T-Bounded WSTS

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Well Structured Transition Systems

$\mathcal{S} = \langle S, s_0, \Sigma, \rightarrow, \leqslant \rangle$:

- $\langle S, s_0, \Sigma, \rightarrow \rangle$ a LTS with labels in Σ, states S, initial state s_0, transitions $\xrightarrow{a} \subseteq S \times S$ for $a \in \Sigma$
- $\leqslant$ a wqo on S:

$$\forall s_0 \cdots s_i \cdots \in S^\omega, \exists i < j \in \mathbb{N}, s_i \leqslant s_j$$

- $\rightarrow$ monotonic wrt. $\leqslant$:

$$\forall s_1, s_2, s_3 \in S, \forall a \in \Sigma, s_1 \leqslant s_2 \wedge s_1 \xrightarrow{a} s_3$$
$$\text{implies } \exists s_4 \geqslant s_3 \in S, s_2 \xrightarrow{a} s_4$$

WSTS Everywhere!

- Petri nets
- Reset Petri nets
- Lossy channel systems
- ...

Decidability

- Generic backward algorithm for coverability for (effective) WSTS
- Also for language emptiness if one uses an upward-closed final set of states
- But undecidable liveness already for reset Petri nets and lossy channel systems

T-Bounded WSTS

A WSTS is T-*bounded* if its *trace set*

$$\mathsf{T}(\mathcal{S}) = \{w \in \Sigma^* \mid \exists s \in S, s_0 \xrightarrow{w} s\}$$

is a bounded language:

Definition (Ginsburg and Spanier, 1964)

A language $L \subseteq \Sigma^*$ is *bounded* if there exists $n \in \mathbb{N}$ and n words $w_1, \ldots, w_n$ in Σ^* such that $L \subseteq w_1^* \cdots w_n^*$. The regular expression $w_1^* \cdots w_n^*$ is then called a *bounded expression* for L.

- T-boundedness is decidable for (some) WSTS
  - T-boundedness is undecidable for nondeterministic WSTS (labeled reset Petri nets)
  - T-boundedness is undecidable for deterministic LTS (2-counter Minsky machines)
  - Post* flattability is undecidable for deterministic WSTS (functional LCS)
  - T-boundedness is not multiply recursive (functional LCS)
  - T-boundedness is ExpSpace-hard for Petri nets
  - the smallest bounded expression can be of non primitive recursive *size* for Petri nets
- ω-regular properties are decidable for (some) T-bounded WSTS

- T-boundedness is decidable for ∞-effective complete deterministic WSTS
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. . . and for free:

- reachability is undecidable for det. T-bounded WSTS (Cortier 2002) (affine counter systems)
- effective computation of the cover for det. T-bounded WSTS (Finkel and Goubault-Larrecq 2009)
- reachability, and CTL*+Presburger counting model checking, decidable on Presburger accelerable well-structured counter systems (Demri et al. 2006)
- regularity and trace inclusion for the same class

Complete Deterministic WSTS

- $(S, \leqslant)$ is a *continuous dcpo*
- each transition function $\xrightarrow{a}$ is a *partial continuous* map f:
  - monotonic
  - *open* domain $\mathsf{dom}\, f$:
    - upward-closed
    - $\forall D$ directed with $\mathsf{lub}(D) \in \mathsf{dom}\, f$, $D \cap \mathsf{dom}\, f \neq \emptyset$
  - $\forall D$ directed $\subseteq \mathsf{dom}\, f, \mathsf{lub}(f(D)) = f(\mathsf{lub}(D))$

Accelerations

- the *lub-acceleration* f^ω of f:

$$\begin{aligned} \mathsf{dom}\, f^\omega &= \{s \in \mathsf{dom}\, f \mid s \leqslant f(s)\} \\ f^\omega(s) &= \mathsf{lub}(\{f^n(s) \mid n \in \mathbb{N}\}) \qquad \text{for } s \text{ in } \mathsf{dom}\, f^\omega \end{aligned}$$

- a comp. det. WSTS is ∞-*effective* if $\xrightarrow{u^\omega}$ is computable for every u in Σ^+
- an *accelerated word* is a sequence of form

$$w = v_0 u_1^\omega v_1 u_2^\omega v_2 \cdots u_n^\omega v_n$$

for some $n \in \mathbb{N}$, $v_i \in \Sigma^*$, $u_i \in \Sigma^+$
- note $\xrightarrow{w}_\infty$; the *accelerated trace set* is then

$$T_{\mathsf{acc}}(\mathcal{S}) = \{w \in \Sigma^{<\omega^2} \mid \exists s \in S, s_0 \xrightarrow{w}_\infty s\}$$

