Algorithmic Complexity of Well-Quasi-Orders

Sylvain Schmitz

based on joint works with D. Figueira, S. Figueira, J. Leroux, and Ph. Schnoebelen

LSV, ENS Paris-Saclay & CNRS, Université Paris-Saclay

MoVeP 2018

Outline

well-quasi-orders (wqo):

- proving algorithm termination

a toolbox for wqo complexity

- upper bounds
- lower bounds
- complexity classes

this talk: focus on one problem

- reachability in vector addition systems

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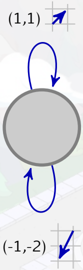
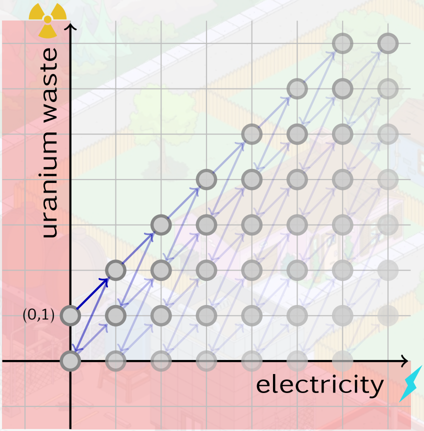
Vector Addition Systems

Vector Addition Systems

Springfield Power Plant

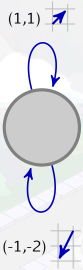


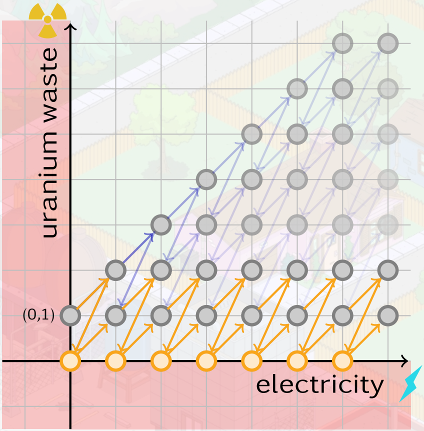


Can we produce unbounded electricity with no left-over uranium waste?

Vector Addition Systems

Springfield Power Plant





Can we produce unbounded electricity with no left-over uranium waste? Yes, $(\infty,0)$ is reachable

Importance of the Problem

Reachability Problem

input: *a vector addition system and two configurations* **source** *and* **target**

question: **source** $\rightarrow^*$ **target**?

Importance of the Problem

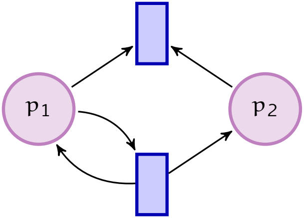
Discrete Resources

- modelling: items, money, energy, molecules, ...
- distributed computing: active threads in thread pool
- data: isomorphism types in data logics and data-centric systems

Importance of the Problem

MoVeP Example: Petri Nets

Petri net

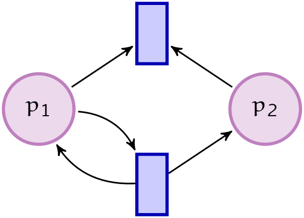


VAS

Importance of the Problem

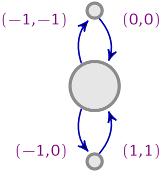
MoVeP Example: Petri Nets

Petri net



$\Longrightarrow$

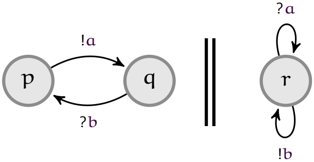
VAS



Importance of the Problem

MoVeP Example: Unordered CFSM

Communicating Finite-State Machine

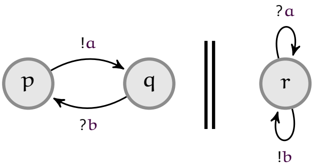


unordered channel:

Importance of the Problem

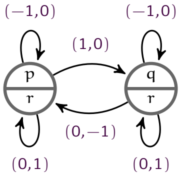
MoVeP Example: Unordered CFSM

Communicating Finite-State Machine



$\Rightarrow$

VAS



unordered channel:

aabab

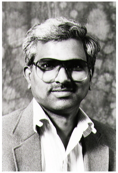
counters:

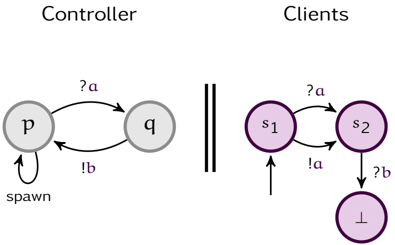
a $\boxed{3}$ $\boxed{2}$ b

Importance of the Problem

MoVeP Example: Asynchronous Rendez-vous

[German & Prasad Sistla'92]





Importance of the Problem

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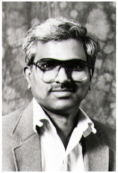
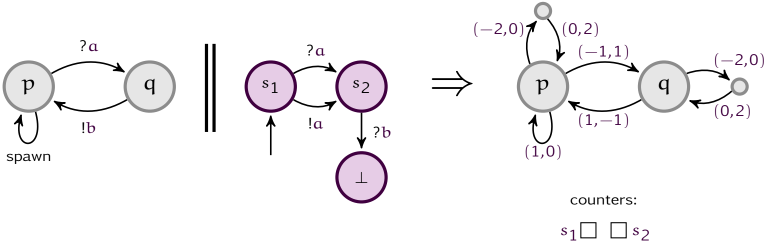


Controller | Clients | VAS



Importance of the Problem

Central Decision Problem [S.'16]

Large number of problems interreducible with reachability in vector addition systems

The Complexity of Reachability in Vector Addition Systems

SYLVAIN SCHMITZ,
LSV, ENS Cachan & CNRS & INRIA, Université Paris-Saclay

The program of the 30th Symposium on Logic in Computer Science held in 2015 in Kyoto included two contributions on the computational complexity of the *reachability problem* for vector addition systems: Blondin, Finkel, Göller, Haase, and McKenzie [2015] attacked the problem by providing the first tight complexity bounds in the case of dimension 2 systems with states, while Leroux and Schmitz [2015] proved the first complexity upper bound in the general case. The purpose of this column is to present the main ideas behind these two results, and more generally survey the current state of affairs.

1. INTRODUCTION

Vector addition systems with states (VASS), or equivalently Petri nets, find a wide range of applications in the modelling of concurrent, chemical, biological, or business processes. Maybe more importantly for this column, their algorithmics, and in particular the decidability of their *reachability problem* [Mayr 1981; Kosaraju 1982; Lambert 1992; Leroux 2011], is the cornerstone of many decidability results in logic, automata, verification, etc.—see Section 5 for a few examples.

In spite of its importance, fairly little is known about the computational complexity of the reachability problem. Regarding the general case, the inclusive surveys on the complexity of decision problems on VASS by Esparza and Nielsen [1994] and Esparza [1998] could only point to the EXPSPACE lower bound of Lipton [1976] and to the fact that the running time of the known algorithms is not primitive recursive: *no* complexity upper bound was known, besides decidability first proven in 1981 by Mayr. When turning to restricted versions of the problem, the 2-dimensional case was only known to be in 2-EXP [Howell, Rosier, Huynh, and Yen 1986] and NP-hard [Rosier and Yen 1986].

This state of affairs has very recently improved with two articles:

— Leroux and Schmitz [2015] have shown that reachability has a 'cubic Ackermann' upper bound, i.e. is in F_{ω^3}, by analysing the complexity of the classical algorithm developed and refined by Mayr [1981], Kosaraju [1982], and Lambert [1992]. Here, F_{ω^3} is a non primitive-recursive complexity class, but among the lower multiply-recursive ones. The main ingredients for this analysis are the fast-growing complexity bounds for termination proofs by well-quasi-orders and ordinal ranking functions from [Figueira et al. 2011; Schmitz 2014].

— Blondin, Finkel, Göller, Haase, and McKenzie [2015] have shown that reachability in 2-dimensional VASS is PSPACE-complete by a careful analysis of the complexity of the 'flattenings' of Leroux and Sutre [2004] for the upper bound, and by applying results on bounded one-counter automata by Fearnley and Jurdziński [2015] ... bound.

The main focus of the column is the complexity of the al... and Lambert [1992]. Section 3 presents it in

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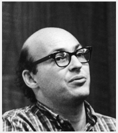
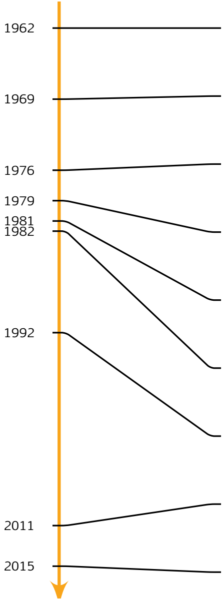
Importance of the Problem

Theorem (Minsky'67)

Reachability is undecidable in 2-dimensional Minsky machines (vector addition systems with zero tests).



Importance of the Problem



 C. A. Petri: Petri nets

  R. M. Karp & R. E. Miller: coverability trees

 R. J. Lipton: EXPSPACE lower bound

  J. E. Hopcroft & J.-J. Pansiot: dim. $\geqslant 3$ not definable in Presburger arithmetic

 E. W. Mayr: decidability by decomposition

 S. R. Kosaraju: decidability by decomposition

 J.-L. Lambert: decidability by decomposition

 J. Leroux: decidability by Presburger inductive invariants

this talk: Leroux & S.'15

Demystifying Reachability in Vector Addition Systems

[Leroux & S.'15]

Ideal Decomposition Theorem

The Decomposition Algorithm computes the ideal decomposition of the set of runs from source to target.

Upper Bound Theorem

Reachability in vector addition systems is in cubic Ackermann.

Demystifying Reachability in Vector Addition Systems

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Demystifying Reachability in Vector Addition Systems

[Leroux & S.'15; S.'17]

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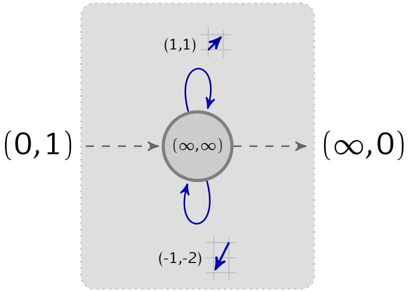
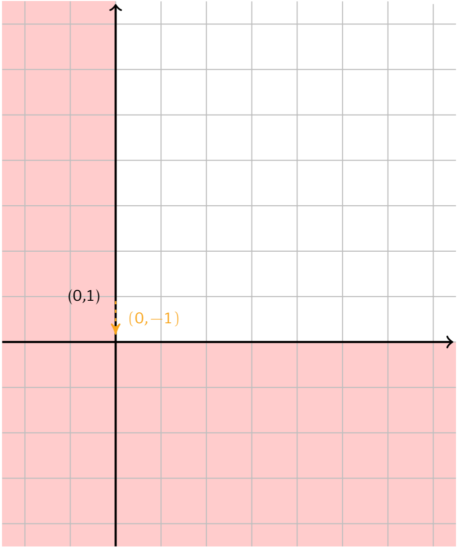
"Simple Runs" (Θ Condition)

[Mayr'81, Kosaraju'82, Lambert'92]



"Simple Runs" (Θ Condition)

[Mayr'81, Kosaraju'82, Lambert'92]



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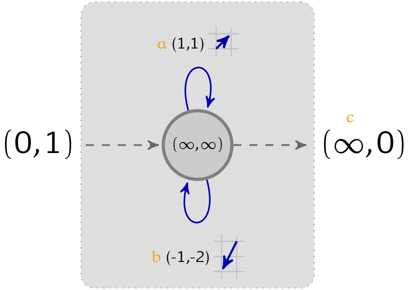
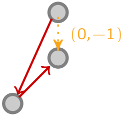


