Algorithmic Theory of WQOs

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LSV, ENS Cachan & CNRS

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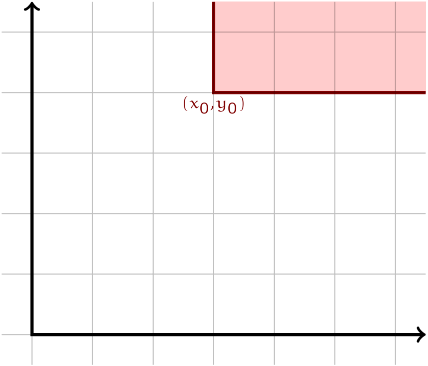
A One-Player Game

- over $\mathbb{Q}_{\geqslant 0} \times \mathbb{Q}_{\geqslant 0}$
- given initially (x_0, y_0)
- Eloise plays (x_j, y_j) s.t. $\forall 0 \leqslant i < j, x_i > x_j$ or $y_i > x_j$



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- If no, how long can she last?

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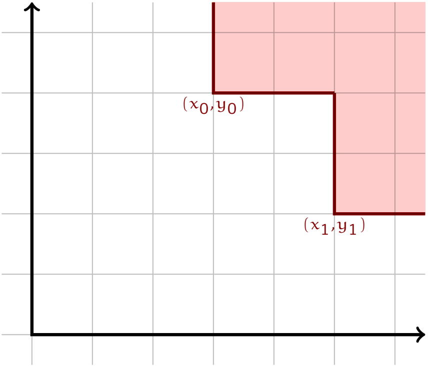

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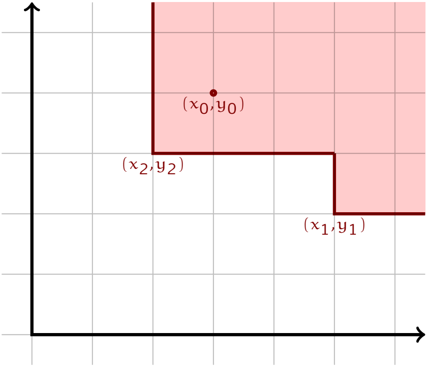

If $(x_0, y_0) \neq (0,0)$, then choosing $(x_j, y_j) = (\frac{x_0}{2^j}, \frac{y_0}{2^j})$ wins.

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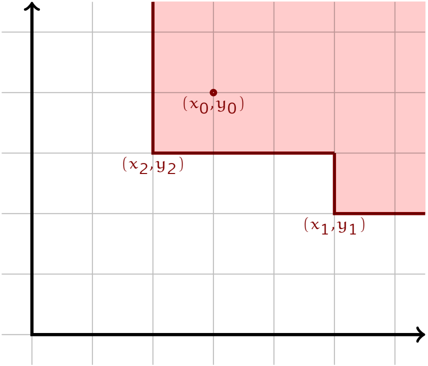

- Can Eloise win, i.e. play indefinitely?
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Assume there exists an infinite sequence $(x_j, y_j)_j$ of moves over $\mathbb{N}^2$. Consider the pairs of indices $i < j$: color (i, j)

- purple if $x_i > x_j$ but $y_i \leqslant y_j$,
- red if $x_i > x_j$ and $y_i > y_j$,
- orange if $y_i > y_j$ but $x_i \leqslant x_j$.

$$(3,4) \qquad (5,2) \qquad (2,3) \qquad \cdots$$

By the infinite Ramsey Theorem, there exists an infinite monochromatic subset of indices. In all cases, it implies the existence of an infinite decreasing sequence in $\mathbb{N}$, a contradiction.

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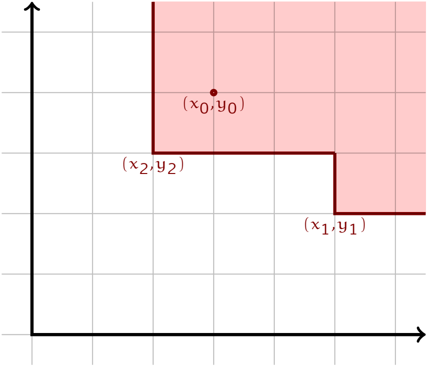

- Can Eloise win, i.e. play indefinitely?
- If no, how long can she last?

