T-Bounded WSTS

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Well Structured Transition Systems

\[ S = \langle S, s_0, \Sigma, \rightarrow, \leq \rangle: \]

\( \langle S, s_0, \Sigma, \rightarrow \rangle \) a LTS with labels in \( \Sigma \), states \( S \), initial state \( s_0 \), transitions \( \rightarrow \subseteq S \times S \) for \( a \in \Sigma \)

\( \leq \) a wqo on \( S \):

\[ \forall s_0 \cdots s_i \cdots \in S^\omega, \exists i < j \in \mathbb{N}, s_i \leq s_j \]

\( \rightarrow \) monotonic wrt. \( \leq \):

\[ \forall s_1, s_2, s_3 \in S, \forall a \in \Sigma, s_1 \leq s_2 \land s_1 \xrightarrow{a} s_3 \]

implies \( \exists s_4 \geq s_3 \in S, s_2 \xrightarrow{a} s_4 \)
WSTS Everywhere!

- Petri nets
- Reset Petri nets
- Lossy channel systems
- ...
Decidability

- Generic backward algorithm for coverability for (effective) WSTS
- Also for language emptiness if one uses an upward-closed final set of states
- But undecidable liveness already for reset Petri nets and lossy channel systems
**T-Bounded WSTS**

A WSTS is \( T \)-bounded if its trace set

\[
T(S) = \{ w \in \Sigma^* \mid \exists s \in S, s_0 \xrightarrow{w} s \}
\]

is a bounded language:

**Definition (Ginsburg and Spanier, 1964)**

A language \( L \subseteq \Sigma^* \) is bounded if there exists \( n \in \mathbb{N} \) and \( n \) words \( w_1, \ldots, w_n \) in \( \Sigma^* \) such that \( L \subseteq w_1^* \cdots w_n^* \). The regular expression \( w_1^* \cdots w_n^* \) is then called a bounded expression for \( L \).
T-boundedness is **decidable** for (some) WSTS

- T-boundedness is **undecidable** for nondeterministic WSTS (labeled reset Petri nets)
- T-boundedness is **undecidable** for deterministic LTS (2-counter Minsky machines)
- Post* flattability is **undecidable** for deterministic WSTS (functional LCS)
- T-boundedness is **not multiply recursive** (functional LCS)
- T-boundedness is ExpSPACE-hard for Petri nets
- the smallest bounded expression can be of **non primitive recursive size** for Petri nets

ω-regular properties are **decidable** for (some) T-bounded WSTS
T-boundedness is **decidable** for $\omega$-effective complete deterministic WSTS

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$\omega$-regular properties are **decidable** for T-bounded $\omega$-effective complete deterministic WSTS
- T-boundedness is **decidable** for $\infty$-effective complete deterministic WSTS
  - T-boundedness is **undecidable** for nondeterministic WSTS (labeled reset Petri nets)
  - T-boundedness is **undecidable** for deterministic LTS (2-counter Minsky machines)
  - **Post* flattability** is undecidable for deterministic WSTS (functional LCS)
  - T-boundedness is not multiply recursive (functional LCS)
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- $\omega$-regular properties are **decidable** for T-bounded $\infty$-effective complete deterministic WSTS
- T-boundedness is **decidable** for \( \infty \)-effective complete deterministic WSTS
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  - T-boundedness is undecidable for **deterministic LTS** (2-counter Minsky machines)
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  - T-boundedness is \( \text{ExpSpace} \)-hard for Petri nets
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- \( \omega \)-regular properties are **decidable** for T-bounded \( \infty \)-effective complete deterministic WSTS
... and for free:

- reachability is undecidable for det. \( T \)-bounded WSTS (Cortier 2002) (affine counter systems)
- effective computation of the cover for det. \( T \)-bounded WSTS (Finkel and Goubault-Larrecq 2009)
- reachability, and CTL\(^*\)+Presburger counting model checking, decidable on Presburger accelerable well-structured counter systems (Demri et al. 2006)
- regularity and trace inclusion for the same class
Complete Deterministic WSTS