Characteristic System

$$0 + 1 \cdot a - 1 \cdot b = c$$
$$1 + 1 \cdot a - 2 \cdot b = 0$$

Solution Path

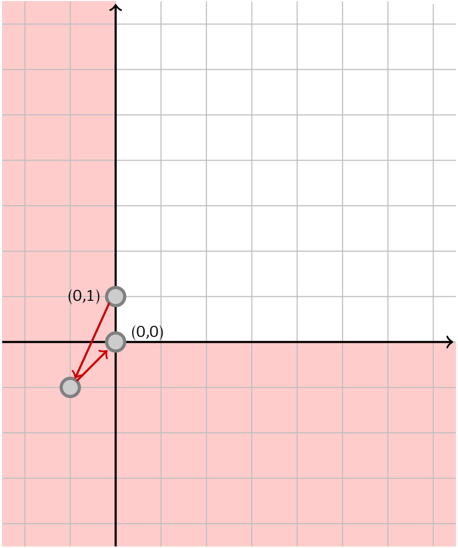


"Simple Runs" (Θ Condition)

[Mayr'81, Kosaraju'82, Lambert'92]

solution path

$\times 1$

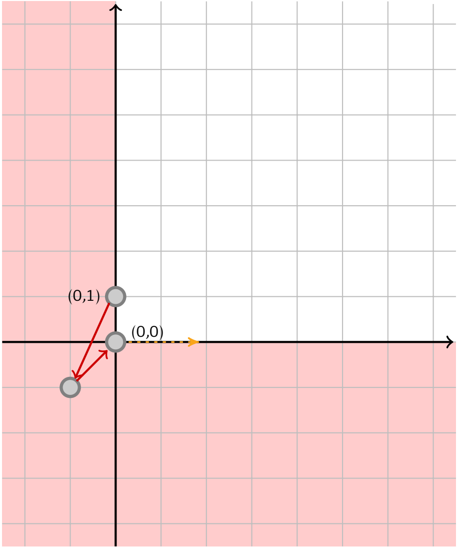


"Simple Runs" (Θ Condition)

[Mayr'81, Kosaraju'82, Lambert'92]

solution path

×1



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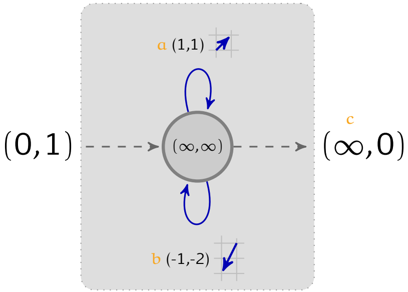
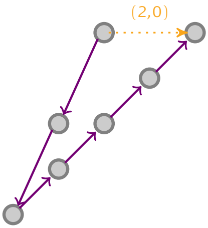


Homogeneous System

$$1 \cdot a - 1 \cdot b = c$$
$$1 \cdot a - 2 \cdot b = 0$$
$$a, b, c > 0$$

Unbounded Path



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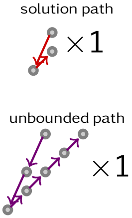
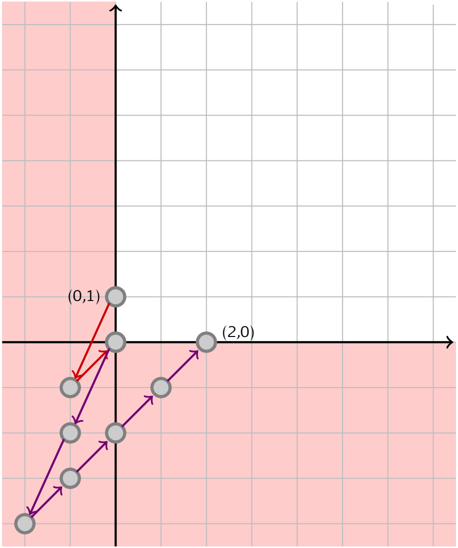
[Mayr'81, Kosaraju'82, Lambert'92]





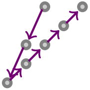
"Simple Runs" (Θ Condition)

[Mayr'81, Kosaraju'82, Lambert'92]

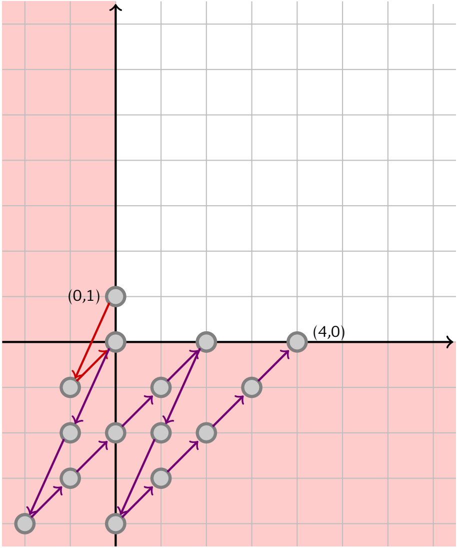
solution path

×1

unbounded path



×2



"Simple Runs" (Θ Condition)

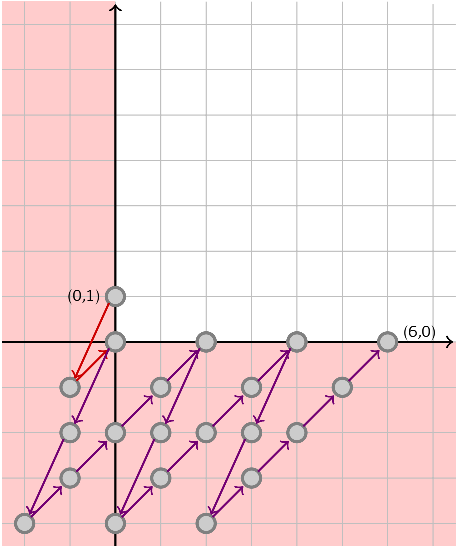
[Mayr'81, Kosaraju'82, Lambert'92]

solution path

$\times 1$

unbounded path

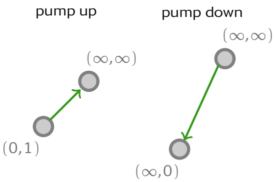
$\times 3$



"Simple Runs" (Θ Condition)

[Mayr'81, Kosaraju'82, Lambert'92]

Pumpable Paths



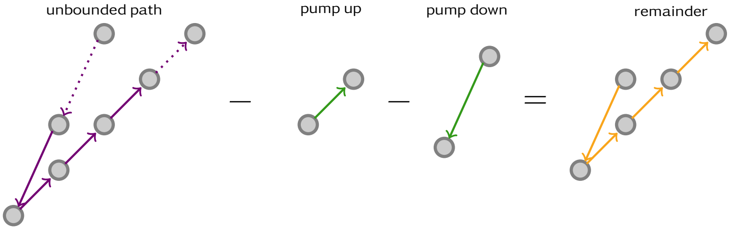
uses *coverability trees* [Karp & Miller'69]

which relies on *Dickson's Lemma* [Dickson, 1913]

"Simple Runs" (Θ Condition)

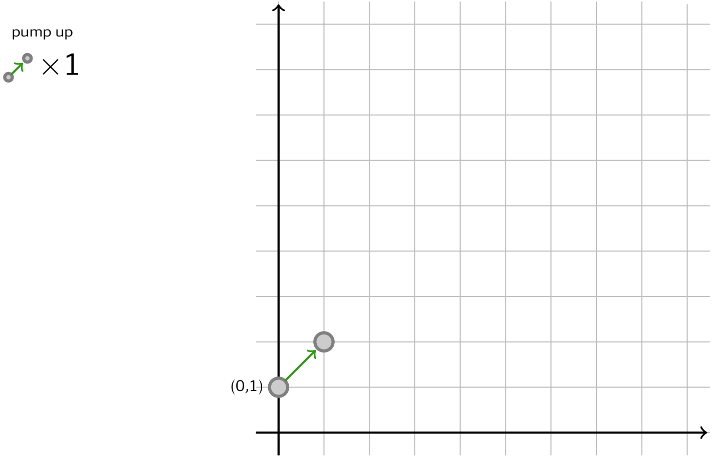
[Mayr'81, Kosaraju'82, Lambert'92]

Pumpable Paths



"Simple Runs" (Θ Condition)

[Mayr'81, Kosaraju'82, Lambert'92]

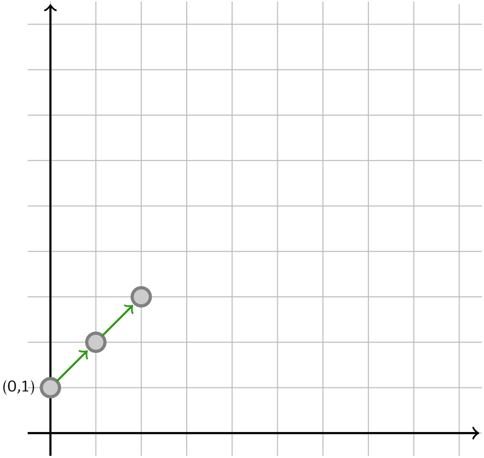


“Simple Runs” (Θ Condition)

[Mayr’81, Kosaraju’82, Lambert’92]

pump up

 $\times 2$



"Simple Runs" (Θ Condition)

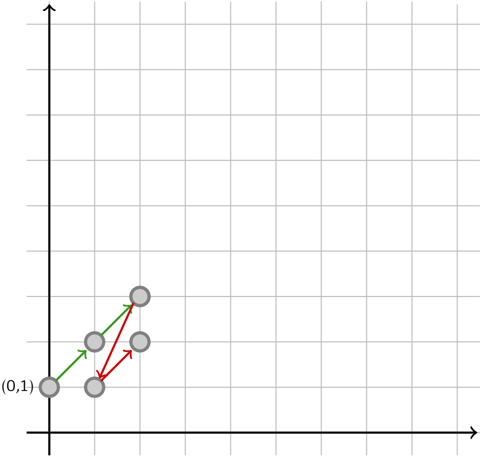
[Mayr'81, Kosaraju'82, Lambert'92]

pump up

$\times 2$

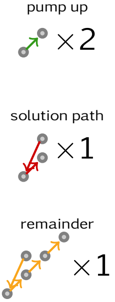
solution path

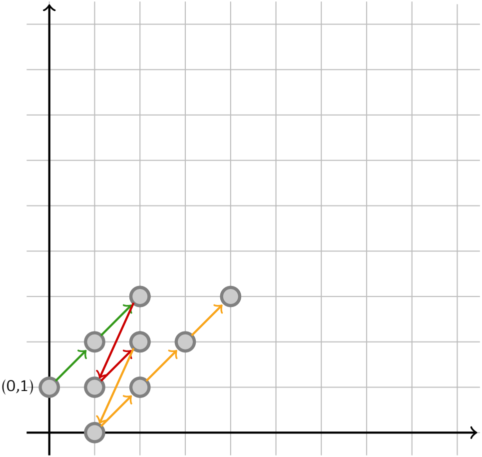
$\times 1$



"Simple Runs" (Θ Condition)

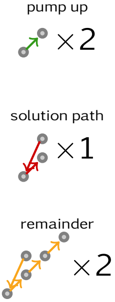
[Mayr'81, Kosaraju'82, Lambert'92]

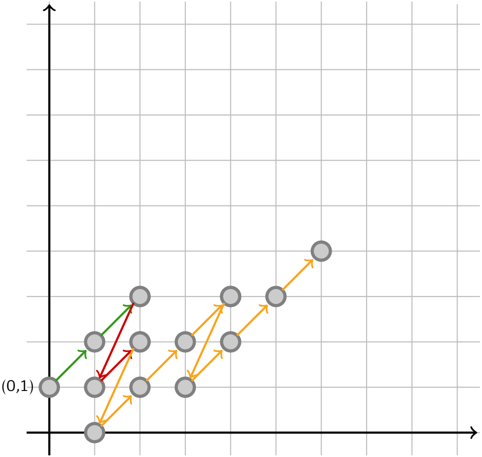




"Simple Runs" (Θ Condition)