- if $(x_0, y_0) = (0,0)$, 0 turns
- otherwise, an arbitrary number of turns N: if $x_0 > 0$:

$$(x_0, y_0), (0, N-1), (0, N-2), \ldots, (0,1), (0,0)$$

Well Quasi Orderings

Definition (wqo)

A wqo is a quasi-order $(A, \leqslant)$ s.t.

$$\forall \mathbf{x} = x_0, x_1, x_2, \ldots \in A^\omega, \exists i_1 < i_2, x_{i_1} \leqslant x_{i_2} .$$

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Example (Basic wqos)

- $(\mathbb{N}, \leqslant)$,
- $(\{0, 1, \ldots, k\}, \leqslant)$ for any $k \in \mathbb{N}$,
- $(\Gamma_p, =)$ for any finite set Γ_p with p elements.

Dickson's Lemma

Lemma

If $(A_1, \leqslant_{A_1})$ *and* $(A_2, \leqslant_{A_2})$ *are two wqos, then* $(A_1 \times A_2, \leqslant_\times)$ *is a wqo,*

where $\leqslant_\times$ *is the product ordering:*

$$\langle a_1, a_2 \rangle \leqslant_\times \langle b_1, b_2 \rangle \overset{\text{def}}{\Leftrightarrow} a_1 \leqslant_{A_1} b_1 \wedge a_2 \leqslant_{A_2} b_2 \,.$$

Example

$(\mathbb{N}^k, \leqslant)$ using the product ordering

Outline

well quasi orderings (wqo)

generic tools for termination arguments

but also

beyond termination: complexity bounds

contents

WQO Algorithms
Length Function Theorems
A Quick Survey

A Rich Theory

- multiple equivalent definitions
- algebraic constructions

A Rich Theory

- multiple equivalent definitions: $(A, \leqslant)$ wqo iff
  - $\leqslant$ is well-founded and has no infinite antichains,
  - every linearization of $\leqslant$ is well-founded,
  - $\leqslant$ has the Ascending Chain Condition,
  - if $\mathbf{x} = x_0 x_1 \cdots \in A^\omega$, then there exists an infinite sequence $i_0 < i_1 < \cdots$ with $x_{i_0} \leqslant x_{i_1} \leqslant \cdots$,
  - etc.
- algebraic constructions

A Rich Theory

- multiple equivalent definitions
- algebraic constructions
  - cartesian products (Dickson's Lemma),
  - finite sequences (Higman's Lemma),
  - disjoint sums,
  - finite sets,
  - finite trees (Kruskal's Tree Theorem),
  - graphs with minors (Robertson and Seymour's Theorem), etc.

Higman's Lemma

Lemma

If $(A, \leqslant)$ is a wqo, then $(A^*, \leqslant_*)$ is a wqo where $\leqslant_*$ is the subword embedding ordering:

$$a_1 \cdots a_m \leqslant_* b_1 \cdots b_n \overset{\text{def}}{\Leftrightarrow} \begin{cases} \exists 1 \leqslant i_1 < \cdots < i_m \leqslant n, \\ \bigwedge_{j=1}^{m} a_j \leqslant_A b_{i_j} \ . \end{cases}$$

Example

$$aba \leqslant_* baaacabbab$$

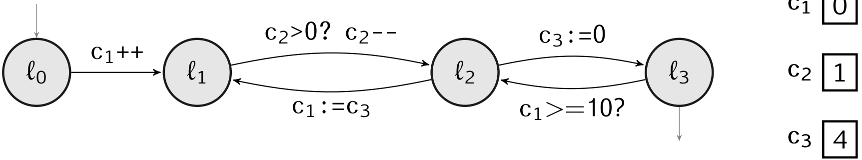
WQOs for Termination

Bad Sequences

- $\mathbf{x} = x_0, x_1, \ldots$ in A^∞ is a good sequence if $\exists i_1 < i_2, x_{i_1} \leqslant x_{i_2}$,
- a bad sequence otherwise,
- if $(A, \leqslant)$ is a wqo: every bad sequence is finite

Example

Monotonic Counter Systems



- A run of M:

$$(\ell_0,0,1,4) \to (\ell_1,1,1,4) \to (\ell_2,1,0,4) \to (\ell_3,1,0,0)$$

- ordering configurations: $(\ell_1,0,0,0) \leqslant (\ell_1,0,1,2)$ but $(\ell_1,0,0,0) \not\leqslant (\ell_2,0,1,2)$

