The Complexity of Coverability in ν-Petri Nets

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Séminaire méthodes formelles, LaBRI, March 8th, 2016
**Outline**

**ν-Petri nets (νPN)**
Petri nets with data management and creation
(Rosa-Velardo and de Frutos-Escrig, 2008, 2011)

- coverability
  - decidable by classical backward coverability algorithm (Abdulla et al., 2000)
  - dual view using downwards-closed sets (Lazić and S., 2015)

**Complexity**
νPN coverability is complete for double Ackermann ($F_{\omega^2}$-complete)
**OUTLINE**

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- decidable by classical **backward coverability** algorithm (Abdulla et al., 2000)

- dual view using **downwards-closed sets**
  (Lazić and S., 2015)

**Complexity**  
νPN coverability is complete for **double Ackermann** ($\mathcal{F}_{\omega,2}$-complete)
**ν-Petri Nets**

**Tokens carry data from an infinite countable domain $\mathbb{D}$**

![Petri net diagram]

**Configurations in $\mathbb{N}_P^\times$: multisets of markings**

$$
\begin{bmatrix}
1 \\ 0 \\ 0 \\
3 \\ 0 \\ 0 \\
1 \\ 1 \\ 0
\end{bmatrix}
$$
**ν-Petri Nets**

Tokens carry data from an infinite countable domain $\mathbb{D}$

![Diagram of ν-Petri Nets]

**Configurations in $(\mathbb{N}^P)^\otimes$: multisets of markings**

$$
\begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{bmatrix}
\xrightarrow{t}
\begin{bmatrix}
1 & 1 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
$$
**Petri Nets as \( \nu \)-Petri Nets**

- \( \alpha \) and \( \bar{\alpha} \) are complementary addressing places.
- \( c \) holds the actual token counts.
Petri Nets as $\nu$-Petri Nets

- $\alpha$ and $\bar{\alpha}$ are complementary addressing places
- $c$ holds the actual token counts
Reset Petri Nets as $\nu$-Petri Nets

- $\alpha$ and $\bar{\alpha}$ are complementary addressing places for active tokens
- $c$ holds both the active and inactive tokens
Reset Petri Nets as ν-Petri Nets

- α and \( \bar{\alpha} \) are complementary addressing places for active tokens
- \( c \) holds both the active and inactive tokens
**Coverability Problem**

verification of safety properties “nothing bad happens”

ordering of configurations by multiset embedding

\[ [u_1, \ldots, u_n] \sqsubseteq [v_1, \ldots, v_p] \]

iff \( \exists f : \{1, \ldots, n\} \rightarrow \{1, \ldots, p\} \) injective,

\[ \forall 1 \leq i \leq n, u_i \leq v_{f(i)} \]

Example:

\[
\begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 1
\end{bmatrix}
\sqsubseteq
\begin{bmatrix}
10 & 1 & 2 \\
1 & 0 & 3 \\
0 & 1 & 1
\end{bmatrix}
\]

input a \( \nu\)-PN, a source configuration \( src \), and a “bad” configuration \( tgt \)

question \( \exists m, tgt \sqsubseteq m \) and \( src \rightarrow^* m \)?
Polyadic $\nu$-Petri Nets

(Rosa-Velardo and Martos-Salgado, 2012)

- hold tuples of tokens in places
- equivalent to the full $\pi$-calculus
- model of dynamic database systems with existential positive guards
- undecidable coverability
**Taxonomy of Petri Net Extensions**

- v-Petri nets (Rosa-Velardo and de Frutos-Escrig, 2008) +order -> ordered data Petri nets (Lazić et al., 2008)
- v-Petri nets (Rosa-Velardo and de Frutos-Escrig, 2008) +fresh -> unordered data Petri nets (Lazić et al., 2008)
- unordered data Petri nets (Lazić et al., 2008) +whole-place -> ordered data nets (Lazić et al., 2008)
- unordered data Petri nets (Lazić et al., 2008) +whole-place -> unordered data nets (Lazić et al., 2008)
- unordered data nets (Lazić et al., 2008) +order -> unordered data nets (Lazić et al., 2008)
- unordered data nets (Lazić et al., 2008) +data -> affine nets (Finkel et al., 2004)
- Petri nets +data -> affine nets (Finkel et al., 2004)
- affine nets (Finkel et al., 2004) +whole-place -> Petri nets
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**Fast-Growing Complexity**