[Mayr'81, Kosaraju'82, Lambert'92]





"Simple Runs" (Θ Condition)

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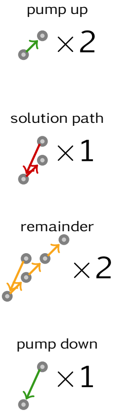
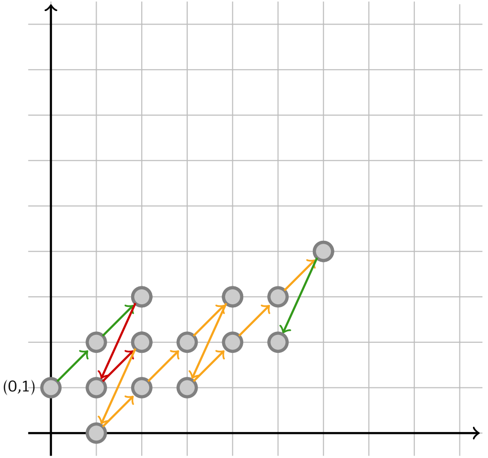
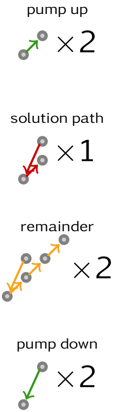
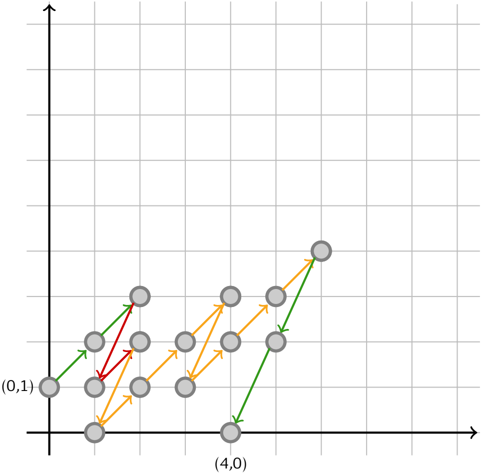




"Simple Runs" (Θ Condition)

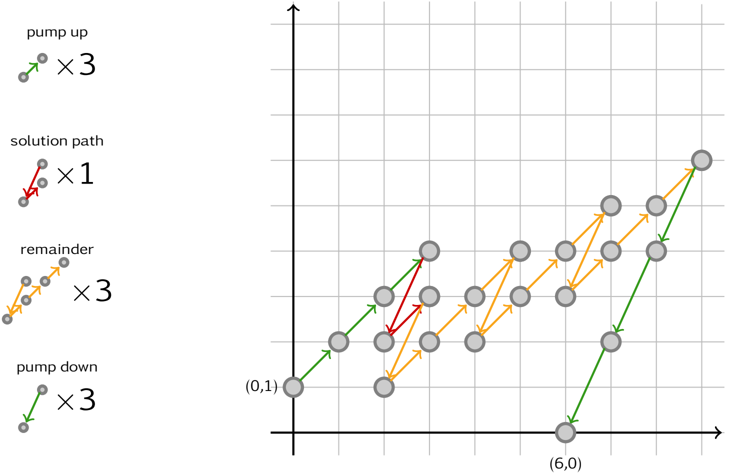
[Mayr'81, Kosaraju'82, Lambert'92]





“Simple Runs” (Θ Condition)

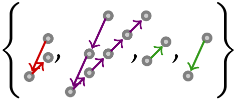
[Mayr’81, Kosaraju’82, Lambert’92]



Decomposition Algorithm

[Mayr'81, Kosaraju'82, Lambert'92]

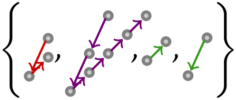
can we build a "simple run"?



Decomposition Algorithm

[Mayr'81, Kosaraju'82, Lambert'92]

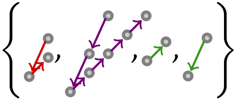
can we build a “simple run”? yes

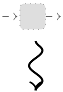


Decomposition Algorithm

[Mayr'81, Kosaraju'82, Lambert'92]

can we build a "simple run"? no





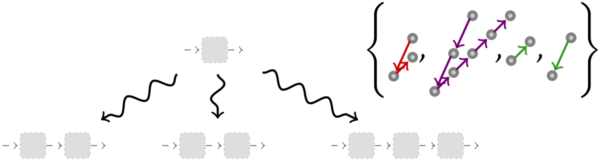
☹

decompose

Decomposition Algorithm

[Mayr'81, Kosaraju'82, Lambert'92]

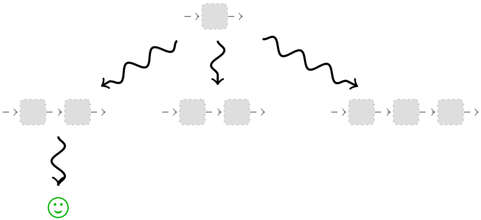
can we build a "simple run"? no



decompose

Decomposition Algorithm

[Mayr'81, Kosaraju'82, Lambert'92]



Decomposition Algorithm

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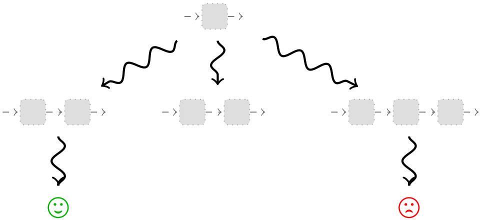
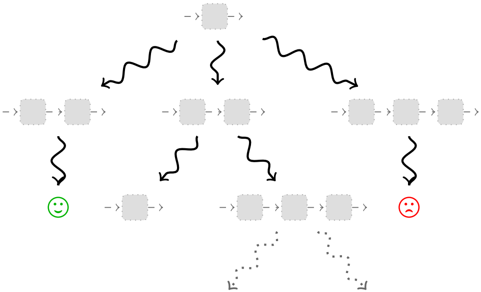


Decomposition Algorithm

[Mayr'81, Kosaraju'82, Lambert'92]



Termination

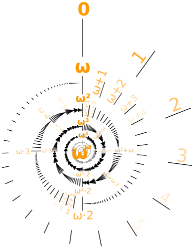
"Finally the checker has to verify that the process comes to an end. Here again he should be assisted by the programmer giving a further definite assertion to be verified. This may take the form of a quantity which is asserted to decrease continually and vanish when the machine stops."

[Turing'49]

Termination

"Finally the checker has to verify that the process comes to an end. Here again he should be assisted by the programmer giving a further definite assertion to be verified. This may take the form of a quantity which is asserted to decrease continually and vanish when the machine stops. To the pure mathematician it is natural to give an ordinal number."

[Turing'49]



Termination of the Decomposition Algorithm

[Mayr'81, Kosaraju'82, Lambert'92]

Ranking Function

$$\omega^{\omega^2}$$

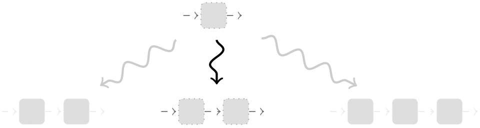
$$\vee$$

$$\alpha_0$$

Termination of the Decomposition Algorithm

[Mayr'81, Kosaraju'82, Lambert'92]

Ranking Function

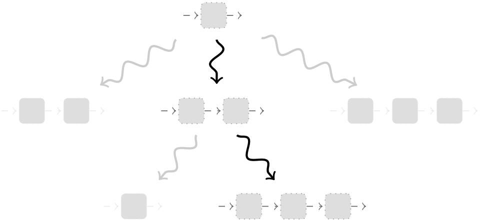


$$\omega^{\omega^2} > \alpha_0 > \alpha_1$$

Termination of the Decomposition Algorithm

[Mayr'81, Kosaraju'82, Lambert'92]

Ranking Function



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Termination of the Decomposition Algorithm

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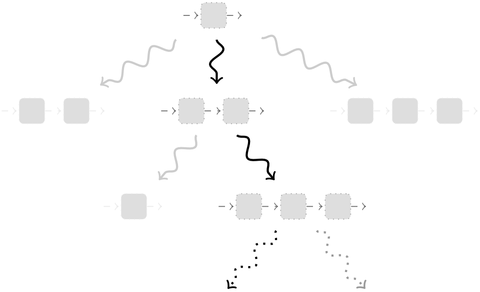


$$\omega^{\omega^2} > \alpha_0 > \alpha_1 > \alpha_2 > \vdots$$

Demystifying Reachability in Vector Addition Systems

[Leroux & S.'15; S.'17]

Ideal Decomposition Theorem

The Decomposition Algorithm computes the ideal decomposition of the set of runs from source to target.

Upper Bound Theorem

Reachability in vector addition systems is in quadratic Ackermann.

Upper Bounds

How to bound the running time of algorithms with ordinal-based termination proofs?

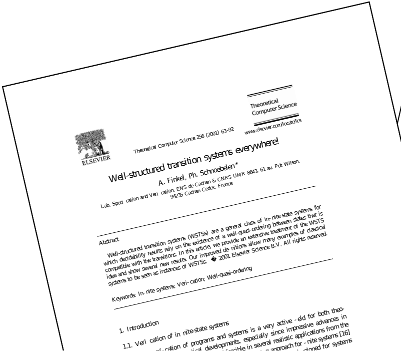
Upper Bounds

How to bound the running time of algorithms with wqo-based termination proofs?

Upper Bounds

How to bound the running time of algorithms with wqo-based termination proofs?

wqos ubiquitous in infinite-state verification


Theoretical Computer Science 256 (2001) 63–92

Theoretical Computer Science

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Well-structured transition systems everywhere!

A. Finkel, Ph. Schnoebelen*

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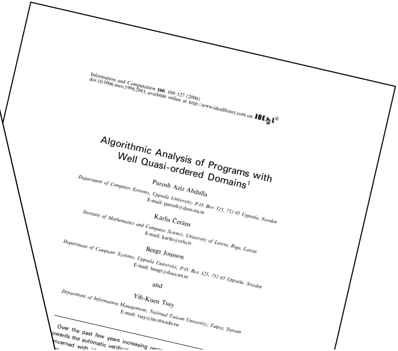
Abstract

Well-structured transition systems (WSTSs) are a general class of in nite-state systems for which decidability results rely on the existence of a well-quasi-ordering between states that is compatible with the transitions. In this article, we provide an extensive treatment of the WSTS idea and show several new results. Our improved de nitions allow many examples of classical systems to be seen as instances of WSTSs. © 2001 Elsevier Science B.V. All rights reserved.

Keywords: In nite systems; Veri cation; Well-quasi-ordering

1. Introduction

1.1. Veri cation of in nite-state systems


Information and Computation 160, 109–127 (2000)
doi:10.1006/inco.1999.2843, available online at http://www.idealibrary.com on IDEAL

Algorithmic Analysis of Programs with Well Quasi-ordered Domains[1]

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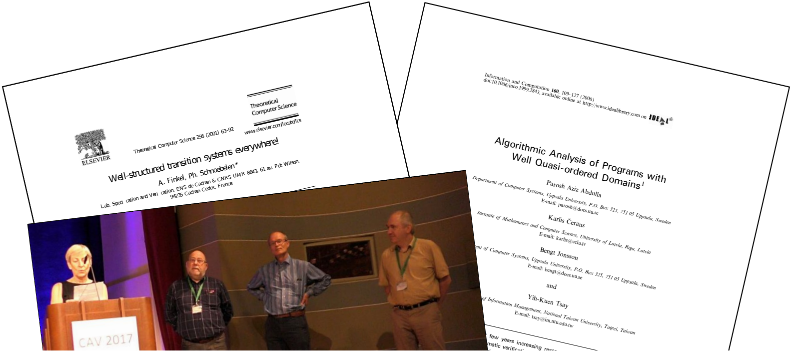
Upper Bounds

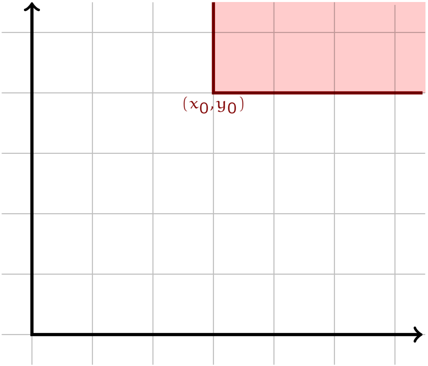
How to bound the running time of algorithms with wqo-based termination proofs?

wqos ubiquitous in infinite-state verification



A One-Player Game

- over $\mathbb{Q}_{\geqslant 0} \times \mathbb{Q}_{\geqslant 0}$
- given initially (x_0, y_0)
- Eloise plays (x_j, y_j) s.t. $\forall 0 \leqslant i < j, x_i > x_j$ or $y_i > y_j$



- Can Eloise win, i.e. play indefinitely?
- If not, how long can she last?