- \((S, \leq)\) is a continuous dcpo
- each transition function \(\xrightarrow{a}\) is a partial continuous map \(f\):
  - monotonic
  - \textit{open} domain \(\text{dom} f\):
    - upward-closed
    - \(\forall D \text{ directed with } \text{lub}(D) \in \text{dom} f, D \cap \text{dom} f \neq \emptyset\)
  - \(\forall D \text{ directed } \subseteq \text{dom} f, \text{lub}(f(D)) = f(\text{lub}(D))\)
Accelerations

- the lub-acceleration $f^\omega$ of $f$:
  \[
  \text{dom } f^\omega = \{ s \in \text{dom } f \mid s \leq f(s) \}
  \]
  \[
  f^\omega(s) = \text{lub}(\{f^n(s) \mid n \in \mathbb{N}\}) \quad \text{for } s \text{ in } \text{dom } f^\omega
  \]

- a comp. det. WSTS is $\infty$-effective if $u^\omega \rightarrow$ is computable for every $u$ in $\Sigma^+$

- an accelerated word is a sequence of form
  \[
  w = v_0 u_1^\omega v_1 u_2^\omega v_2 \cdots u_n^\omega v_n
  \]
  for some $n \in \mathbb{N}$, $v_i \in \Sigma^*$, $u_i \in \Sigma^+$

- note $w \rightarrow \infty$; the accelerated trace set is then
  \[
  T_{\text{acc}}(S) = \{ w \in \Sigma^{<\omega^2} \mid \exists s \in S, s_0 \overset{w}{\rightarrow} \infty s \}
Decidability

1. find a witness for T-boundedness
2. find a witness for T-unboundedness
Witness for T-Boundedness

Enumerate bounded expressions $E = \omega_1^* \cdots \omega_n^*$: $T(S) \subseteq L(E)$ is decidable:

1. $L(E)$ is regular
2. compute a DFA for $\Sigma^* \setminus L(E)$
3. intersect with $S$: this is a WSTS with an upward-closed set of final states
4. language emptiness is decidable for such WSTS
**Witness for T-Unboundedness**

Explore *accelerated runs* of $S$ in search of an *increasing fork*:

![Diagram](https://example.com/diagram.png)

**Definition**

A comp. WSTS has an *increasing fork* if there exist $a \neq b$ in $\Sigma$, $u$ in $\Sigma^{<\omega^2}$, $v$ in $\Sigma^*$, and $s$, $s_a \geq s$, $s_b \geq s$ in $S$ such that $s_0 \rightarrow \infty s$, $s \xrightarrow{au} \infty s_a$, and $s \xrightarrow{bv} s_b$. 
Witness for $T$-Unboundedness

Example

\[
T(N(1, 0, 0)) = a^* \cup a^n b\{c, d\} \leq n
\]
T-Unboundedness ⇒ Fork

Lemma
Let $L \subseteq \Sigma^*$ be an unbounded language. There exists $a$ in $\Sigma$ such that $a^{-1}L$ is also unbounded.

Definition
Let be $L \subseteq \Sigma^*$ and $w \in \Sigma^+$. The removal of $w$ from $L$ is the language $\overline{w}L = (w^*)^{-1}L \setminus w\Sigma^*$.

Lemma
If a det. comp. WSTS $S$ has an unbounded $L \subseteq T(S)$, then there are two words $v$ in $\Sigma^*$ and $u$ in $\Sigma^+$ such that $vu^\omega \in T_{\text{acc}}(S)$, $vu \in \text{Pref}(L)$ and $\overline{u}(v^{-1}L)$ is also unbounded.
\( T \)-Unboundedness \( \Rightarrow \) Fork

Define \((v_i, u_i)_{i \geq 0}, (L_i)_{i \geq 0}\) with \(L_0 = T(S)\), and \((s_i)_{i \geq 0}\):

- \(|v_{i+1}.u_{i+1}| \geq |u_i|
- s_i \xrightarrow{v_{i+1}u_{i+1}} s_{i+1}
- L_{i+1} = \overline{u_{i+1}}(v_{i+1}^{-1}L_i)\) is unbounded

then

\[ \overline{u_{i+1}}(v_{i+1}^{-1}L_i) \subseteq T(S(s_{i+1})) \]
T-Unboundedness $\Rightarrow$ Fork