Decidability

1. find a witness for T-boundedness
2. find a witness for T-unboundedness

Witness for T-Boundedness

Enumerate bounded expressions $E = w_1^* \cdots w_n^*$:
$T(\mathcal{S}) \subseteq L(E)$ is decidable:

1. $L(E)$ is regular
2. compute a DFA for $\Sigma^* \backslash L(E)$
3. intersect with $\mathcal{S}$: this is a WSTS with an upward-closed set of final states
4. language emptiness is decidable for such WSTS

Witness for T-*Un*boundedness

Explore *accelerated runs* of $\mathcal{S}$ in search of an *increasing fork*:

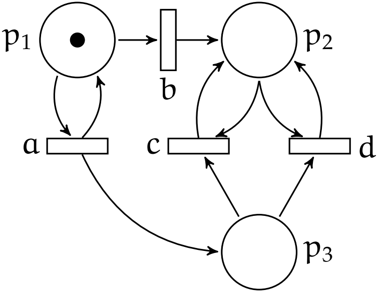
$$\begin{tikzcd}
 & & s_a \\
s_0 \arrow[r, "\infty" near end] & s \arrow[ur, "au", "\leqslant"', "\infty" near end] \arrow[dr, "bv", "\leqslant"'] & \\
 & & s_b
\end{tikzcd}$$

Definition

A comp. WSTS has an *increasing fork* if there exist $a \neq b$ in Σ, u in $\Sigma^{<\omega^2}$, v in Σ^*, and s, $s_a \geqslant s$, $s_b \geqslant s$ in S such that $s_0 \rightarrow^\infty s$, $s \xrightarrow{au}{}^{\infty} s_a$, and $s \xrightarrow{bv} s_b$.

Witness for T-*Un*boundedness

Example



$$T(\mathcal{N}(1,0,0)) = a^* \cup a^n b\{c,d\}^{\leqslant n}$$

T-Unboundedness $\Rightarrow$ Fork

Lemma

Let $L \subseteq \Sigma^*$ *be an unbounded language. There exists* a *in* Σ *such that* $a^{-1}L$ *is also unbounded.*

Definition

Let be $L \subseteq \Sigma^*$ and $w \in \Sigma^+$. The *removal* of w from L is the language $\overline{w}L = (w^*)^{-1}L \backslash w\Sigma^*$.

Lemma

If a det. comp. WSTS $\mathcal{S}$ *has an unbounded* $L \subseteq T(\mathcal{S})$*, then there are two words* v *in* Σ^* *and* u *in* Σ^+ *such that* $vu^\omega \in T_{\mathsf{acc}}(\mathcal{S}), vu \in \mathsf{Pref}(L)$ *and* $\overline{u}(v^{-1}L)$ *is also unbounded.*

T-Unboundedness $\Rightarrow$ Fork

Define $(v_i, u_i)_{i>0}$, $(L_i)_{i\geqslant 0}$ with $L_0 = T(\mathcal{S})$, and $(s_i)_{i\geqslant 0}$:

- $|v_{i+1}.u_{i+1}| \geqslant |u_i|$
- $s_i \xrightarrow{v_{i+1}u_{i+1}^{\omega}} s_{i+1}$
- $L_{i+1} = \overline{u_{i+1}}(v_{i+1}^{-1}L_i)$ is unbounded

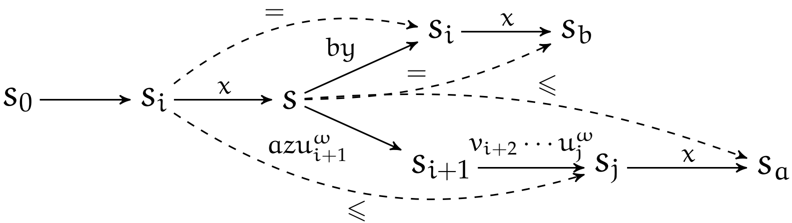
then

$$\overline{u_{i+1}}(v_{i+1}^{-1}L_i) \subseteq T(\mathcal{S}(s_{i+1}))$$

T-Unboundedness $\Rightarrow$ Fork

1. $\exists i < j, s_i \leqslant s_j$
2. u_i is not a prefix of $v_{i+1}u_{i+1}$ and $|v_{i+1}u_{i+1}| \geqslant |u_i|$
3. $\exists a \neq b \in \Sigma, \exists x \in \Sigma^*,$

$$u_i = xby \qquad\qquad v_{i+1}u_{i+1} = xaz$$



$\square$

Fork $\Rightarrow$ T-Unboundedness

Lemma (Continuity)