(S., 2016)

\[ F_{\omega \cdot 2} \overset{\text{def}}{=} \bigcup_{p \in \mathcal{F}_{\omega \cdot 2}} \text{DTime}(A_{\omega \cdot 2}(p(n))) \]

- **Elementary**
  
  \[ \bigcup_{k} F_k = \text{PRIMITIVE-RECURSIVE} \]

- **Primitive-Recursive**
  
  \[ \bigcup_{\alpha < \omega} \alpha \omega F_{\alpha} = \text{MULTIPLY-RECURSIVE} \]

- **Multiply-Recursive**
  
  \[ F_{\omega \cdot 2} \]

- **Ackermann**: “Ackermannian in” \( x \mapsto 2x \)

  \[ A_1(x) \overset{\text{def}}{=} 2x \quad A_{k+2}(x) \overset{\text{def}}{=} A_{k+1}^x(1) \quad A_{\omega}(x) \overset{\text{def}}{=} A_{x+1}(x) \]

- **Double Ackermann**: “Ackermannian in” \( A_{\omega}(x) \)

  \[ A_{\omega+k+1}(x) \overset{\text{def}}{=} A_{\omega+k}^x(1) \quad A_{\omega \cdot 2}(x) \overset{\text{def}}{=} A_{\omega+x+1}(x) \]
**Fast-Growing Complexity**

(S., 2016)

\[ F_{\omega \cdot 2} \overset{\text{def}}{=} \bigcup_{p \in \mathcal{F}_{\leq \omega \cdot 2}} \text{DTime}(A_{\omega \cdot 2}(p(n))) \]

\[ \bigcup_{\alpha < \omega} F_{\alpha} = \text{MULTIPLY-RECURSIVE} \]

\[ \bigcup_{k} F_{k} = \text{PRIMITIVE-RECURSIVE} \]

\[ F_3 = \text{TOWER} \]

\[ F_{\omega} F_{\omega + 1} \ldots F_{\omega \cdot 2} \ldots F_{\omega \omega} \]

- Ackermann: "Ackermannian in" \( x \mapsto 2x \)

\[ A_1(x) \overset{\text{def}}{=} 2x \quad A_{k+2}(x) \overset{\text{def}}{=} A_{k+1}^x(1) \quad A_\omega(x) \overset{\text{def}}{=} A_{x+1}(x) \]

- Double Ackermann: "Ackermannian in" \( A_\omega(x) \)

\[ A_{\omega + k + 1}(x) \overset{\text{def}}{=} A_{\omega + k}^x(1) \quad A_{\omega \cdot 2}(x) \overset{\text{def}}{=} A_{\omega + x + 1}(x) \]
**Fast-Growing Complexity**

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- Ackermann: “Ackermannian in” \( x \mapsto 2x \)

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- double Ackermann: “Ackermannian in” \( A_\omega(x) \)

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A_{\omega+k+1}(x) \overset{\text{def}}{=} A_{\omega+k}^x(1) \quad A_{\omega \cdot 2}(x) \overset{\text{def}}{=} A_{\omega+x+1}(x)
\]
Main Result

Theorem

Coverability in νPNs is $\mathbb{F}_{\omega^2}$-complete.

lower bound extends Lipton’s “object-oriented” programming in Petri nets

- basic block: Ackermann counters using Schnoebelen’s construction for reset Petri nets
- pushed to double Ackermann: composition and iteration operations

upper bound analyses a dual view of the backward coverability algorithm
**Taxonomy of Petri Net Extensions**

*complexity of coverability*

- ι-Petri nets (Rosa-Velardo and de Frutos-Escrig, 2008)
  - $F_\omega \leq ? \leq F_\omega \omega$ (Rosa-Velardo, 2014)
- unordered data Petri nets (Lazić et al., 2008)
  - $F_3 \leq ? \leq F_\omega \omega$ (Lazić et al., 2008, Rosa-Velardo, 2014)
- unordered data nets (Lazić et al., 2008)
  - $F_\omega \omega \omega$-complete (Haddad et al., 2012)
- ordered data Petri nets (Lazić et al., 2008)
  - $F_\omega \omega \omega$-complete (Haddad et al., 2012)
- unordered data nets (Lazić et al., 2008)
  - $F_\omega \omega \omega$-complete (Rosa-Velardo, 2014)
- affine nets (Finkel et al., 2004)
  - $F_{\omega}$-complete
    - (Schnoebelen, 2010; Figueira et al., 2011)
- Petri nets
  - EXPSPACE-complete (Lipton, 1976; Rackoff, 1978)
Taxonomy of Petri Net Extensions