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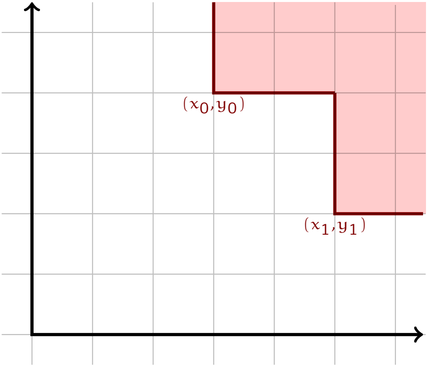

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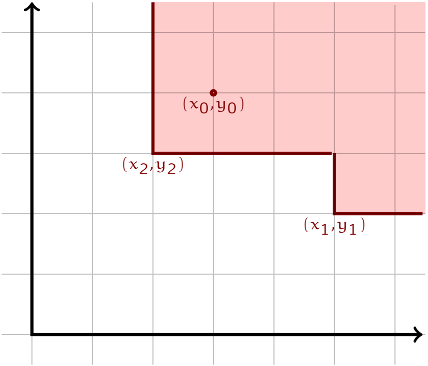

- Can Eloise win, i.e. play indefinitely?
- If not, how long can she last?

If $(x_0, y_0) \neq (0,0)$, then choosing $(x_j, y_j) = (\frac{x_0}{2^j}, \frac{y_0}{2^j})$ wins.

A One-Player Game

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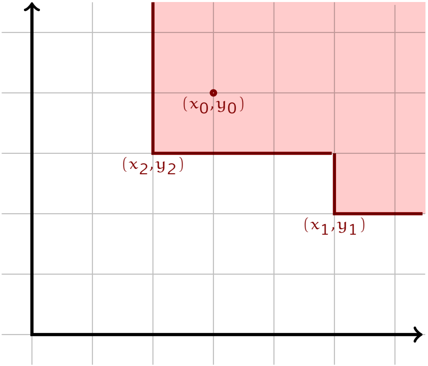
- given initially (x_0, y_0)
- Eloise plays (x_j, y_j) s.t. $\forall 0 \leqslant i < j, x_i > x_j$ or $y_i > y_j$



- Can Eloise win, i.e. play indefinitely?
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Assume there exists an infinite sequence $(x_j, y_j)_j$ of moves over $\mathbb{N}^2$.

Assume there exists an infinite sequence $(x_j, y_j)_j$ of moves over $\mathbb{N}^2$. Consider the pairs of indices $i < j$: color (i,j)

purple if $x_i > x_j$ but $y_i \leqslant y_j$,

red if $x_i > x_j$ and $y_i > y_j$,

orange if $y_i > y_j$ but $x_i \leqslant x_j$.

$(3,4) \qquad (5,2) \qquad (2,3) \qquad \cdots$

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By the infinite Ramsey Theorem, there exists an infinite monochromatic subset of indices.

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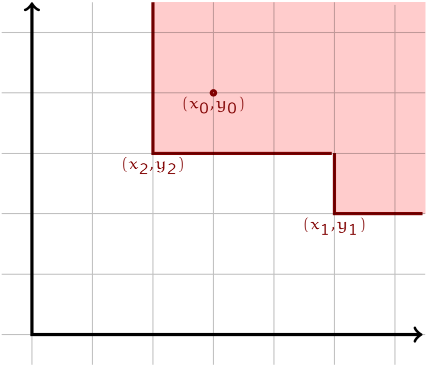
$(3,4) \qquad (5,2) \qquad (2,3) \qquad \cdots$

By the infinite Ramsey Theorem, there exists an infinite monochromatic subset of indices. In all cases, it implies the existence of an infinite decreasing sequence in $\mathbb{N}$, a contradiction.

A One-Player Game

- over $\mathbb{N} \times \mathbb{N}$
- given initially (x_0, y_0)
- Eloise plays (x_j, y_j) s.t. $\forall 0 \leqslant i < j, x_i > x_j$ or $y_i > y_j$



- Can Eloise win, i.e. play indefinitely?
- If not, how long can she last?

- if $(x_0, y_0) = (0,0)$, 0 turns
- otherwise, an arbitrary number of turns N: if $x_0 > 0$:

$$(x_0, y_0), (0, N-1), (0, N-2), \ldots, (0,1), (0,0)$$

Bad Sequences

Over a qo $(X, \leqslant)$

- $x_0, x_1, \ldots$ is bad if $\forall i < j \,.\, x_i \not\leqslant x_j$
- $(X, \leqslant)$ wqo iff all bad sequences are finite
-

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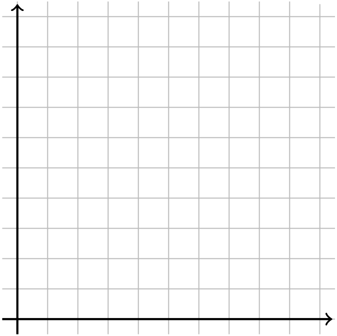

Example (over $\mathbb{N}^2$)

WQOs for Algorithm Termination

SIMPLE (a,b)
$c \longleftarrow 1$
while $a > 0 \wedge b > 0$
    $\langle a,b,c\rangle \longleftarrow \langle a-1,b,2c\rangle$
  or
    $\langle a,b,c\rangle \longleftarrow \langle 2c,b-1,1\rangle$
end

$$\langle a_0,b_0,c_0\rangle$$
$$\langle a_1,b_1,c_1\rangle$$
$$\vdots$$
$$\langle a_i,b_i,c_i\rangle$$
$$\vdots$$
$$\langle a_j,b_j,c_j\rangle$$

- in any execution, $\langle a_0,b_0\rangle,\ldots,\langle a_n,b_n\rangle$ is a bad sequence over $(\mathbb{N}^2,\leqslant_\times)$,
- $(\mathbb{N}^2,\leqslant_\times)$ is a wqo: all the runs are finite
- How long can SIMPLE run?

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A Rich Theory

- multiple equivalent definitions
- algebraic constructions

A Rich Theory

- multiple equivalent definitions: $(X, \leqslant)$ wqo iff
  - $\leqslant$ is well-founded and has no infinite antichains,
    - thus every ordinal is a wqo
  - every linearisation of $\leqslant$ is well-founded,
  - $\leqslant$ has the Ascending Chain Condition,
  - if $x_0, x_1, \cdots \in X^\omega$, then there exists an infinite sequence $i_0 < i_1 < \cdots$ with $x_{i_0} \leqslant x_{i_1} \leqslant \cdots$,
  - etc.
- algebraic constructions

A Rich Theory

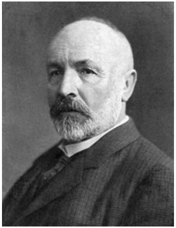
- multiple equivalent definitions
- algebraic constructions
  - Cartesian products (Dickson's Lemma),
  - finite sequences (Higman's Lemma),
  - disjoint sums,
  - finite sets with Hoare's quasi-ordering,
  - finite trees (Kruskal's Tree Theorem),
  - graphs with minors (Robertson and Seymour's Graph Minor Theorem), etc.

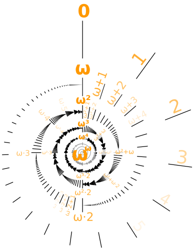
Example: Ordinals

ordinal: well-founded linear order

bad sequences are descending sequences:

$$\alpha \not\leq \beta \text{ iff } \alpha > \beta$$





Example: Dickson's Lemma

Lemma (Dickson 1913)

If $(X, \leqslant_X)$ and $(Y, \leqslant_Y)$ are two wqos, then $(X \times Y, \leqslant_\times)$ is a wqo, where $\leqslant_\times$ is the product ordering:

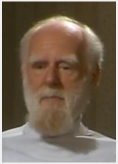
$$\langle x, y \rangle \leqslant_\times \langle x', y' \rangle \stackrel{\text{def}}{\Leftrightarrow} x \leqslant_X x' \wedge y \leqslant_Y y' .$$

Example

$(\mathbb{N}^d, \leqslant)$ using the product ordering

Example: Higman's Lemma

Lemma (Higman 1952)



If $(X, \leqslant)$ is a wqo, then $(X^*, \leqslant_*)$ is a wqo where $\leqslant_*$ is the subword embedding ordering:

$$a_1 \cdots a_m \leqslant_* b_1 \cdots b_n \overset{\text{def}}{\Leftrightarrow} \begin{cases} \exists 1 \leqslant i_1 < \cdots < i_m \leqslant n, \\ \bigwedge_{j=1}^{m} a_j \leqslant_A b_{i_j} . \end{cases}$$

Example

$$aba \leqslant_* baaacabbab$$

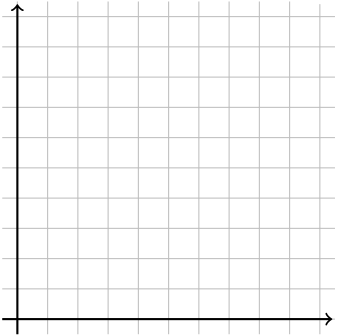
Bad Sequences

Over a qo $(X, \leqslant)$

$x_0, x_1, \ldots$ is bad if $\forall i < j \,.\, x_i \not\leqslant x_j$

$(X, \leqslant)$ wqo iff all bad sequences are finite

but can be of arbitrary length



Example (over $\mathbb{N}^2$)

Controlled Bad Sequences

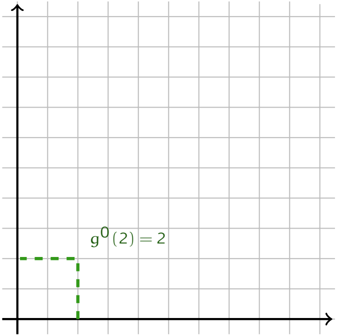
Over a qo $(X, \leqslant)$ with norm $\|\cdot\|$

$x_0, x_1, \ldots$ is bad if $\forall i < j \,.\, x_i \not\leqslant x_j$

$(X, \leqslant)$ wqo iff all bad sequences are finite

controlled by $g\colon \mathbb{N} \to \mathbb{N}$ monotone and inflationary and $n_0 \in \mathbb{N}$ if $\forall i \,.\, \|x_i\| \leqslant g^i(n_0)$

[Cichoń & Tahhan Bittar'98]



Example (over $\mathbb{N}^2$ with $n_0 = 2$ and $g(n) = n + 1$)

Controlled Bad Sequences

Over a qo $(X, \leqslant)$ with norm $\|\cdot\|$

- $x_0, x_1, \ldots$ is bad if $\forall i < j \,.\, x_i \not\leqslant x_j$
- $(X, \leqslant)$ wqo iff all bad sequences are finite
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[Cichoń & Tahhan Bittar'98]

Proposition

Over $(X, \leqslant)$, assuming $\forall n \, \{x \in X \mid \|x\| \leqslant n\}$ finite, (g, n_0)-controlled bad sequences have a maximal length, noted $L_{g,X}(n_0)$.

