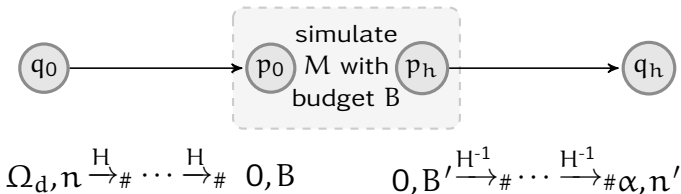


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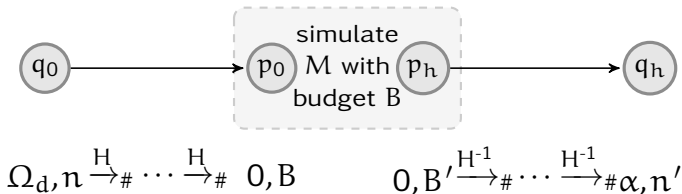


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CONCLUDING REMARKS

model priority channel systems

ordering priority embedding

Perspectives

verifying PCSs regular model checking and acceleration

using PCSs reducing problems about other models (e.g. manipulating bounded depth trees and graphs)



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FUNDAMENTAL SEQUENCES

Definition (Fundamental Sequences)

For limit ordinals in $\varepsilon_0 + 1$:

$$(\gamma + \omega^{\beta+1})(x) \stackrel{\text{def}}{=} \gamma + \omega^{\beta} \cdot (x + 1)$$

$$(\gamma + \omega^{\lambda})(x) \stackrel{\text{def}}{=} \gamma + \omega^{\lambda(x)}$$

$$(\varepsilon_0)(x) \stackrel{\text{def}}{=} \Omega_{x+1} \stackrel{\text{def}}{=} \omega^{\omega^{\dots^{\omega}}} \}_{x+1 \text{ stacked } \omega\text{'s}}$$



ENCODINGS

$s_d: T_{d+1} \rightarrow \Sigma_d^*$ by induction on d :

$$s_d(\bullet(t_1 \cdots t_n)) \stackrel{\text{def}}{=} \begin{cases} \varepsilon & \text{if } n = 0, \\ s_{d-1}(t_1)d \cdots s_{d-1}(t_n)d & \text{otherwise.} \end{cases}$$

$s_d: \Omega_{d+1} \rightarrow \Sigma_d^*$ by induction on d :

$$s_d\left(\sum_{i=1}^n \gamma_i\right) \stackrel{\text{def}}{=} s_d(\gamma_1) \cdots s_d(\gamma_n), \quad s_d(\omega^\alpha) \stackrel{\text{def}}{=} s_{d-1}(\alpha)d.$$