**complexity of coverability**

- Petri nets (Rosa-Velardo and de Frutos-Escrig, 2008)
  - $F_\omega \cdot 2$-complete (this talk)

- Ordered data Petri nets (Lazić et al., 2008)
  - $F_{\omega \cdot \omega}$-complete (Haddad et al., 2012)

- Unordered data Petri nets (Lazić et al., 2008)
  - $F_3 \leq F_{\omega \cdot 2}$ (Lazić et al., 2008, this talk)

- Unordered data nets (Lazić et al., 2008)
  - $F_{\omega \cdot \omega}$-complete (Rosa-Velardo, 2014)

- Affine nets (Finkel et al., 2004)
  - $F_{\omega}$-complete
    - (Schnoebelen, 2010; Figueira et al., 2011)

- Petri nets
  - EXPSPACE-complete (Lipton, 1976; Rackoff, 1978)

- $\nu$-Petri Nets

- Fast-Growing Complexity

- Backward Coverability

- Upper Bound
**γ-Petri Nets are Well-Structured**

*(Finkel and Schnoebelen, 2001; Abdulla et al., 2000)*

1. \((\mathbb{N}^P)^\otimes, \sqsubseteq\) is a well-quasi-order (wqo), which entails

   - finite bad sequences: any sequence \(m_0, m_1, m_2, \ldots\) with \(\forall i < j, m_i \nsubseteq m_j\), is finite

   - finite basis property: any upwards-closed subset \(U\) has a finite basis \(B\) such that \(U = \uparrow B\)

   - ascending chain property: all the ascending chains \(U_0 \subset U_1 \subset U_2 \subset \cdots\) of upwards-closed subsets are finite

2. **compatibility:** if \(m_1 \sqsubseteq m_1'\) and \(m_1 \rightarrow m_2\), then there exists \(m_2', m_2 \sqsubseteq m_2'\) and \(m_1' \rightarrow m_2'\)
"CLASSICAL" BACKWARD COVERABILITY

(ABDULLA et al., 2000)

compute $U_k = \{ m' \mid \exists m \supseteq \text{tgt}, m' \rightarrow^\leq_k m \}$; $U_* = \bigcup_k U_k$:

initially $U_0 \overset{\text{def}}{=} \uparrow \text{tgt}$

step $U_{k+1} \overset{\text{def}}{=} \text{Pre}_\exists(U_k) \cup U_k$

where

$\text{Pre}_\exists(S) \overset{\text{def}}{=} \{ m \mid \exists s \in S, m \rightarrow s \}$

representation of upwards-closed subsets $\mathcal{U}$ through their minimal elements thanks to finite basis property

termination guaranteed by ascending chain property
“Classical” Backward Coverability

(ABDULLA et al., 2000)

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representation of upwards-closed subsets $U$ through their minimal elements thanks to finite basis property

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representation of upwards-closed subsets \( \mathcal{U} \) through their minimal elements thanks to finite basis property

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**Ideal Decompositions for a wqo** \((X, \leq)\)

(Bonnet, 1975; Finkel and Goubault-Larrecq, 2009; Goubault-Larrecq et al., 2016)

- a subset \(\Delta \subseteq X\) is directed iff \(\Delta \neq \emptyset\) and \(\forall x, y \in \Delta, \exists z \in \Delta, x \leq z\) and \(y \leq z\)

- an ideal \(I\) is a downwards-closed and directed subset

  - equivalently, \(I\) is downwards-closed and irreducible: if \(I \subseteq D_1 \cup D_2\) for \(D_1, D_2\) downwards-closed, then \(I \subseteq D_1\) or \(I \subseteq D_2\)

  - every downwards-closed subset \(D \subseteq X\) is the union of a unique finite family of incomparable ideals: \(D = I_1 \cup \cdots \cup I_n\), called its canonical ideal decomposition

- finite ideal representations for many wqos
**Ideal Decompositions for a wqo** \((X, \leq)\)

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- a subset \(\Delta \subseteq X\) is **directed** iff \(\Delta \neq \emptyset\) and
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- Every downwards-closed subset $D \subseteq X$ is the union of a unique finite family of incomparable ideals: $D = I_1 \cup \cdots \cup I_n$, called its **canonical ideal decomposition**.