Controlled Bad Sequences

Proposition

Over a wqo $(X, \leqslant)$, assuming $\{x \in X \mid \|x\| \leqslant n\}$ to be finite $\forall n$, (g, n_0)-controlled bad sequences have a maximal length, noted $L_{g,X}(n_0)$.

Proof Idea

Controlled Bad Sequences

Proposition
Over a wqo $(X, \leqslant)$*, assuming* $\{x \in X \mid \|x\| \leqslant n\}$ *to be finite* $\forall n$*,* (g, n_0)*-controlled bad sequences have a maximal length, noted* $L_{g,X}(n_0)$.

Objective
Provide upper bounds for $L_{g,X}(n_0)$.

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 or
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end

$$
\begin{array}{c}
\langle a_0, b_0, c_0\rangle \\
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\end{array}
$$

- in any execution, $\langle a_0, b_0\rangle, \ldots, \langle a_n, b_n\rangle$ is a bad sequence over $(\mathbb{N}^2, \leqslant_\times)$,
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- How long can SIMPLE run?

A Computation of simple$(2,3)$

simple (a,b)
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$\langle a,b,c\rangle$	loop iterations
$\langle 2,3,2^0\rangle$	0

A COMPUTATION OF SIMPLE$(2,3)$

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$\langle 1,3,2^1\rangle$	1

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end

$\langle a,b,c\rangle$	loop iterations
$\langle 2,3,2^0\rangle$	0
$\langle 1,3,2^1\rangle$	1
$\langle 2^2,2,2^0\rangle$	2

A Computation of simple$(2,3)$

simple (a,b)
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end

$\langle a,b,c\rangle$	loop iterations
$\vdots$	$\vdots$
$\langle 2^2,2,2^0\rangle$	2
$\vdots$	$\vdots$
$\langle 1,2,2^{2^2-1}\rangle$	$2+2^2-1$

A COMPUTATION OF SIMPLE$(2,3)$

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$\langle 2^{2^2},1,1\rangle$	$2+2^2$

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$\langle a,b,c\rangle$	loop iterations
$\vdots$	$\vdots$
$\langle 2^{2^2},1,1\rangle$	$2+2^2$
$\vdots$	$\vdots$
$\langle 1,1,2^{2^{2^2}-1}\rangle$	$2+2^2+2^{2^2}-1$

A COMPUTATION OF SIMPLE$(2,3)$

SIMPLE (a,b)
$c \longleftarrow 1$
while $a > 0 \wedge b > 0$
  $\langle a,b,c\rangle \longleftarrow \langle a-1,b,2c\rangle$
 or
  $\langle a,b,c\rangle \longleftarrow \langle 2c,b-1,1\rangle$
end

$\langle a,b,c\rangle$	loop iterations
$\vdots$	$\vdots$
$\langle 1,1,2^{2^{2^2}-1}\rangle$	$2+2^2+2^{2^2}-1$
$\langle 0,1,2^{2^{2^2}}\rangle$	$2+2^2+2^{2^2}$

A COMPUTATION OF SIMPLE$(2,3)$

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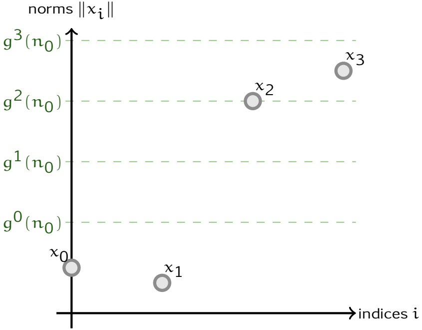
- non-elementary complexity
- derive (matching) upper bounds for termination arguments based on $(\mathbb{N}^2,\leqslant_\times)$ being a wqo

Descent Equation

(g, n_0)-controlled bad sequence $x_0, x_1, x_2, x_3, \ldots$ over a wqo $(X, \leqslant)$:



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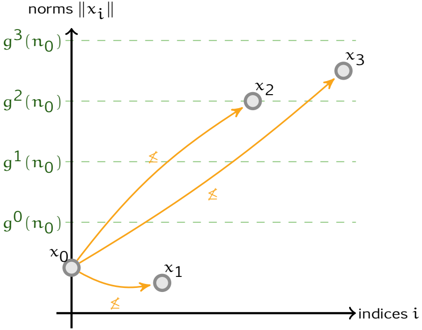

over the suffix $x_1,x_2,x_3,\ldots,\forall i>0$,

$x_0 \not\leq x_i$

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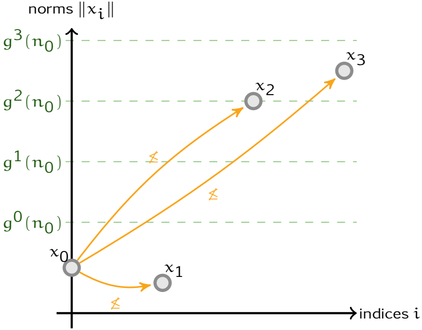

over the suffix $x_1, x_2, x_3, \ldots, \forall i > 0$,

$$x_i \in X \setminus \uparrow x_0 \stackrel{\text{def}}{=} \{x \in X \mid x_0 \not\leqslant x\}$$

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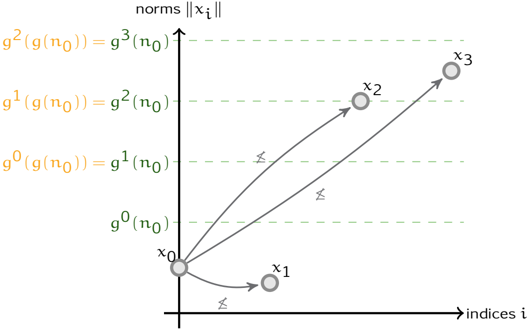

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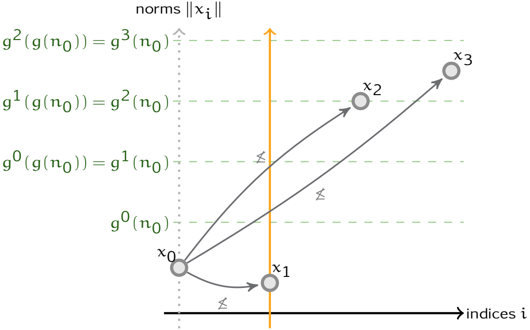

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$$L_{g,X}(n_0) = \max_{x_0 \in X, \|x_0\| \leqslant n_0} 1 + L_{g, X \backslash \uparrow x_0}(g(n_0))$$

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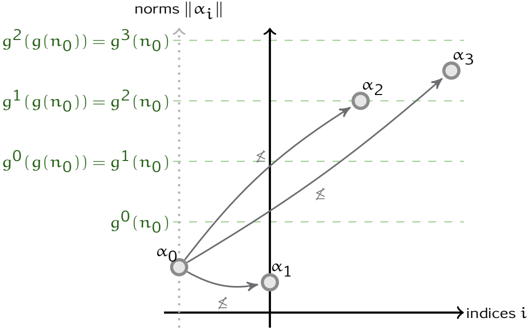

over the suffix $\alpha_1,\alpha_2,\alpha_3,\ldots, \forall i > 0$,

$\alpha_i \in \alpha_0 \stackrel{\text{def}}{=} \{\beta \in \alpha \mid \beta \not\leq \alpha_0\}$

$\|\alpha_i\| \leqslant g^{i-1}(g(n_0))$

$$L_{g,\alpha}(n_0) = \max_{\alpha_0 \in \alpha, \|\alpha_0\| \leqslant n_0} 1 + L_{g,\alpha_0}(g(n_0))$$

The Case of Ordinals

[S.'14]

- Cantor Normal Form (CNF) for ordinals $\alpha < \varepsilon_0$:

$$\alpha = \omega^{\alpha_1} \cdot c_1 + \cdots + \omega^{\alpha_k} \cdot c_k$$
$$\alpha > \alpha_1 > \cdots > \alpha_k \text{ in CNF}, \qquad 0 < c_1, \ldots, c_k < \omega$$

- Norm of ordinals $\alpha < \varepsilon_0$: "maximal constant"

$$\|\alpha\| \stackrel{\text{def}}{=} \max_{1 \leqslant i \leqslant k} \left(\max(\|\alpha_i\|, c_i) \right)$$

Example

$$\|\omega^{\omega^2}\| = 2$$
$$\|\omega^{\omega \cdot 5} + \omega^2 \cdot 3\| = 5$$

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Recall the descent equation:

$$L_{g,\alpha}(n_0) = \max_{\alpha_0 \in \alpha, \|\alpha_0\| \leqslant n_0} 1 + L_{g,\alpha_0}(g(n_0))$$

Proposition (variant of [Buchholtz, Cichoń & Weiermann'94])

Let $0 < \alpha < \varepsilon_0$ *and* $\|\alpha\| \leqslant n_0$. *Then*

$$L_{g,0}(n_0) = 0 \qquad L_{g,\alpha}(n_0) = 1 + L_{g,P_{n_0}(\alpha)}(g(n_0))$$

$P_x(\alpha)$ denotes the predecessor at x of $\alpha > 0$: "maximal ordinal $\beta < \alpha$ s.t. $\|\beta\| \leqslant x$"

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$$P_3(\omega^2) = \omega \cdot 3 + 3$$

$$\begin{aligned} P_3(\omega^{\omega^2}) = {} & \omega^{\omega\cdot 3+3}\cdot 3 + \omega^{\omega\cdot 3+2}\cdot 3 + \omega^{\omega\cdot 3+1}\cdot 3 + \omega^{\omega\cdot 3}\cdot 3 \\ & + \omega^{\omega\cdot 2+3}\cdot 3 + \omega^{\omega\cdot 2+2}\cdot 3 + \omega^{\omega\cdot 2+1}\cdot 3 + \omega^{\omega\cdot 2}\cdot 3 \\ & + \omega^{\omega+3}\cdot 3 + \omega^{\omega+2}\cdot 3 + \omega^{\omega+1}\cdot 3 + \omega^{\omega}\cdot 3 \\ & + \omega^{3}\cdot 3 + \omega^{2}\cdot 3 + \omega\cdot 3 + 3 \end{aligned}$$

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&\quad + \omega^{\omega\cdot 2+3}\cdot 3 + \omega^{\omega\cdot 2+2}\cdot 3 + \omega^{\omega\cdot 2+1}\cdot 3 + \omega^{\omega\cdot 2}\cdot 3 \\
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This function was already known in the literature!