- a wqo as a product of wqos: $(Loc,=) \times (\mathbb{N}^3,\leqslant_\times)$

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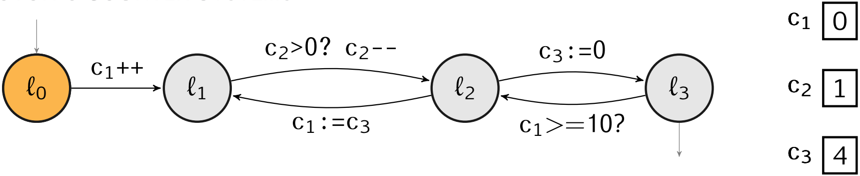

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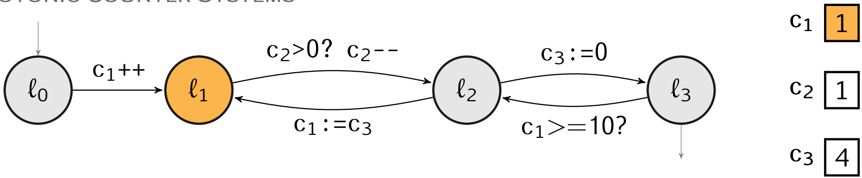

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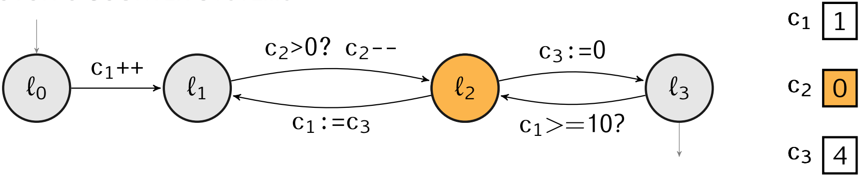

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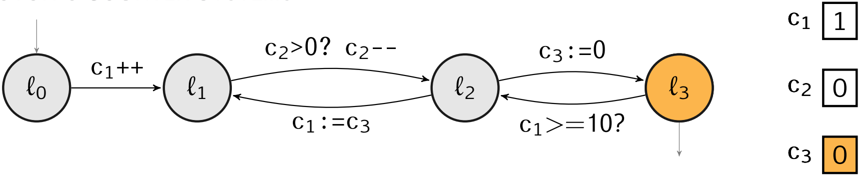

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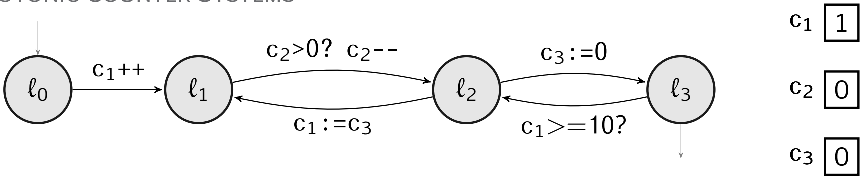

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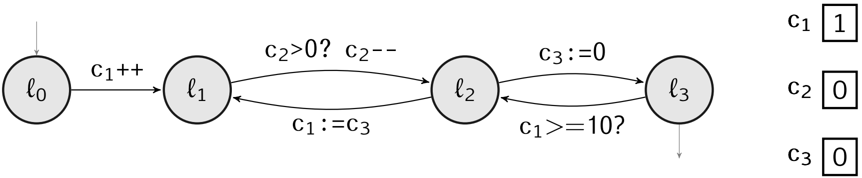

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- a wqo as a product of wqos: $(Loc,=) \times (\mathbb{N}^3,\leqslant_\times)$
- compatibility between configurations: if $c \leqslant d$ and $c \to c'$, then $\exists d' \geqslant c'$, $d \to d'$

Example

Deciding whether a MCS Terminates

input: a MCS and an initial configuration c_0 in $Loc \times \mathbb{N}^k$

question: are all the runs starting from c_0 finite?

- termination is semi-decidable: explore all runs
- non-termination is semi-decidable: finite witness

$$
\begin{tikzcd}
c_0 \arrow[r, dotted] & c \arrow[r, dotted] \arrow[r, dashed, bend right, "\leqslant"'] & c'
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- generalizes to many classes of systems: well-structured transition systems (WSTS)

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From Termination to Complexity

- wqos for termination: bad sequences are finite
- ... but how long can they be?