- Finite ideal representations for many wqos.
**Ideal Decompositions for a wqo \((X, \preceq)\)**

(Bonnet, 1975; Finkel and Goubault-Larrecq, 2009; Goubault-Larrecq et al., 2016)

- A subset \(\Delta \subseteq X\) is **directed** iff \(\Delta \neq \emptyset\) and
  \[ \forall x, y \in \Delta, \exists z \in \Delta, x \preceq z \text{ and } y \preceq z \]

- An **ideal** \(I\) is a downwards-closed and directed subset

- Equivalently, \(I\) is downwards-closed and **irreducible**: if \(I \subseteq D_1 \cup D_2\) for \(D_1, D_2\) downwards-closed,
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- finite ideal representations for many wqos
  - extended markings: \(\text{Idl}(\mathbb{N}^P) = \{ \downarrow u \mid u \in \mathbb{N}_\omega^P \overset{\text{def}}{=} (\mathbb{N} \cup \{\omega\})^P\}\)

  - extended configurations: \(\text{Idl}((\mathbb{N}^P)^\otimes) = \{ \downarrow (B, S) \mid B \in (\mathbb{N}_\omega^P)^\otimes, S \subseteq_f \mathbb{N}_\omega^P\}\)
    - where \(m \subseteq (B, S)\) iff \(\exists m' \in S^\otimes, m \subseteq B \oplus m'\)
    - \((B, S)\) is **reduced** iff \(S\) is an antichain and \(\forall u \in \text{Support}(B), \forall v \in S, u \not\leq v\)
**Ideal Decompositions for a wqo** \((X, \leq)\)

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- **finite ideal representations** for many wqos
  - extended markings: \(\text{Idl}(\mathbb{N}_P^\omega) = \{\downarrow u \mid u \in \mathbb{N}_P^\omega \overset{\text{def}}{=} (\mathbb{N} \cup \{\omega\})^P\}\)
  - extended configurations: 
    \(\text{Idl}((\mathbb{N}_P^\omega)^\circledast) = \{\downarrow (B, S) \mid B \in (\mathbb{N}_P^\omega)^\circledast, S \subseteq_f \mathbb{N}_P^\omega\}\)
    - where \(m \sqsubseteq (B, S)\) iff \(\exists m' \in S^\circledast, m \sqsubseteq B \oplus m'\)
    - \((B, S)\) is **reduced** iff \(S\) is an antichain and 
      \(\forall u \in \text{Support}(B), \forall v \in S, u \not\leq v\)
**Dual Backward Coverability**

(Lazić and S., 2015)

compute $D_k = \{ m' \mid \forall m \supseteq \text{tgt}, m' \rightarrow^* m \}$; $D_* = \bigcap_k D_k$:

initially $D_0 \overset{\text{def}}{=} (\mathbb{N}P)^\oplus \setminus (\uparrow \text{tgt})$

step $D_{k+1} \overset{\text{def}}{=} \text{Pre}_\forall(D_k) \cap D_k$

where

$\text{Pre}_\forall(S) \overset{\text{def}}{=} \{ m \mid \forall s, m \rightarrow s \implies s \in S \}$

representation of downwards-closed subsets $D$ through finite representations of their ideal decompositions

termination guaranteed by descending chain property
**Dual Backward Coverability**

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**Dual Backward Coverability: Example**

\[ p_0 \xrightarrow{2} p_1 \]

\[ \text{tgt} = (0,5) \]

\[ D_0 = \downarrow(\omega,4) \]
**Dual Backward Coverability: Example**

\[ \text{tgt} = (0, 5) \]

\[ D_1 = \downarrow (1, 4) \cup \downarrow (\omega, 3) \]
**Dual Backward Coverability: Example**

\[ D_2 = \downarrow(1,4) \cup \downarrow(3,3) \cup \downarrow(\omega,2) \]

\[ \text{tgt} = (0,5) \]
**Dual Backward Coverability: Example**

\[\text{tgt} = (0, 5)\]

\[D_3 = \downarrow(1, 4) \cup \downarrow(3, 3) \cup \downarrow(5, 2) \cup \downarrow(\omega, 1)\]
**Dual Backward Coverability: Example**

