Ackermann and Primitive-Recursive Bounds with Dickson's Lemma

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Outline

well quasi orderings

generic tools for termination arguments

this talk

beyond termination: complexity upper bounds

contents

Dickson's Lemma

- wqo $(S, \leqslant)$: well-founded quasi-ordering with no infinite antichains
- every infinite sequence $\mathbf{x} = x_0, x_1, x_2, \ldots$ of S^ω contains an infinite increasing subsequence $x_{i_1} \leqslant x_{i_2} \leqslant x_{i_3} \leqslant \cdots$ for some $i_1 < i_2 < i_3 < \cdots$
- Dickson's Lemma: $(\mathbb{N}^k, \leqslant)$ with $\leqslant$ denoting the product ordering is a wqo.

Good and Bad Sequences

- a sequence of S^∞ is r-*good* if it contains an increasing subsequence $x_{i_1} \leqslant x_{i_2} \leqslant \cdots \leqslant x_{i_{r+1}}$ of length $r+1$
- otherwise it is r-*bad*
- for $r = 1$: *good* and *bad* sequences
- wqo: every bad sequence is finite

Well-structured transition systems

- transition systems $(Q, \to, q_0)$ with a wqo $\leqslant$ on Q compatible with transitions:

$$\forall p, q, p' \in Q, (p \xrightarrow{a} q \wedge p \leqslant p') \Rightarrow \exists q', (q \leqslant q' \wedge p' \xrightarrow{a} q')$$

- a generic framework for decidability results: safety, termination, EF model checking, ...
- many classes of concrete systems are WSTS:
  - over $(\mathbb{N}^k, \leqslant)$: vector addition systems, resets/transfer Petri nets, increasing counter systems, ...
  - over $(\Sigma^*, \sqsubseteq)$: lossy channel systems, ...
  - beyond: data nets, ...

Example: (Non) Termination

- given $(Q, \to, q_0)$, decide whether there exists an infinite run $q_0 \to q_1 \to \cdots$
- holds iff there exists $q_i \leqslant q_j$ with $q_0 \to^* q_i \to^+ q_j$
- thanks to wqo, termination is both r.e. and co-r.e.
- what is the complexity?

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- thanks to wqo, termination is both r.e. and co-r.e.
- **what is the complexity?**

Controlled Sequences

- bound the length of bad sequences
- but: choose any N, and consider the bad sequence $N, N-1, \ldots, 0$ over $\mathbb{N}$
- similarly: $\langle 3,3\rangle, \langle 3,2\rangle, \langle 3,1\rangle, \langle 3,0\rangle, \langle 2,N\rangle, \langle 2,N-1\rangle, \ldots$

Controlled Sequences

- bound the length of bad sequences

- fix a *control function* $f : \mathbb{N} \to \mathbb{N}$
- $\mathbf{x} = x_0, x_1, \ldots$ over $\mathbb{N}^k$ is f-*controlled* if

$$\forall i = 0, 1, \ldots, \forall 1 \leqslant j \leqslant k, x_i[j] < f(i)$$

- for fixed r, k, f, there are *finitely* many r-bad f-controlled sequences over $\mathbb{N}^k$

Controlled Sequences

- bound the length of *controlled* bad sequences
- distinguish x_0: consider $(\lambda x.f(x+t))$-controlled sequences instead (thus $x_i[j] < f(i+t)$)
- bound the maximal length of a r-bad $(\lambda x.f(x+t))$-controlled sequence over $\mathbb{N}^k$ as a function of t:

$$L_{r,k,f}(t)$$

Example

$k = 2, t = 1, f(x) = x + 3$

i	0	1	2	3	4	5	$\cdots$	10	11	12	13	$\cdots$	26	27	28	29	$\cdots$	58	59
$x_i[1]$	3	3	3	3	2	2	$\cdots$	2	2	1	1	$\cdots$	1	1	0	0	$\cdots$	0	0
$x_i[2]$	3	2	1	0	7	6	$\cdots$	1	0	15	14	$\cdots$	1	0	31	30	$\cdots$	1	0
$f(i+t)$	4	5	6	7	8	9	$\cdots$	14	15	16	17	$\cdots$	30	31	32	33	$\cdots$	62	63

Easy Cases

$$
\begin{aligned}
L_{0,k,f}(t) &= 0 && \text{(by convention)}\\
L_{1,0,f}(t) &= 1\\
L_{r,0,f}(t) &= r\\
L_{1,1,f}(t) &= f(t)
\end{aligned}
$$

the latter sequence being

$$f(t)-1,\ f(t)-2,\ \ldots,\ 1,\ 0$$

A More General Problem

- *disjoint sums* $A_1 \oplus A_2$
- wqo for the sum ordering :

$$\begin{aligned} x \leqslant x' \overset{\text{def}}{\Leftrightarrow} & \left(x, x' \in A_1 \wedge x \leqslant_1 x'\right) \\ & \vee \left(x, x' \in A_2 \wedge x \leqslant_2 x'\right) \end{aligned}$$