Definition (Cichoń Hierarchy [Cichoń & Tahhan Bittar'98])
For $g : \mathbb{N} \to \mathbb{N}$, define $(g_\alpha : \mathbb{N} \to \mathbb{N})_\alpha$ by

$$g_0(x) \stackrel{\text{def}}{=} 0 \qquad g_\alpha(x) \stackrel{\text{def}}{=} 1 + g_{P_x(\alpha)}(g(x)) \text{ for } \alpha > 0$$

The Case of Ordinals

[S.'14]

Length Function Theorem (for Ordinals)

Let $\alpha < \varepsilon_0$ and $n_0 \geqslant \|\alpha\|$. Then the longest (g, n_0)-controlled descending sequence over α is of length $L_{g,\alpha}(n_0) = g_\alpha(n_0)$

Relating Norm and Length

[Cichoń & Tahhan Bittar'98]

Recall the definition of the Cichoń Hierarchy:

$$g_0(x) \stackrel{\text{def}}{=} 0 \qquad g_\alpha(x) \stackrel{\text{def}}{=} 1 + g_{P_x(\alpha)}(g(x)) \text{ for } \alpha > 0$$

Definition (Hardy Hierarchy)

For $g : \mathbb{N} \to \mathbb{N}$, define $(g^\alpha : \mathbb{N} \to \mathbb{N})_\alpha$ by

$$g^0(x) \stackrel{\text{def}}{=} x \qquad g^\alpha(x) \stackrel{\text{def}}{=} g^{P_x(\alpha)}(g(x)) \text{ for } \alpha > 0$$

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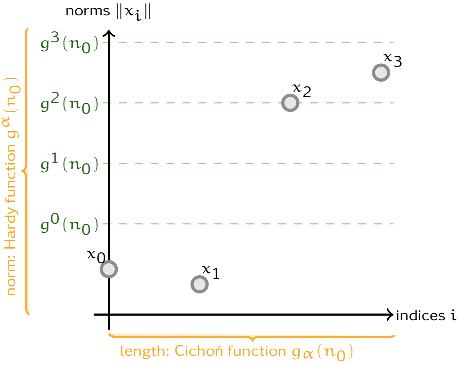

$$g^\alpha(x) = g^{g_\alpha(x)}(x)$$

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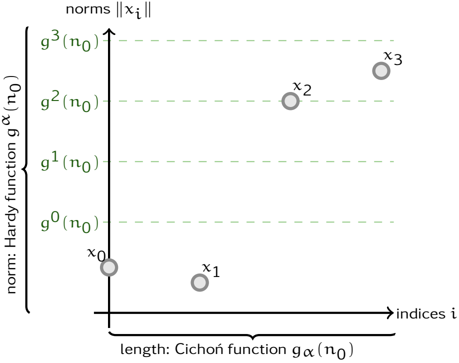

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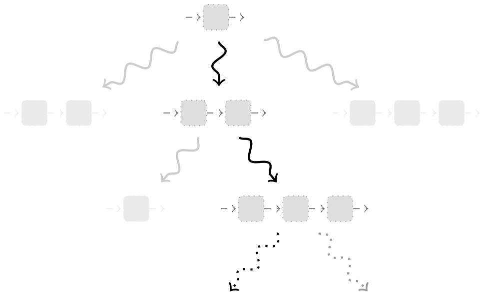
The Length of Decomposition Branches



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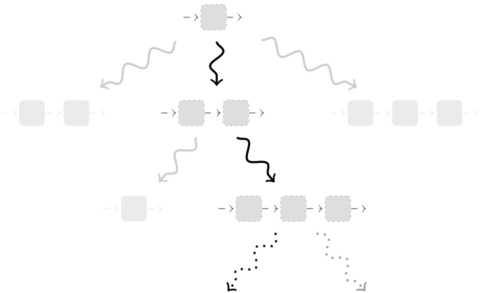

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Corollary

Assume $n_0 \geqslant 2$ and $g : \mathbb{N} \to \mathbb{N}$ are such that the sequence of ordinal ranks computed by the decomposition algorithm is (g, n_0)-controlled. The algorithm runs in SPACE$\left(g^{\omega^{\omega^2}}(n_0)\right)$.

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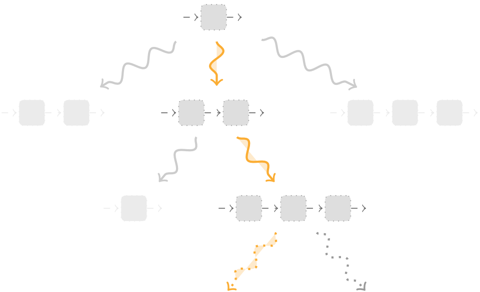

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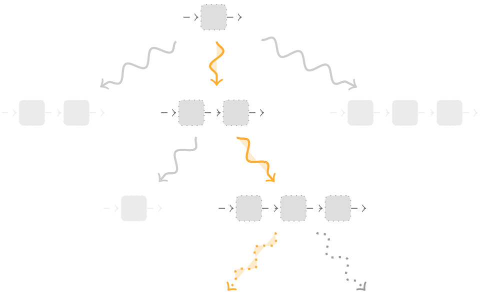

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Consequence of (Figueira, Figueira, S. & Schnoebelen'11)

The control $g(x) \stackrel{def}{=} H^{\omega^\omega}(e(x))$ *for* $H(x) \stackrel{def}{=} x+1$ *and an elementary function* e*, and* n_0 *the size of the reachability instance fit. Thus the decomposition algorithm runs in* $\mathit{SPACE}\big((H^{\omega^\omega} \circ e)^{\omega^{\omega^2}}(n)\big)$.

Restating the Result

"$\mathrm{SPACE}\Big((H^{\omega^\omega} \circ e)^{\omega^{\omega^2}}(n)\Big)$" is unreadable!

1. give names
   - H^{ω^ω} is the Ackermann function
   - $H^{\omega^{\omega^2}}$ is the "quadratic Ackermann" function
2. define coarse-grained complexity classes

$$\mathscr{F}_{<\alpha} \stackrel{\text{def}}{=} \bigcup_{\gamma<\omega^\alpha} \mathrm{FDTIME}(H^\gamma(n)) \qquad \mathbf{F}_\alpha \stackrel{\text{def}}{=} \bigcup_{f\in\mathscr{F}_{<\alpha}} \mathrm{DTIME}(H^{\omega^\alpha}(f(n)))$$

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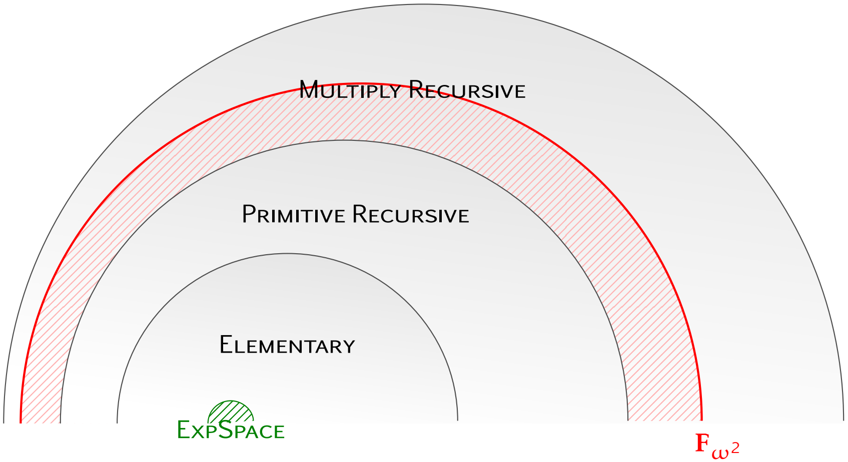
Complexity Classes Beyond Elementary

[S.'16]



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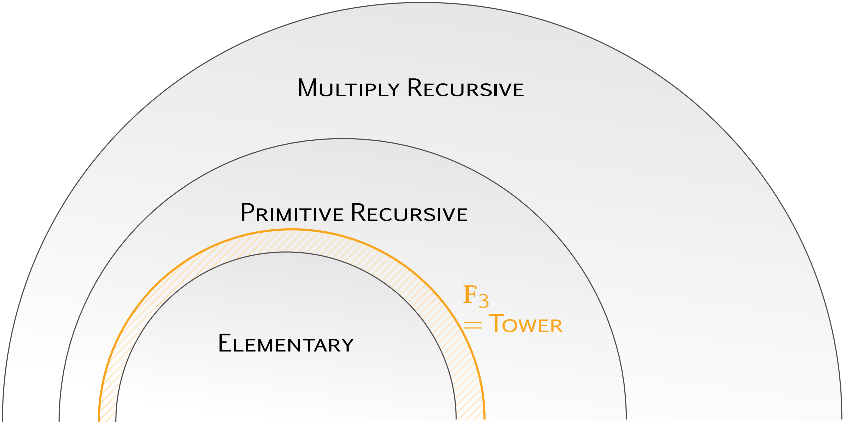

$$\mathbf{F}_3 \stackrel{\text{def}}{=} \bigcup_{e \text{ elementary}} \mathrm{DTime}\big(\mathrm{tower}(e(n))\big)$$

Complexity Classes Beyond Elementary

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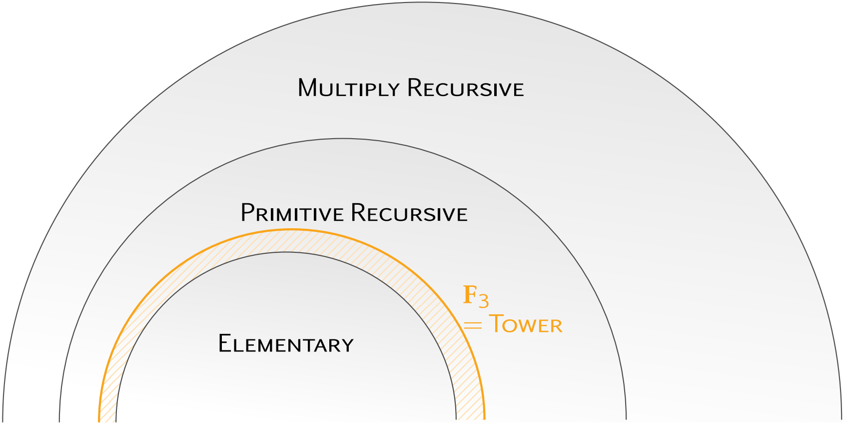

Examples of Tower-Complete Problems:

- satisfiability of first-order logic on words [Meyer'75]
- β-equivalence of simply typed λ terms [Statman'79]
- model-checking higher-order recursion schemes [Ong'06]

Complexity Classes Beyond Elementary

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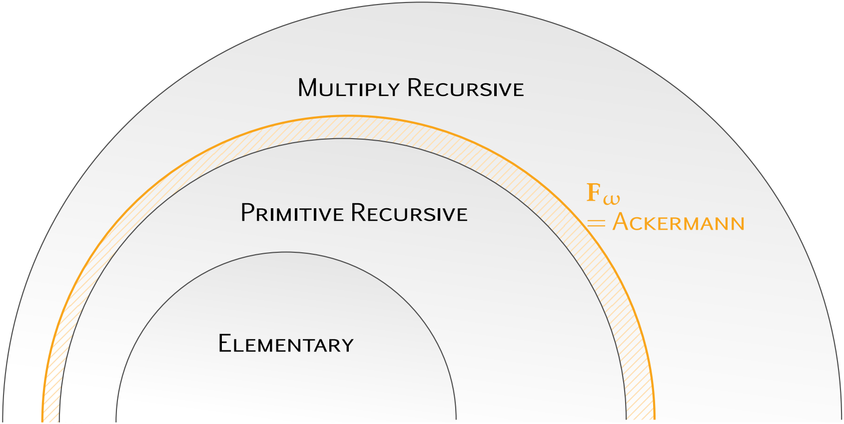

$$\mathbf{F}_\omega \stackrel{\text{def}}{=} \bigcup_{p \text{ primitive recursive}} \mathrm{DTime}\big(\mathrm{ackermann}(p(n))\big)$$

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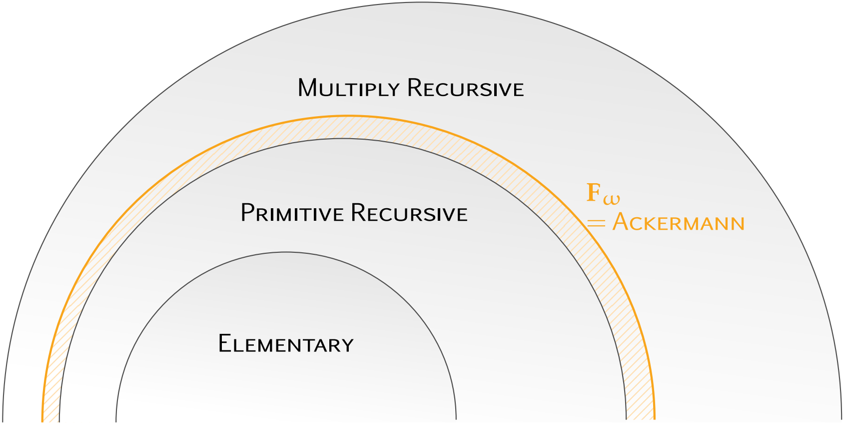