\[ D_4 = \downarrow(1,4) \cup \downarrow(3,3) \cup \downarrow(5,2) \cup \downarrow(7,1) \cup \downarrow(\omega,0) \]
**Dual Backward Coverability: Example**

\[ \begin{align*}
p_0 & \to p_1 \quad 2 \\
\text{tgt} & = (0, 5)
\end{align*} \]

\[ D_5 = \downarrow(1, 4) \cup \downarrow(3, 3) \cup \downarrow(5, 2) \cup \downarrow(7, 1) \cup \downarrow(9, 0) = D_* \]
Consider a norm $\| \cdot \| : X \rightarrow \mathbb{N}$ with:

$$\forall n, X_{\leq n} \overset{\text{def}}{=} \{ x \in X \mid \| x \| \leq n \}$$

finite:

$$\| u \| \overset{\text{def}}{=} \max_{p \in P} u(p)$$

for $u \in \mathbb{N}_\omega^P$

$$\| B, S \| \overset{\text{def}}{=} \max_{u \in \text{Support}(B), v \in S} (\| B \|, \| u \|, \| v \|)$$

for $\downarrow (B, S) \in \text{Idl}((\mathbb{N}_\omega^P)\otimes)$

$$\| D \| \overset{\text{def}}{=} \max_{1 \leq i \leq n} \| B_i, S_i \|$$

for $D = \downarrow (B_1, S_1) \cup \cdots \cup \downarrow (B_n, S_n)$

Consider a control function $g : \mathbb{N} \rightarrow \mathbb{N}$ strictly monotone and an initial norm $n \in \mathbb{N}$.

A sequence $x_0, x_1, \ldots$ of elements of $X$ is $(g, n)$-controlled if $\forall i, \| x_i \| \leq g^i(n)$.

Strongly $(g, n)$-controlled if $\| x_0 \| \leq n$ and $\forall i, \| x_{i+1} \| \leq g(\| x_i \|)$.
**Controlled Sequences**

- consider a norm $\| \cdot \| : X \rightarrow \mathbb{N}$ with
  $\forall n, X_{\leq n} \overset{\text{def}}{=} \{ x \in X \mid \| x \| \leq n \}$ finite:

  $\| u \| \overset{\text{def}}{=} \max_{p \in P} u(p)$ \quad for $u \in \mathbb{N}_\omega^P$

  $\| B, S \| \overset{\text{def}}{=} \max_{u \in \text{Support}(B), v \in S} (|B|, \| u \|, \| v \|)$ \quad for $\downarrow(B, S) \in \text{Idl}((\mathbb{N}_\omega^P) \otimes)$

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**Controlled Sequences**

▶ consider a norm \( \| \cdot \| : X \to \mathbb{N} \) with
\[
\forall n, X_n \overset{\text{def}}{=} \{ x \in X \mid \| x \| \leq n \} \text{ finite:}
\]
\[
\| u \| \overset{\text{def}}{=} \max_{p \in P} u(p) \quad \text{for } u \in \mathbb{N}_P^\omega
\]
\[
\| B, S \| \overset{\text{def}}{=} \max_{u \in \text{Support}(B), v \in S} (|B|, \| u \|, \| v \|) \quad \text{for } \downarrow (B, S) \in \text{Idl}((\mathbb{N}_P^\omega)^\otimes)
\]
\[
\| D \| \overset{\text{def}}{=} \max_{1 \leq i \leq n} \| B_i, S_i \| \quad \text{for } D = \downarrow (B_1, S_1) \cup \cdots \cup \downarrow (B_n, S_n)
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▶ consider a control function \( g : \mathbb{N} \to \mathbb{N} \) strictly monotone and an initial norm \( n \in \mathbb{N} \)

▶ a sequence \( x_0, x_1, \ldots \) of elements of \( X \) is \((g, n)\)-controlled if \( \forall i, \| x_i \| \leq g^i(n) \)

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**Length Function Theorems (1/3)**

(Figueira et al., 2011; S. and Schnoebelen, 2012)

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**Fact (Length Function Theorem for Bad Sequences in \( \mathbb{N}^P_\omega \))**

Let \( n > 0 \). Any \((g, n)\)-controlled bad sequence \( e_0, e_1, \ldots, e_\ell \) of extended markings in \((\mathbb{N}^P_\omega, \leq)\) has length at most “Ackermannian in” \( g(\max(n, |P|)) \).
Length Function Theorems (2/3)