An Example

SIMPLE (a,b)
$c \longleftarrow 1$
while $a>0 \wedge b>0$
  $\langle a,b,c\rangle \longleftarrow \langle a-1,b,2c\rangle$
 or
  $\langle a,b,c\rangle \longleftarrow \langle 2c,b-1,1\rangle$
end

- in any run, $\langle a_0,b_0\rangle,\ldots,\langle a_n,b_n\rangle$ is a bad sequence over $(\mathbb{N}^2,\leqslant_\times)$,
- $(\mathbb{N}^2,\leqslant_\times)$ is a wqo: all the runs are finite
- How long can SIMPLE run?

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A Computation of simple$(2,3)$

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$\langle a,b,c\rangle$	loop iterations
$\langle 2,3,2^0\rangle$	0

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$\langle 2^2,2,2^0\rangle$	2

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$\vdots$	$\vdots$
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$\vdots$	$\vdots$
$\langle 1,2,2^{2^2-1}\rangle$	$2+2^2-1$

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$\langle 2^{2^2}, 1, 1 \rangle$	$2 + 2^2$
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- non-elementary complexity
- derive (matching) upper bounds for termination arguments based on $(\mathbb{N}^2,\leqslant_\times)$ being a wqo

Controlled Sequences

- bound the length of bad sequences over $(A, \leqslant)$

Controlled Sequences

- bound the length of bad sequences over $(A, \leqslant)$
- recall our game: choose any N, and consider the bad sequence $\langle 3,3\rangle, \langle 3,2\rangle, \langle 3,1\rangle, \langle 3,0\rangle, \langle 2,N\rangle, \langle 2,N-1\rangle, \ldots$

Controlled Sequences

- bound the length of bad sequences over $(A, \leqslant\, ; |.|_A)$
- associate a norm function $|.|_A : A \to \mathbb{N}$ to each wqo $(A, \leqslant)$
- assume $|.|_A$ is proper $\overset{\text{def}}{\Leftrightarrow}$ for all n

$$A_{<n} \overset{\text{def}}{=} \{x \in A \mid |x|_A < n\} \text{ is finite}$$

Example (Normed wqos)

$$|k|_{\mathbb{N}} \overset{\text{def}}{=} k \qquad |a_i|_{\Gamma_p} \overset{\text{def}}{=} 0 \qquad |\langle a, b\rangle|_{A\times B} \overset{\text{def}}{=} \max(|a|_A, |b|_B)$$

Controlled Sequences

- bound the length of controlled bad sequences over $(A, \leqslant; |.|_A)$
- fix a control function $g : \mathbb{N} \to \mathbb{N}$ (strictly increasing)
- $\mathbf{x} = x_0, x_1, \ldots$ over A is (g, n)-controlled iff

$$\forall i,\ |x_i|_A < g^i(n)$$

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$$\forall i, |x_i|_A < g^i(n)$$

Example (SIMPLE(2,3))

$A = \mathbb{N}^2, n = 4, g(x) = 2x$

Controlled Sequences

- bound the length of controlled bad sequences over $(A, \leqslant; |.|_A)$
- for fixed A, g, n, there are finitely many bad (g, n)-controlled sequences over A
- maximal length function

$$L_{A,g}(n)$$

Length Function Theorems

Bound $L_{A,g}$ by some functions for various A.

Example

- for $(\Gamma_p, =; |.|_{\Gamma_p})$:
- for $(\mathbb{N}, \leqslant; |.|_{\mathbb{N}})$:
- recall SIMPLE: not every length function is that 'small'…

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- recall SIMPLE: not every length function is that 'small'...

Length Function Theorems

Bound $L_{A,g}$ by some functions for various A.

Example

- for $(\Gamma_p, =; |.|_{\Gamma_p})$: $L_{\Gamma_p,g}(n) = p$
- for $(\mathbb{N}, \leqslant; |.|_{\mathbb{N}})$: $L_{\mathbb{N},g}(n) = n$:

$$n-1, n-2, \ldots, 1, 0$$

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Length Function Theorems

Bound $L_{A,g}$ by some functions for various A.