- multiset notation: $\tau = \{\!\{k_1, k_2, \ldots\}\!\}$, $\mathbb{N}^\tau = \bigoplus_i \mathbb{N}^{k_i}$
- shift to $L_{\tau,f}(t)$

Decomposition for $\mathbb{N}^k$

A bad sequence $\mathbf{x} = x_0, x_1, \ldots, x_l$ over $\mathbb{N}^k$:

- control: $x_0 \leqslant \langle f(t) - 1, \ldots, f(t) - 1 \rangle$
- badness: $\forall i > 0, \exists 1 \leqslant j \leqslant k, x_i[j] < x_0[j]$
- $\forall i > 1, \exists 1 \leqslant j \leqslant k, 0 \leqslant s \leqslant f(t) - 1, x_i \in R_{j,s}$ where $R_{j,s} = \{x \in \mathbb{N}^k \mid x[j] = s\}$
- there are $N_{k,f}(t) \stackrel{\text{def}}{=} k \cdot (f(t) - 1)$ regions in total

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- control: $x_0 \leqslant \langle f(t)-1, \ldots, f(t)-1 \rangle$
- badness: $\forall i > 0,\ \exists 1 \leqslant j \leqslant k, x_i[j] < x_0[j]$
- $\forall i > 1, \exists 1 \leqslant j \leqslant k, 0 \leqslant s \leqslant f(t)-1, x_i \in R_{j,s}$ where $R_{j,s} = \{x \in \mathbb{N}^k \mid x[j] = s\}$
- there are $N_{k,f}(t) \stackrel{\text{def}}{=} k \cdot (f(t)-1)$ regions in total

Decomposition for $\mathbb{N}^k$

Example

$$\mathbf{x} = \langle 2,2\rangle,\ \langle 1,5\rangle,\ \langle 4,0\rangle,\ \langle 1,1\rangle,\ \langle 0,100\rangle,\ \langle 0,99\rangle,\ \langle 3,0\rangle$$

$$\langle 2,2\rangle, \left[\begin{array}{llllll} & & & \langle 0,100\rangle,\langle 0,99\rangle, & & (R_{1,0} : x[1] = 0) \\ \langle 1,5\rangle, & & \langle 1,1\rangle, & & & (R_{1,1} : x[1] = 1) \\ & \langle 4,0\rangle, & & & \langle 3,0\rangle & (R_{2,0} : x[2] = 0) \\ & & & & & (R_{2,1} : x[2] = 1) \end{array} \right]$$

Decomposition for $\mathbb{N}^k$

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$$\mathbf{x} = \langle 2,2\rangle,\ \langle 1,5\rangle,\ \langle 4,0\rangle,\ \langle 1,1\rangle,\ \langle 0,100\rangle,\ \langle 0,99\rangle,\ \langle 3,0\rangle$$

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\langle *,5\rangle, & & \langle *,1\rangle, & & & (R_{1,1} : x[1] = 1) \\
 & \langle 4,*\rangle, & & & \langle 3,*\rangle & (R_{2,0} : x[2] = 0) \\
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\end{array}\right]$$

Decomposition for $\mathbb{N}^k$

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 & \langle 4,*\rangle, & & & \langle 3,*\rangle & (R_{2,0} : x[2] = 0) \\
 & & & & & (R_{2,1} : x[2] = 1)
\end{array}\right]$$

$$L_{\{k\},f}(t) \leqslant 1 + L_{N_{k,f}(t)\times\{k-1\},f}(t+1)$$

Decomposition for $\bigoplus_i \mathbb{N}^{k_i}$

Example

$\tau = \{\!|1,2,2|\!\}$:

$$\left[\langle 2,2\rangle, \langle 5\rangle, \langle 1,5\rangle, \langle 4,0\rangle, \langle 3\rangle\ \langle 12,1\rangle, \langle 1,1\rangle, \langle 3,5\rangle\ \langle 0,100\rangle, \langle 0,99\rangle, \langle 3,0\rangle\right]$$

$$\langle 2,2\rangle \left[\langle 5\rangle, \langle *,5\rangle, \langle 4,*\rangle, \langle 3\rangle\ \langle 12,1\rangle, \langle *,1\rangle, \langle 3,5\rangle\ \langle *,100\rangle, \langle *,99\rangle, \langle 3,*\rangle\right]$$

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Decomposition for $\bigoplus_i \mathbb{N}^{k_i}$