Let $\mathcal{S}$ be a det. comp. WSTS and $n \geqslant 0$. If

$$w_n = v_{n+1} u_n^{\omega} v_n \cdots u_1^{\omega} v_1 \in T_{acc}(\mathcal{S})$$

with the u_i in Σ^+ and the v_i in Σ^*, then there exist $k_1, \ldots, k_n$ in $\mathbb{N}$, such that

$$w'_n = v_{n+1} u_n^{k_n} v_n \cdots u_1^{k_1} v_1 \in T(\mathcal{S}) .$$

Proof by induction. For $n = 0$, $w_0 = v_{n+1} \in T(\mathcal{S})$.

Fork $\Rightarrow$ T-Unboundedness

$$s_0 \xrightarrow{v_{n+1}u_n^\omega} s \xrightarrow{w_{n-1}=v_n u_{n-1}^\omega v_{n-1}\cdots u_1^\omega v_1} s_f$$

$w_{n-1} \in T_{acc}(\mathcal{S}(s))$: $\exists k_1, \ldots, k_{n-1}$

$$w'_{n-1} = v_n u_{n-1}^{k_{n-1}} v_{n-1} \cdots u_1^{k_1} v_1 \in T(\mathcal{S}(s))$$

1. $\xrightarrow{w'_{n-1}}$ partial continuous
2. $D\{s_m \mid s_0 \xrightarrow{v_n u_n^m} s_m\}$ directed with $s = \mathsf{lub}(D)$
3. $\exists s' \in D \cap \mathsf{dom} \xrightarrow{w'_{n-1}}$
4. define k_n s.t. $s_0 \xrightarrow{v_n u_n^{k_n}} s'$ $\square$

Fork $\Rightarrow$ T-Unboundedness

Let $\mathcal{S}$ have an increasing fork, and suppose $\mathsf{T}(\mathcal{S})$ is bounded. Then there exists a DFA $\mathcal{A} = \langle Q, q_0, \Sigma, \delta, F \rangle$ s.t. $L(\mathcal{A}) = w_1^* \cdots w_n^*$ and $\mathsf{T}(\mathcal{S}) \subseteq L(\mathcal{A})$.
Set $N = |Q| + 1$.

Fork $\Rightarrow$ T-Unboundedness

Increasing fork and monotonicity:

$$w\{au, bv\}^* \subseteq T_{acc}(\mathcal{S})$$

In particular

$$w(bv)^N au(bv)^N au \cdots au(bv)^N \in T_{acc}(\mathcal{S}) \quad \text{(N times)}$$

By the Continuity Lemma, there exist w' and $(au_i)_{1 \leqslant i < N}$ in Σ^+ such that

$$w'(bv)^N au_1(bv)^N au_2 \cdots au_{N-1}(bv)^N \in T(\mathcal{S})$$

Fork $\Rightarrow$ T-Unboundedness

In $\mathcal{A}$, for each $(bv)^N$ factor, there exists a state q_i s.t. $\delta(q_i, (bv)^{k_i}) = q_i$ for some $k_i > 0$:

$$q_0 \xrightarrow{w'(bv)^{N-k_1-k'_1}} q_1 \xrightarrow{(bv)^{k_1}} q_1$$

$$q_1 \xrightarrow{(bv)^{k'_1} au_1 (bv)^{N-k_2-k'_2}} q_2 \xrightarrow{(bv)^{k_2}} q_2$$

$$q_2 \xrightarrow{(bv)^{k'_2} au_2 \cdots au_{N-1} (bv)^{N-k_N-k'_N}} q_N \xrightarrow{(bv)^{k_N}} q_N \xrightarrow{(bv)^{k'_N}} q_f \in F$$

But there are N such factors: $\exists i < j$ s.t. $q_i = q_j$. Then

$$\delta(q_i, (bv)^{k'_i} au_i \cdots au_j (bv)^{N-k_j-k'_j}) = q_i$$

which contradicts $L(\mathcal{A})$ bounded. $\square$

Questions?