Examples of Ackermann-Complete Problems:

- reachability in lossy Minsky machines [Urquhart'98, Schnoebelen'02]
- satisfiability of safety Metric Temporal Logic [Lazić et al.'16]
- satisfiability of Vertical XPath [Figueira and Segoufin'17]

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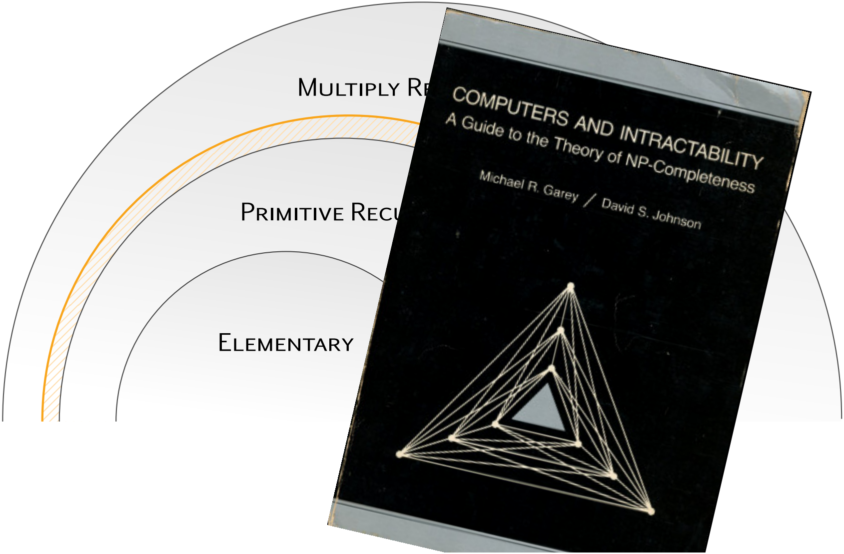

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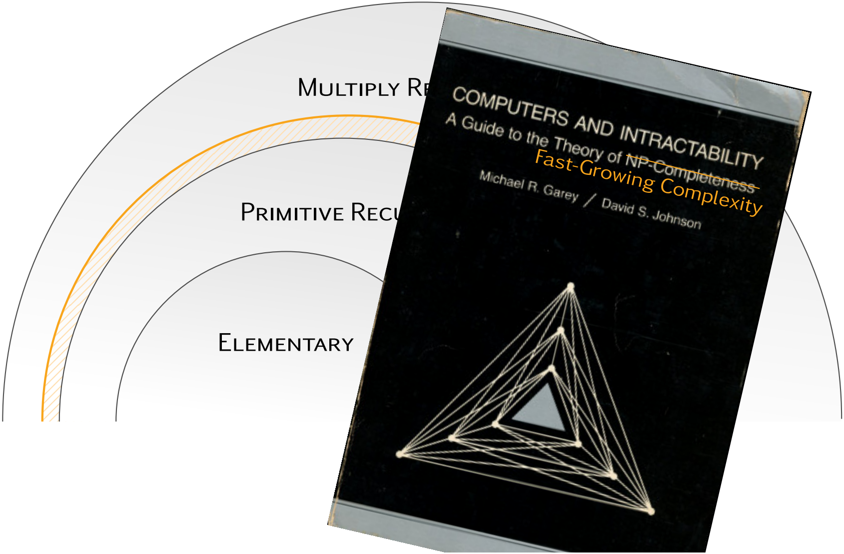

A Related Problem

labelled VAS transitions carry labels from some alphabet

$L(\mathcal{V}, \mathbf{source}, \mathbf{target})$ the language of labels in runs from **source** to **target**

$\downarrow L$ the set of subwords (for $\leqslant_*$) of the words in the language L

Downwards Language Inclusion Problem

input: *two labelled VAS $\mathcal{V}$ and $\mathcal{V}'$ and configurations* $\mathbf{source}, \mathbf{target}, \mathbf{source}', \mathbf{target}'$

question: $\downarrow L(\mathcal{V}, \mathbf{source}, \mathbf{target}) \subseteq \downarrow L(\mathcal{V}', \mathbf{source}', \mathbf{target}')$?

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Given a labelled VAS $\mathcal{V}$ and configurations $\mathbf{source}$ *and* $\mathbf{target}$ *and its decomposition, one can construct a finite automaton for* $\downarrow L(\mathcal{V}, \mathbf{source}, \mathbf{target})$ *in polynomial time.*

Corollary

The Downwards Language Inclusion problem is in quadratic Ackermann.

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Theorem (Zetzsche'16)

The Downwards Language Inclusion problem is Ackermann*-hard.*

Summary

well-quasi-orders (wqo):

- proving algorithm termination

a toolbox for wqo-based complexity

- upper bounds: length function theorems (for ordinals, Dickson's Lemma, Higman's Lemma, and combinations)
- lower bounds
- complexity classes: $(\mathbf{F}_\alpha)_\alpha$

this talk: focus on one problem

- reachability in vector addition systems in $\mathbf{F}_{\omega^2}$

Perspectives

1. complexity gap for VAS reachability
   - ExpSpace-hard [Lipton'76]
     better lower bounds?
   - decomposition algorithm: at least $\mathbf{F}_\omega$ (Ackermannian) time [Zetzsche'16]
2. reachability in VAS extensions
   - decidable in VAS with hierarchical zero tests [Reinhardt'08]
   - what about
     - branching VAS
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     - pushdown VAS

Perspectives

1. complexity gap for VAS reachability
   - ExpSpace-hard [Lipton'76]
     better lower bounds?
   - decomposition algorithm: at least $\mathbf{F}_\omega$ (Ackermannian) time
     [Zetzsche'16]
2. reachability in VAS extensions
   - decidable in VAS with hierarchical zero tests [Reinhardt'08]
   - what about
     - branching VAS
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Demystifying Reachability in Vector Addition Systems

[Leroux & S.'15]

Ideal Decomposition Theorem

The Decomposition Algorithm computes the ideal decomposition of the set of runs from source to target.

Upper Bound Theorem

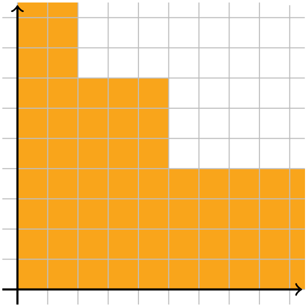
Reachability in vector addition systems is in cubic Ackermann.

Ideals of Well-Quasi-Orders $(X, \leqslant)$

- Canonical decompositions
  [Bonnet'75]
  if $D \subseteq X$ is $\downarrow$-closed, then

$$D = I_1 \cup \cdots \cup I_n$$

  for (maximal) ideals $I_1, \ldots, I_n$



Example (over $\mathbb{N}^2$)

$$D = (\{0,\ldots,2\} \times \mathbb{N}) \cup (\{0,\ldots,5\} \times \{0,\ldots,7\}) \cup (\mathbb{N} \times \{0,\ldots,4\})$$

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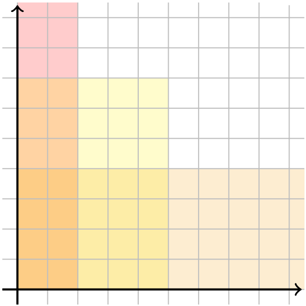

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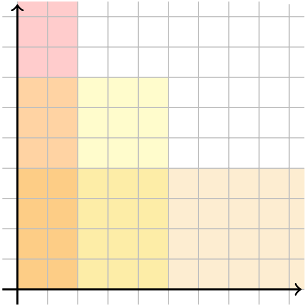
- Effective representations
  [Goubault-Larrecq et al.'17]



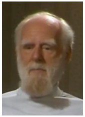
Example (over $\mathbb{N}^2$)

$$D = [\![(2, \infty)]\!] \cup [\![(5, 7)]\!] \cup [\![(\infty, 4)]\!]$$

Decomposition Theorem

Well-Quasi-Order on Runs

combination of Dickson's and Higman's lemmata



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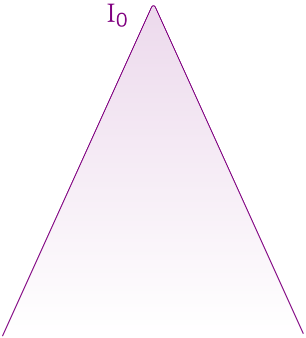

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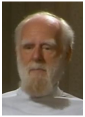

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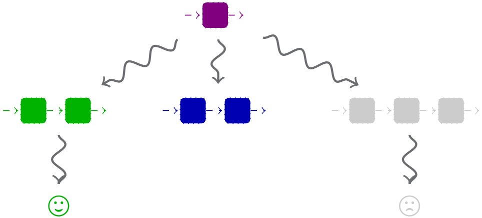

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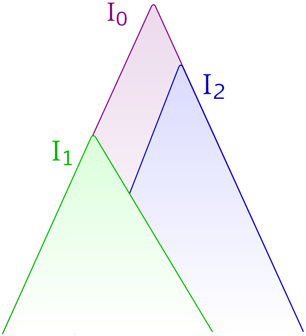

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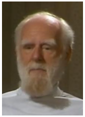

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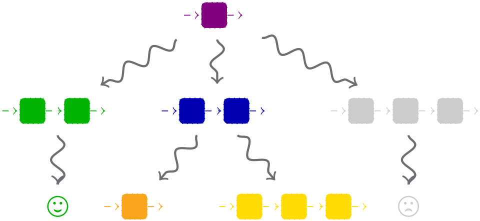

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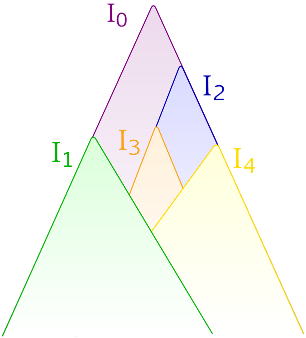

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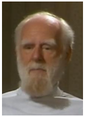

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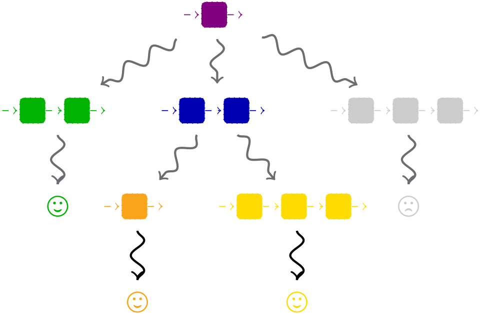

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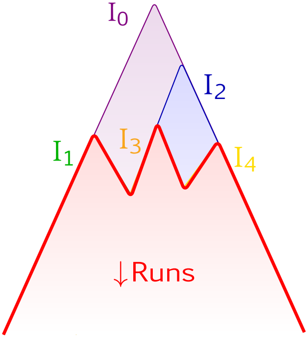

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- I is adherent to Runs if
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- semantic equivalent to
  Θ condition
- undecidable for arbitrary ideals
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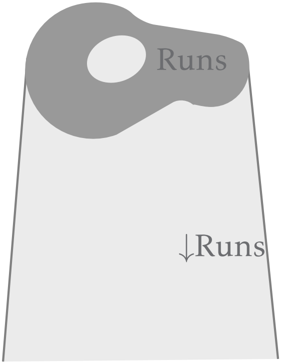

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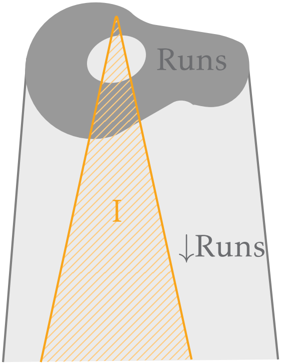