Example

$$\langle 2,2\rangle \left[\begin{array}{l} \langle 5\rangle, \quad \langle 3\rangle \quad \langle *,100\rangle, \langle *,99\rangle, \\ \langle *,5\rangle, \quad \langle *,1\rangle, \\ \langle 4,*\rangle, \quad \langle 3,*\rangle \\ \langle 12,1\rangle, \quad \langle 3,5\rangle \end{array}\right]$$

$$\begin{aligned} \tau_{\langle k,t,f\rangle} &\stackrel{\text{def}}{=} \tau - \{k\} + N_{k,f}(t) \times \{k-1\} \\ &= \tau - \{k\} + k(f(t)-1) \times \{k-1\} \end{aligned}$$

$$L_{\tau,f}(t) \leqslant \max_{k\in\tau} \left\{ 1 + L_{\tau_{\langle k,t,f\rangle},f}(t+1) \right\}$$

A Bounding Function

$$M_{\tau,f}(t) \stackrel{\text{def}}{=} \max_{k\in\tau}\{1 + M_{\tau_{\langle k,t,f\rangle},f}(t+1)\}\,.$$

Then for all τ and t

$$L_{\tau,f}(t) \leqslant M_{\tau,f}(t)$$

Proposition (Maximizing Strategy)

For all $\tau \neq \emptyset$, $t \geqslant 0$, $M_{\tau,f}(t) = 1 + M_{\tau_{\langle \min\tau,t,f\rangle},f}(t+1)$.

A Bounding Function

Proposition (Maximizing Strategy)

For all $\tau \neq \emptyset$, $t \geqslant 0$, $M_{\tau,f}(t) = 1 + M_{\tau_{\langle \min \tau, t, f\rangle}, f}(t+1)$.

Example

$t = 1$, $f(x+t) = x+t$, $\tau = \{\!|2, 1|\!\}$: compare

$$\langle 0,0\rangle, \langle 1\rangle, \langle 0\rangle$$

$$\langle 0\rangle, \langle 1,1\rangle, \langle 1,0\rangle, \langle 0,3\rangle, \langle 0,2\rangle, \langle 0,1\rangle, \langle 0,0\rangle$$

Complexity Results

Proposition

Let $k, r \geqslant 1$ *be natural numbers and* $\gamma \geqslant 1$ *an ordinal. If* f *is a monotone unary function of* $\mathfrak{F}_\gamma$ *with* $f(x) \geqslant \max(1, x)$ *for all* x*, then* $M_{r\times\{k\},f}$ *is in* $\mathfrak{F}_{\gamma+k-1}$.

Proposition

Let $k, r \geqslant 1$ *be natural numbers and* $\gamma \geqslant 0$ *an ordinal with* $\gamma + k \geqslant 3$*. Then* $L_{r\times\{k\},F_\gamma}$ *is bounded below by a function which is* ***not*** *in* $\mathfrak{F}_{\gamma+k-2}$.

Fast Growing Hierarchy: $(F_\alpha)_\alpha$

Hierarchy of functions $(F_\alpha)_\alpha$ indexed by ordinals; we only need the *finite* fragment.

$$F_0(x) \stackrel{\text{def}}{=} x + 1$$

$$F_{n+1}(x) \stackrel{\text{def}}{=} F_n^{x+1}(x)$$

$$F_1(x) = 2x + 1$$

$$F_2(x) = (x+1) \cdot 2^{x+1} - 1$$

F_3 is non elementary

$F_\omega \stackrel{\text{def}}{=} \lambda x.F_x(x)$ is non primitive-recursive

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Fast Growing Hierarchy: $(\mathfrak{F}_\alpha)_\alpha$

Hierarchy of *classes* of functions: $\mathfrak{F}_\alpha$ is the closure of $\{\lambda x.0, \lambda xy.x + y, \lambda \bar{x}.x_i\} \cup \{F_\beta \mid \beta \leqslant \alpha\}$ under

substitution if $h_0, h_1, \ldots, h_n \in \mathfrak{F}_\alpha$ then so does f if

$$f(x_1, \ldots, x_n) = h_0(h_1(x_1, \ldots, x_n), \ldots, h_n(x_1, \ldots, x_n))$$

limited recursion if $h_1, h_2, h_3 \in \mathfrak{F}_\alpha$ then so does f if

$$\begin{aligned} f(0, x_1, \ldots, x_n) &= h_1(x_1, \ldots, x_n) \\ f(y+1, x_1, \ldots, x_n) &= h_2(y, x_1, \ldots, x_n, f(y, x_1, \ldots, x_n)) \\ f(y, x_1, \ldots, x_n) &\leqslant h_3(y, x_1, \ldots, x_n) \end{aligned}$$