Example

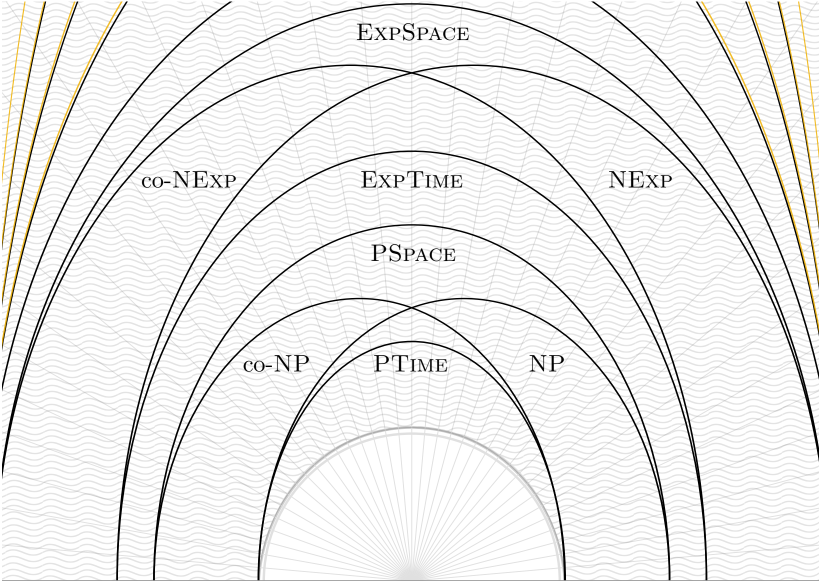
- for $(\mathbb{N}^k, \leqslant; |.|_{\mathbb{N}^k})$: let g be in $\mathfrak{F}\mathbf{F}_\gamma$ for $\gamma \geqslant 1$, then $L_{\mathbb{N}^k,g}(n)$ is bounded by a function in $\mathfrak{F}\mathbf{F}_{\gamma+k}$
- for $(\Gamma_p^*, \leqslant_*; |.|_{\Gamma_p^*})$: let g be primitive-recursive and $p \geqslant 2$, then $L_{\Gamma_p^*,g}(n)$ is bounded by a function in $\mathfrak{F}\mathbf{F}_{\omega^{p-1}}$
- more on $\mathfrak{F}\mathbf{F}_\alpha$ soon...

Using the Length Function

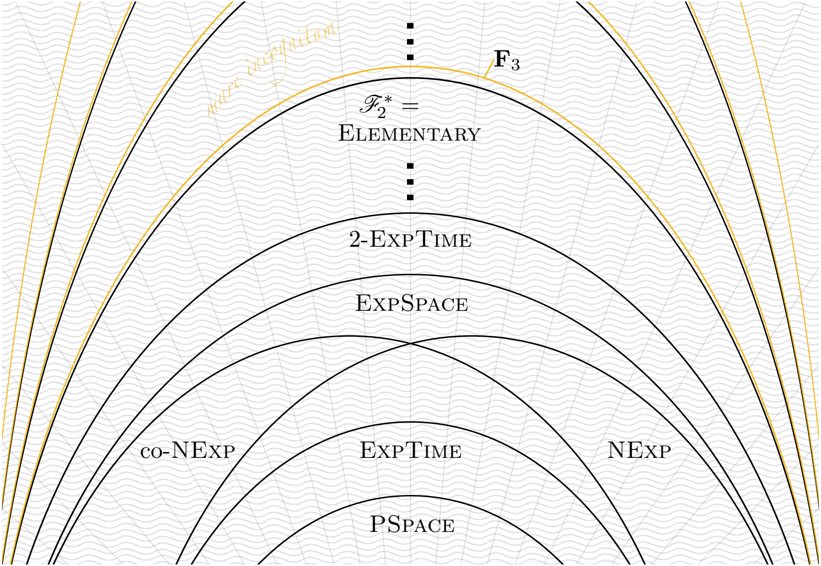
Example (MCS Termination)

- control function $g(n) = n + 1$ in $\mathfrak{F}\mathbf{F}_1$
- any run of length $L_{Loc \times \mathbb{N}^k, g}(n_0) + 1$ is good
- new algorithm for termination: try to find such a run
- complexity in $\mathbf{F}_{k+2}$ when k is fixed, $\mathbf{F}_\omega$ when not

