Perfect Half Space Games

Thomas Colcombet, Marcin Jurdziński, Ranko Lazić, and Sylvain Schmitz

LSV, ENS Paris-Saclay & CNRS & Inria

LICS 2017, June 23rd, 2017
What to do this week-end?

Reykjavik
Landmannalaugar
Thórsmörk
Hrútafjördur
Vatnajökull
Mývatn

Maximal dry temperature as a parity objective
Uncontrolled events as a two-players game
Discrete resources as a multi-energy objective

(−1,0)
(−4,−3)
(0,0)
(−1,0)
(0,0)
(−1,0)
(−2,−1)
(−4,−5)
(−1,0)
(−1,0)
(1,0)

2/10
What to do this weekend?

- Reykjavik
- Hrútafjördur
- Landmannalaugar
- Thórsmörk
- Vatnajökull
- Mývatn
WHAT TO DO THIS WEEK-END?

MAXIMAL DRY TEMPERATURE

- Reykjavik: 10°C
- Landmannalaugar: 9°C
- Þórsmörk: 6°C
- Hrútafjörður: 4°C
- Vatnajökull: 9°C
- Mývatn: 12°C
- Maximal dry temperature as a parity objective
- Uncontrolled events as a two-players game
- Discrete resources as a multi-energy objective

(−1,0) (−4,−3) (0,0) (−1,0) (0,0) (−1,0) (1,0) (−2,−1) (−4,−5) (−1,0) (−1,0)
WHAT TO DO THIS WEEK-END?

**Maximal dry temperature**
as a parity objective

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![Game Diagram](image)
What to do this week-end?

Maximal dry temperature
as a parity objective
What to do this week-end?

Uncontrolled events
**What to do this week-end?**

**Maximal dry temperature as a parity objective**

**Uncontrolled events as a two-players game**
WHAT TO DO THIS WEEK-END?

Maximal dry temperature as a parity objective
Uncontrolled events as a two-players game
What to do this week-end?

Discrete resources

Energy Parity Games
Extended Energy Games
Bounding Games
Perfect Half Space Games
What to do this week-end?

Maximal dry temperature as a parity objective

Uncontrolled events as a two-players game

Discrete resources as a multi-energy objective
What to do this week-end?

Maximal dry temperature as a parity objective

Uncontrolled events as a two-players game

Discrete resources as a multi-energy objective
Player 1 wins a play if both

- **energy** objective: no component goes negative
- **parity** objective: the maximal priority is odd

**Example**

\[
R(0,0) \xrightarrow{(1,0)} R(1,0) \xrightarrow{(1,0)} R(2,0) \xrightarrow{(-1,0)} H(1,0) \xrightarrow{(0,0)} R(1,0) \rightarrow \cdots
\]
Multi-Dimensional Energy Parity Games

Applications

- contractive \((\oplus,!)\)-Horn linear logic
  (Kanovich, APAL ’95)

- (weak) simulation of finite-state systems by Petri nets
  (Abdulla et al., Concur ’13)

- model-checking Petri nets with a fragment of \(\mu\)-calculus
  (Abdulla et al., Concur ’13)

- resource-bounded agent temporal logic \(\text{RB}^\pm\text{ATL}^*\)
  (Alechina et al., RP ’16 & AI ’17)
**Multi-Dimensional Energy Parity Games**

**Complexity**

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<tr>
<th>lower bound</th>
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**Multi-Dimensional Energy Parity Games**

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<td>EXPSPACE</td>
<td>(Lasota, IPL ’09)</td>
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Multi-Dimensional Energy Parity Games

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## Multi-Dimensional Energy Parity Games

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upper bound

2-EXP

this talk

coNP

(Chatterjee et al., Concur ‘12)
**Fixed Dimensional Energy Fixed Parity Games**

**Complexity**

- **lower bound**
  - EXP for $d \geq 4$
  - (Courtois and S., MFCS ’14)

- **upper bound**
  - pseudoP
  - this talk

- **∃ initial credit**

- **w. initial credit**
  - EXP for $d \geq 4$
  - this talk

- **pseudoP**
  - this talk
Outline

- multi-dimensional energy parity games
  (Jančar, RP ’15)
  
  extended multi-dimensional energy games (Brázdil et al., ICALP ’10)
  
  bounding games (Jurdziński et al., ICALP ’15)
  
  perfect half space games (this paper)
  
  lexicographic energy games (Colcombet and Niwiński)
  
  mean-payoff games (Comin and Rizzi, Algorithmica ’16)
**Extended Multi-Dimensional Energy Games**

**Encode Priorities as Energy** (Jančar, RP '15)

Two new dimensions: tolerance to humid low/high temperature
Bounding Games

Player 1’s Objective

energy

bounding
Bounding Games

Player 1’s Objective

energy

bounding
**Bounding Games**

**Player 1’s Objective**

- **Energy**
- **Bounding**
Bounding Games

Player 1’s Objective

energy

bounding
Bounding Games

Player 1’s Objective

energy

bounding
**Bounding Games**

**Encoding Extended Energy Games**

- **Bin excess energy**
  - ![Diagram](image)
  - 
    - 
    - 
    - 
    - 
    - 
    - 
    - 
    - 

- **Unbounded replenishing**
  - ![Diagram](image)
  - 
    - 
    - 
    - 
    - 
    - 
    - 
    - 
    - 

\[(\ldots, 0, \ldots) \rightarrow (0, 1, 0)\]
**Bounding Games**

**Theorem (Jurdziński et al., ICALP ’15)**

*Bounding games on multi-weighted game graphs* $(V, E, d)$ *are solvable in* $(|V| \cdot \|E\|)^{O(d^4)}$.