Fast Growing Hierarchy: $(\mathfrak{F}_\alpha)_\alpha$

- for $k \geqslant 1$, $\mathfrak{F}_k \subsetneq \mathfrak{F}_{k+1}$, because $F_{k+1} \notin \mathfrak{F}_k$
- $\mathfrak{F}_0 = \mathfrak{F}_1$: linear functions, like $\lambda x.x + 3$ or $\lambda x.2x$,
- $\mathfrak{F}_2$: elementary functions, like $\lambda x.2^{2^x}$,
- $\mathfrak{F}_3$: tetration functions, like $\lambda x.\ \underbrace{2^{2^{\cdot^{\cdot^{\cdot^2}}}}}_{x \text{ times}}$, etc.
- $\bigcup_k \mathfrak{F}_k$: primitive-recursive functions

Affine Counter Systems

- $\mathcal{C} = \langle Q, k, \delta, m_0 \rangle$
- transitions (q, g, q') where $g(x) = Ax + B$ an affine function, $A \in \mathbb{N}^{k \times k}$, $B \in \mathbb{Z}^k$
- $m_0 \in \mathbb{N}^k$
- generalize reset/transfer Petri nets, broadcast protocols,...

Termination for ACS

Given $\langle \mathcal{C} \rangle$ a k-ACS, does every run of $\mathcal{C}$ terminate?

- exponential control in $\mathfrak{F}_2$
- $t < |m_0| < |\mathcal{C}|$
- upper bound: $\mathfrak{F}_{k+1}$
- lower bound: $\mathfrak{F}_{k-O(1)}$ (Schnoebelen, 2010)
- if k is not fixed, non-primitive recursive, with an upper bound in $\mathfrak{F}_\omega$

Concluding Remarks

- practical applications of wqo's yield upper bounds!
- out-of-the-box upper bounds
- "essentially" matching lower bounds for decision problems on monotone counter systems (lossy counter systems, reset or transfer Petri nets)
- the future: Higman's Lemma

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References: Upper Bounds for WQO

Dickson's Lemma

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Lower Bound

Specific sequence, bad for $(\mathbb{N}^k, \leqslant_{\text{lex}})$, of length $\ell_{k,f}(t)$.

Example

$k = 2, t = 1, f(x) = x + 3$:

i	0	1	2	3	4	5	$\cdots$	10	11	12	13	$\cdots$	26	27	28	29	$\cdots$	58	59
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$$
\begin{aligned}
5 &= 1 + 4 &&= 1 + \ell_{1,f}(1) \\
13 &= 5 + 8 &&= 5 + \ell_{1,f}(5) \\
29 &= 16 + 13 &&= 13 + \ell_{1,f}(13)
\end{aligned}
$$

Lower Bound

Specific sequence, bad for $(\mathbb{N}^k, \leqslant_{\text{lex}})$, of length $\ell_{k,f}(t)$.
In general, on the $k+1$th coordinate:

$$\underbrace{f(t)-1\; f(t)-1\; \cdots\; f(t)-1}_{\ell_{k,f}(t)\text{ times}} \quad \underbrace{f(t)-2,\, f(t)-2,\, \cdots,\, f(t)-2}_{\ell_{k,f}(o_{k,f}(t))\text{ times}}$$

$$\cdots \qquad \underbrace{0,\, 0,\, \cdots,\, 0}_{\ell_{k,f}\left(o_{k,f}^{f(t)-1}(t)\right)\text{ times}}$$

$$o_{k,f}(t) \stackrel{\text{def}}{=} t + \ell_{k,f}(t)$$

$$\ell_{k+1,f}(t) = \sum_{j=1}^{f(t)} \ell_{k,f}\left(o_{k,f}^{j-1}(t)\right)$$

Lower Bound

Specific sequence, bad for $(\mathbb{N}^k, \leqslant_{\mathrm{lex}})$, of length $\ell_{k,f}(t)$. One can have $\ell_{k,f}(t) < L(\{k\}, t)$: let $f(x) = 2x$ and $t = 1$,

$$\langle 1,1\rangle, \langle 1,0\rangle, \langle 0,5\rangle, \langle 0,4\rangle, \langle 0,3\rangle, \langle 0,2\rangle, \langle 0,1\rangle, \langle 0,0\rangle$$
$$\langle 1,1\rangle, \langle 0,3\rangle, \langle 0,2\rangle, \langle 0,1\rangle, \langle 9,0\rangle, \langle 8,0\rangle, \langle 7,0\rangle, \langle 6,0\rangle, \langle 5,0\rangle, \ldots, \langle 0,0\rangle$$

$$\ell_{2,f}(1) = 8 \qquad\qquad L_{\{2\},f}(1) \geqslant 14$$