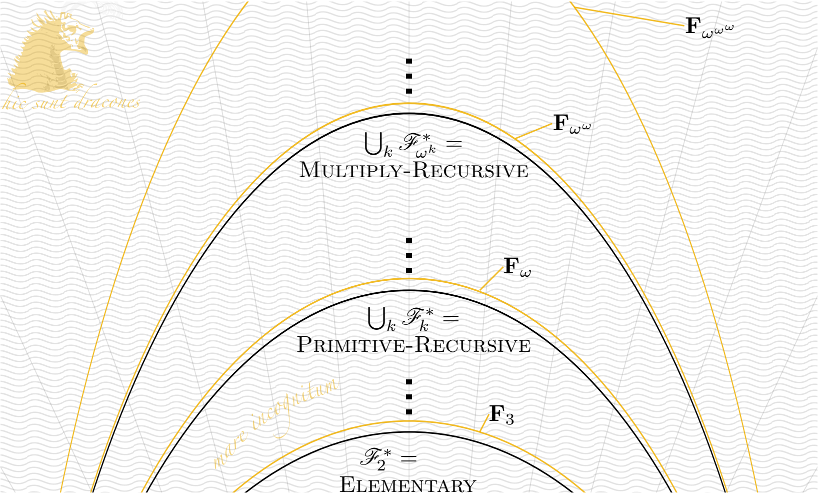
Beyond Elementary



Beyond Elementary



Beyond Elementary



Fast-Growing Functions

(Löb and Wainer, 1970)

$$F_0(x) \stackrel{\text{def}}{=} x+1, \quad F_{\alpha+1}(x) \stackrel{\text{def}}{=} F_\alpha^x(x), \quad F_\lambda \stackrel{\text{def}}{=} F_{\lambda_x}(x).$$

where λ_x is the xth element of a fundamental sequence $(\lambda_x)_x$ for the limit ordinal λ

Example

$$F_1(x) = 2x$$

$$F_2(x) = x \cdot 2^x$$

F_3 is non elementary

F_ω is non primitive-recursive

F_{ω^ω} is non multiply-recursive

Fast-Growing Complexities

$$\alpha \geqslant 2: \qquad \mathscr{F}_\alpha \stackrel{\text{def}}{=} \bigcup_{c<\omega} \mathrm{FSpace}\big(F_\alpha^c(n)\big)$$

$$\alpha \geqslant 3: \qquad \mathbf{F}_\alpha \stackrel{\text{def}}{=} \bigcup_{p \in \bigcup_{\beta<\alpha} \mathscr{F}_\beta} \mathrm{Space}\big(F_\alpha(p(n))\big)$$

Example

$\mathbf{F}_3$ complexity bounded by a tower of exponentials of elementary height,

$\mathbf{F}_\omega$ Ackermannian complexity of some primitive-recursive function,

Some $\mathbf{F}_\omega$-Complete Problems

Decision of problems on

- monotonic counter systems (Finkel and Schnoebelen, 2001), e.g.
  - finite VASS containment (Mayr and Meyer, 1981; Jančar, 2001)
  - lossy counter systems termination (Schnoebelen, 2010),
- relevance logics (Urquhart, 1999),
- data logics (Demri and Lazić, 2009; Figueira and Segoufin, 2009), ...

Some $\mathbf{F}_{\omega^\omega}$-Complete Problems

Decision of problems on

- lossy channel systems (Chambart and Schnoebelen, 2008),
- Post embedding problem RatEP (Chambart and Schnoebelen, 2007),
- 1-clock alternating timed automata (Lasota and Walukiewicz, 2008),
- Metric temporal logic (Ouaknine and Worrell, 2007),
- finite concurrent programs under weak (TSO/PSO) memory models (Atig et al., 2010)
- alternating register automata over ordered domains (Figueira et al., 2010),...

Summary

- wqos for termination of algorithms
- length function theorems: out-of-the-box complexity upper bounds
- matching lower bounds for many problems

Course material at

http://www.lsv.ens-cachan.fr/~schmitz/teach/2012_esslli/

Perspectives

Some big challenges ahead, for instance:

- complexity of VASS reachability
- decidability of BVASS reachability

Also easier problems: internships within ANR ReacHard:

http://www.lsv.ens-cachan.fr/projects/anr-reachard

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