1. $\exists i < j, s_i \leq s_j$
2. $u_i$ is not a prefix of $v_{i+1}u_{i+1}$ and $|v_{i+1}u_{i+1}| \geq |u_i|$
3. $\exists a \neq b \in \Sigma, \exists x \in \Sigma^*$,

$$u_i = xby$$
$$v_{i+1}u_{i+1} = xaz$$
Fork $\Rightarrow$ T-Unboundedness

Lemma (Continuity)

Let $S$ be a det. comp. WSTS and $n \geq 0$. If

$$w_n = v_{n+1}u_0^\omega v_n \cdots u_1^\omega v_1 \in T_{acc}(S)$$

with the $u_i$ in $\Sigma^+$ and the $v_i$ in $\Sigma^*$, then there exist $k_1, \ldots, k_n$ in $\mathbb{N}$, such that

$$w'_n = v_{n+1}u_n^{k_n}v_n \cdots u_1^{k_1}v_1 \in T(S).$$

Proof by induction. For $n = 0$, $w_0 = v_{n+1} \in T(S)$.
Fork $\Rightarrow$ T-Unboundedness

\[ s_0 \xrightarrow{v_{n+1}u_n^\omega} s \xrightarrow{w_{n-1} = v_n u_{n-1}^\omega v_{n-1} \cdots u_1^\omega v_1} s_f \]

\( w_{n-1} \in T_{\text{acc}}(S(s)) : \exists k_1, \ldots, k_{n-1} \)

\( w'_{n-1} = v_n u_{n-1}^{k_{n-1}} v_{n-1} \cdots u_1^{k_1} v_1 \in T(S(s)) \)

1. \( \xrightarrow{w'_{n-1}} \) partial continuous

2. \( D\{s_m \mid s_0 \xrightarrow{v_0 u_{n}^m} s_m\} \) directed with \( s = \text{lub}(D) \)

3. \( \exists s' \in D \cap \text{dom} \xrightarrow{w'_{n-1}} \)

4. define \( k_n \) s.t. \( s_0 \xrightarrow{v_n u_n^{k_n}} s' \)
Fork $\Rightarrow$ $T$-Unboundedness

Let $S$ have an increasing fork, and suppose $T(S)$ is bounded. Then there exists a DFA $A = \langle Q, q_0, \Sigma, \delta, F \rangle$ s.t. $L(A) = w_1^* \cdots w_n^*$ and $T(S) \subseteq L(A)$. Set $N = |Q| + 1.$
Fork $\Rightarrow$ T-Unboundedness

Increasing fork and monotonicity:

$$\omega\{au, bv\}^* \subseteq T_{acc}(S)$$

In particular

$$\omega(bv)^N au(bv)^N au \cdots au(bv)^N \in T_{acc}(S) \quad (N \text{ times})$$

By the Continuity Lemma, there exist $\omega'$ and $(au_i)_{1 \leq i < N}$ in $\Sigma^+$ such that

$$\omega'(bv)^N au_1(bv)^N au_2 \cdots au_{N-1}(bv)^N \in T(S)$$
Fork $\Rightarrow$ $T$-Unboundedness

In $\mathcal{A}$, for each $(bv)^N$ factor, there exists a state $q_i$ s.t. $\delta(q_i, (bv)^{k_i}) = q_i$ for some $k_i > 0$:

$$q_0 \xrightarrow{w'(bv)^{N-k_1-k_1'}} q_1 \xrightarrow{(bv)^{k_1}} q_1$$

$$q_1 \xrightarrow{(bv)^{k_1}au_1(bv)^{N-k_2-k_2'}} q_2 \xrightarrow{(bv)^{k_2}} q_2$$

$$q_2 \xrightarrow{(bv)^{k_2}au_2\cdots au_{N-1}(bv)^{N-k_N-k_N'}} q_N \xrightarrow{(bv)^{k_N}} q_N \xrightarrow{(bv)^{k_N'}} q_f \in F$$

But there are $N$ such factors: $\exists i < j$ s.t. $q_i = q_j$. Then

$$\delta(q_i, (bv)^{k_i}au_i\cdots au_j(bv)^{N-k_j-k_j'}) = q_i$$

which contradicts $L(\mathcal{A})$ bounded.

$\square$
Questions?