**Corollary**

*The given initial credit problem with credit $c$ for energy parity games on multi-weighted game graphs* $(V, E, d)$ *with $p$ even priorities is solvable in*

$$O(|V| \cdot \|E\|)^{2^{O(d \log (d + p))}} + O(d \cdot \log \|c\|).$$
Bounding Games

Theorem (Jurdziński et al., ICALP ’15)
Bounding games on multi-weighted game graphs \((V,E,d)\) are solvable in \((|V| \cdot \|E\|)^{O(d^4)}.\)

Corollary
The given initial credit problem with credit \(c\) for energy parity games on multi-weighted game graphs \((V,E,d)\) with \(p\) even priorities is solvable in

\[O(|V| \cdot \|E\|)^{2^{O(d \log(d+p))}} + O(d \cdot \log \|c\|).\]
Bounding Games

**Theorem (this paper)**

Bounding games on multi-weighted game graphs \((V,E,d)\) are solvable in \((|V| \cdot ||E||)^O(d^3)\).

**Corollary**

The given initial credit problem with credit \(c\) for energy parity games on multi-weighted game graphs \((V,E,d)\) with \(p\) even priorities is solvable in

\[
O(|V| \cdot ||E||)^{2O(d \log (d+p))} + O(d \cdot \log ||c||) .
\]
**Perfect Half Space Games**

**Player 2’s Objective in a Bounding Game**

Key Intuition
Player 2 can escape in a perfect half space
Perfect Half Space Games

Player 2’s Objective in a Bounding Game

Key Intuition
Player 2 can escape in a perfect half space
Perfect Half Space Games

Perfect Half Space

\[ \{(x, y) : x + y < 0\} \]
Perfect Half Space Games

Perfect Half Space

\[ \{(x,y) : x + y < 0\} \]

boundary: \[ \{(x,y) : x + y = 0\} \]
**Perfect Half Space Games**

**Perfect Half Space**

\[
\{(x, y) : x + y < 0 \} \cup \{(x, y) : x + y = 0 \land x < 0 \}
\]
Perfect Half Space Games

Plays

- pairs of vertices and perfect half spaces:
  \[(v_0, H_0) \xrightarrow{w_1} (v_1, H_1) \xrightarrow{w_2} (v_2, H_2) \ldots\]

- in his vertices, Player 2 chooses the current perfect half space

- Player 2 wins if \(\exists i \text{ s.t. } \sum_{j \geq 0} w_j \text{ diverges into } \bigcap_{j > i} H_j\)
Perfect Half Space Games

- Player 2 wins if $\exists i$ s.t. $\sum_{j \geq 0} w_j$ diverges into $\bigcap_{j > i} H_j$

Example

$H_L \cap H_R = H_Y$
**SOLVING PERFECT HALF SPACE GAMES**

**Theorem**

Perfect half space games on multi-weighted game graphs 
\((V, E, d)\) are solvable in \(\mathcal{O}(d^3)\).

**Proof Idea**

- reduce to a lexicographic energy game (Colcombet and Niwiński)
- \(\approx\) perfect half space game with a single fixed \(H\)
- itself reduced to a mean-payoff game
**SOLVING PERFECT HALF SPACE GAMES**

**THEOREM**

Perfect half space games on multi-weighted game graphs 
$$(V, E, d)$$ are solvable in 
$$(|V| \cdot \|E\|)^{O(d^3)}$$. 

**PROOF IDEA**

- reduce to a *lexicographic energy game* (Colcombet and Niwiński)

- $\approx$ perfect half space game with a single fixed $H$

- itself reduced to a mean-payoff game
Player 2 Strategies

Oblivious Strategy
Player 2 chooses the same $H_v$ every time it visits vertex $v$.

Theorem
If Player 2 has a winning strategy in a perfect half space game, then it has an oblivious one.

"Counterless" Strategy

Corollary (Brázdil et al., ICALP ’10)
If Player 2 has a winning strategy in a multi-dimensional energy parity game, then it has a positional one.
**Player 2 Strategies**

**Oblivious Strategy**
Player 2 chooses the same $H_v$ every time it visits vertex $v$.

**Theorem**
If Player 2 has a winning strategy in a perfect half space game, then it has an oblivious one.

**“Counterless” Strategy**

**Corollary** (Brázdil et al., ICALP ’10)
If Player 2 has a winning strategy in a multi-dimensional energy parity game, then it has a positional one.
CONCLUDING REMARKS

- tight 2-EXP bounds for multi-energy parity games
- impacts numerous problems
- fine understanding of Player 2’s strategies:
  Player 2 can win by announcing in which perfect half space he will escape
The Icelandic Met Office does not endorse any of the information provided during this talk, and cannot be held liable for a ruined week-end subsequent to foolishly trusting these fabricated forecasts.
REFERENCES