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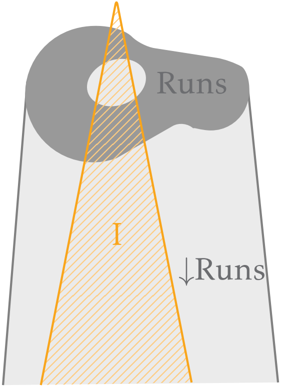

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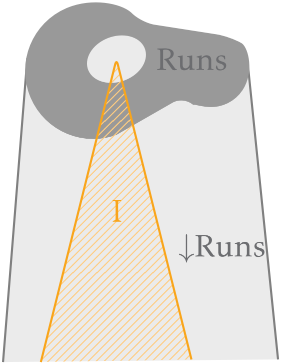

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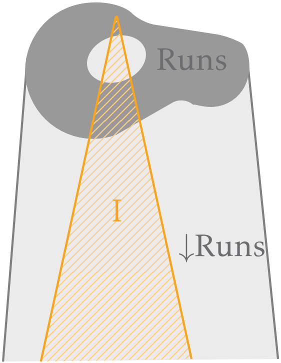

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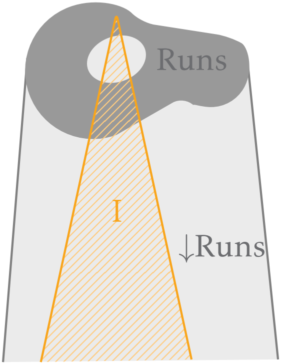

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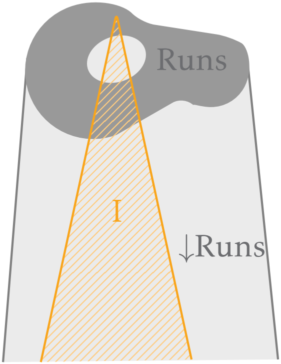

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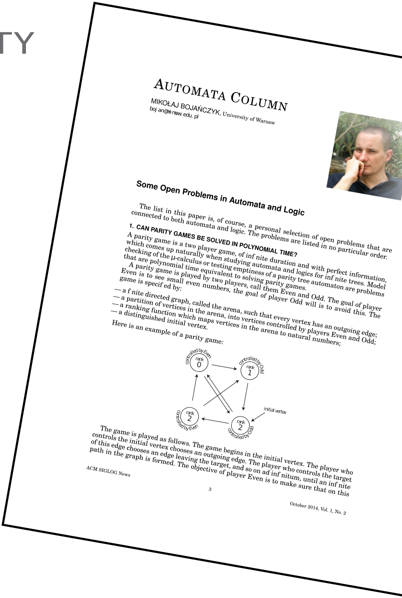

AUTOMATA COLUMN

MIKOŁAJ BOJAŃCZYK, University of Warsaw
boj an@mimuw. edu. pl

Some Open Problems in Automata and Logic

The list in this paper is, of course, a personal selection of open problems that are connected to both automata and logic. The problems are listed in no particular order.

1. CAN PARITY GAMES BE SOLVED IN POLYNOMIAL TIME?

A parity game is a two player game, of inf nite duration and with perfect information, which comes up naturally when studying automata and logics for inf nite trees. Model checking of the μ-calculus or testing emptiness of a parity tree automaton are problems that are polynomial time equivalent to solving parity games.

A parity game is played by two players, call them Even and Odd. The goal of player Even is to see small even numbers, the goal of player Odd will is to avoid this. The game is specif ed by:

— a f nite directed graph, called the arena, such that every vertex has an outgoing edge;
— a partition of vertices in the arena, into vertices controlled by players Even and Odd;
— a ranking function which maps vertices in the arena to natural numbers;
— a distinguished initial vertex.

Here is an example of a parity game:

The game is played as follows. The game begins in the initial vertex. The player who controls the initial vertex chooses an outgoing edge. The player who controls the target of this edge chooses an edge leaving the target, and so on ad inf nitum, until an inf nite path in the graph is formed. The objective of player Even is to make sure that on this

ACM SIGLOG News 3 October 2014, Vol. 1, No. 2

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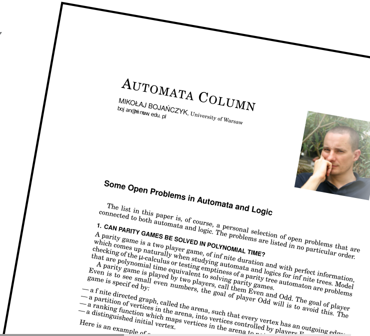

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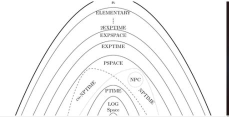

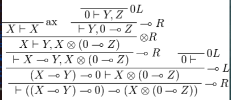

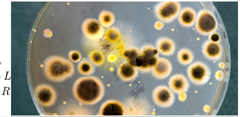

Programming Languages	Database Theory	Security	Computational Linguistics

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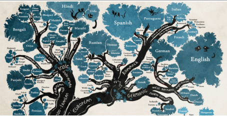

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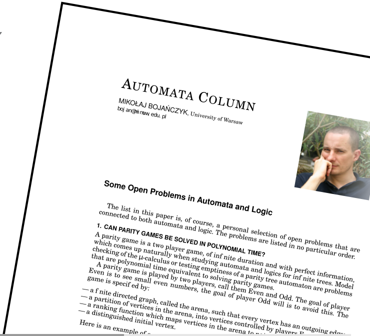

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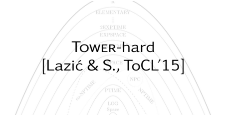
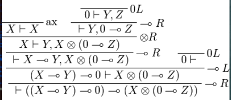
Complexity Theory



Tower-hard
[Lazić & S., ToCL'15]

Distributed Computing

Proof Theory



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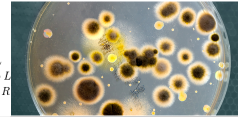

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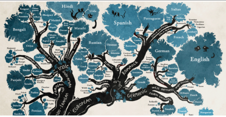

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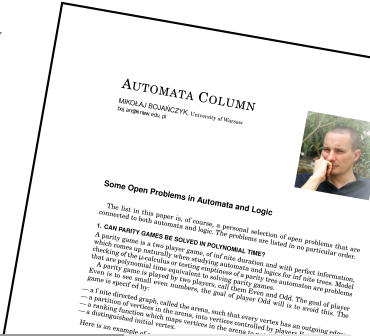
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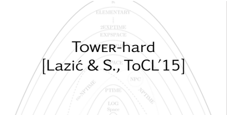

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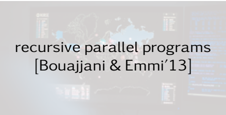
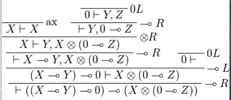
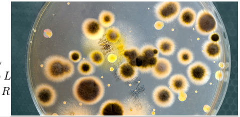


recursive parallel programs
[Bouajjani & Emmi'13]

Proof Theory



Computational Biology



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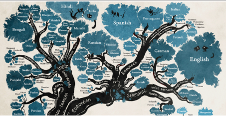

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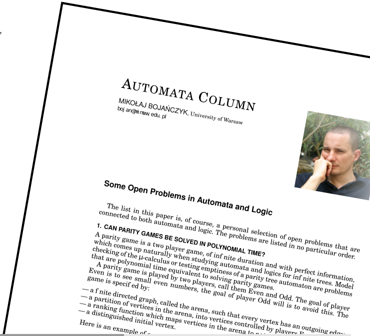
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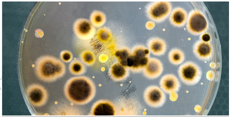
Tower-hard [Lazić & S., ToCL'15]	recursive parallel programs [Bouajjani & Emmi'13]	linear and relevance logics [de Groote et al.'04 Lazić & S., ToCL'15 S., JSL'16]	
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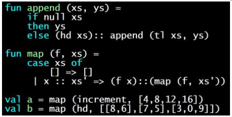			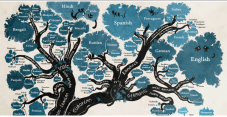

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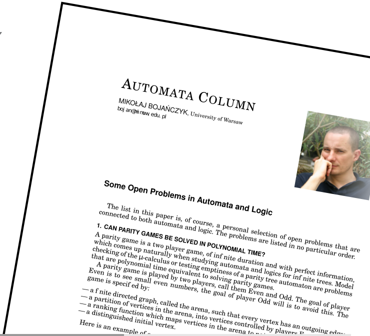
Automata Column

MIKOŁAJ BOJAŃCZYK, University of Warsaw

Some Open Problems in Automata and Logic

The list in this paper is, of course, a personal selection of open problems that are connected to both automata and logic. The problems are listed in no particular order.

1. CAN PARITY GAMES BE SOLVED IN POLYNOMIAL TIME?

A parity game is a two player game, of infinite duration and with perfect information, which comes up naturally when studying automata and logics for infinite trees. Model checking of the μ-calculus or testing emptiness of a parity tree automaton are problems that are polynomial time equivalent to solving parity games.

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Here is an example of ...

Complexity Theory	Distributed Computing	Proof Theory	Computational Biology
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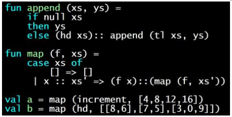			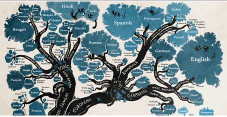

Branching VAS Reachability

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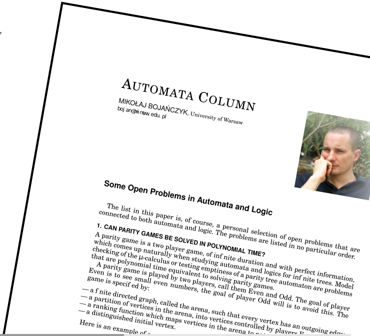

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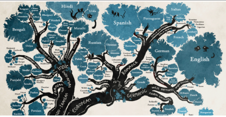
observational equivalence [Cotton-Barratt et al.'17]			

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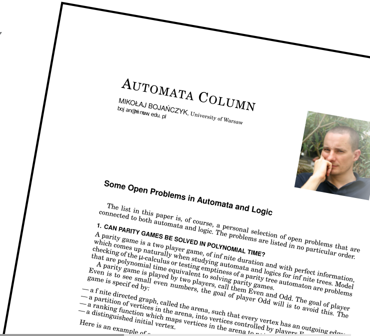
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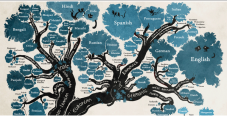
observational equivalence [Cotton-Barratt et al.'17]	data logics [Bojańczyk et al.'09, Abriola et al.'17]		

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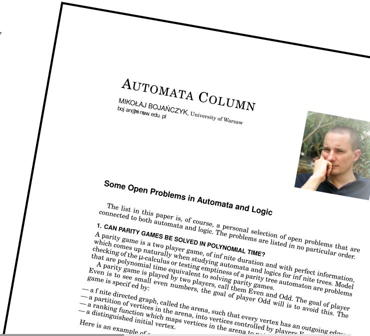
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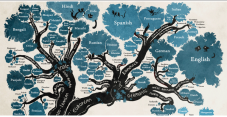
observational equivalence [Cotton-Barratt et al.'17]	data logics [Bojańczyk et al.'09, Abriola et al.'17]	security protocols [Verma & Goubault-Larrecq'05]	

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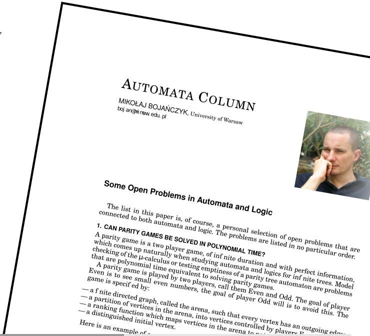
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