(Lazić and S., 2015)

- consider a descending chain $D_0 \supsetneq D_1 \supsetneq \cdots \supsetneq D_\ell$

- extract at each step $0 \leq k < \ell$ a proper ideal $I_k$ from the canonical decomposition of $D_k$, s.t. $I_k \not\subseteq D_{k+1}$

- bad sequence of proper ideals $I_0, I_1, \ldots, I_{\ell-1}$

- in particular, for descending chains $\downarrow S_0 \supsetneq \downarrow S_1 \supsetneq \cdots \supsetneq \downarrow S_\ell$

of antichains

Corollary (Length Function Theorem for Hoare-Descending Chains over $\mathbb{N}_\omega^P$)

Let $n > 0$. Any $(g, n)$-controlled descending chain $\downarrow S_0 \supsetneq \downarrow S_1 \supsetneq \cdots \supsetneq \downarrow S_\ell$ of antichains of $(\mathbb{N}_\omega^P, \subseteq)$ has length at most “Ackermannian in” $g(\max(n, |P|))$. 
Length Function Theorems (2/3)

(Lazić and S., 2015)

- consider a descending chain $D_0 \supseteq D_1 \supseteq \cdots \supseteq D_\ell$

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**Length Function Theorems (2/3)**

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- consider a descending chain $D_0 \supseteq D_1 \supseteq \cdots \supseteq D_\ell$

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**Corollary (Length Function Theorem for Hoare-Descending Chains over $\mathbb{N}_\omega^P$)**

Let $n > 0$. Any $(g, n)$-controlled descending chain $\downarrow S_0 \supseteq \downarrow S_1 \supseteq \cdots \supseteq \downarrow S_\ell$ of antichains of $(\mathbb{N}_\omega^P, \subseteq)$ has length at most "Ackermannian in" $g(\max(n, |P|))$. 
Length Function Theorems (3/3)

- a descending chain $D_0 \supseteq D_1 \supseteq \cdots \supseteq D_\ell$ over $(\mathbb{N}^P)^\otimes$ is **star-monotone** if $\forall 0 \leq k < \ell - 1$, $\forall I_{k+1} = \downarrow (B_{k+1}, S_{k+1})$ proper ideal from the canonical decomposition of $D_{k+1}$, $\exists I_k = \downarrow (B_k, S_k)$ proper ideal from the canonical decomposition of $D_k$ s.t. $\downarrow S_{k+1} \subseteq \downarrow S_k$

**Theorem (Length Function Theorem for Star-Monotone Descending Chains over $(\mathbb{N}_P^\omega)^\otimes$)**

Let $n > 0$. Any strongly $(g, n)$-controlled star-monotone descending chain $D_0 \supseteq D_1 \supseteq \cdots \supseteq D_\ell$ of configurations in $(\mathbb{N}_P^\omega)^\otimes$ has length at most “double Ackermannian in” $g(\max(n, |P|))$. 
WRAPPING UP

**Lemma (Strong Control for νPNs)**

The descending chain computed by the backward algorithm for a νPN $N$ and target tgt is strongly $(g, n)$-controlled for $g(x) \overset{\text{def}}{=} x + |N|$ and $n \overset{\text{def}}{=} ||tgt||$.

**Lemma (νPN Descending Chains are Star-Monotone)**

The descending chains computed by the backward coverability algorithm for νPNs are star-monotone.

**Theorem (Upper Bound)**

The coverability problem for νPNs is in $\mathcal{F}_{\omega^2}$. 
Concluding Remarks

- first “natural” decision problem complete for $\text{F}_{\omega^2}$
- ideals and downwards-closed sets as algorithmic tools
  - here, backward analysis (Lazić and S., 2015)
    - forward analysis (Finkel and Goubault-Larrecq, 2009, 2012)
    - reachability in Petri nets (Leroux and S., 2015)
    - formal languages (Zetzsche, 2015; Hague et al., 2016)
    - invariant inference (Padon et al., 2016)
    - piecewise testable separability (Goubault-Larrecq and S., 2016)
**Concluding Remarks**

- first “natural” decision problem complete for $F_{\omega^2}$
- ideals and downwards-closed sets as *algorithmic* tools
  - here, backward analysis (Lazić and S., 2015)
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