The Ordinal-Recursive Complexity of Timed-Arc Petri Nets, Data Nets, and Other Enriched Nets

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Complexity of wqo Algorithms

- well-quasi orders (wqo): tools for proving termination
- generic complexity upper bounds
- enormous complexities
- what about "natural" lower bounds?

Overview

Theorem

Coverability and termination in enriched nets are $F_{\omega^{\omega^{\omega}}}(n + O(1))$*-complete.*

Contents

Enriched Nets

- classes of colored Petri nets:

  timed-arc Petri nets: clocks in $\mathbb{R}$

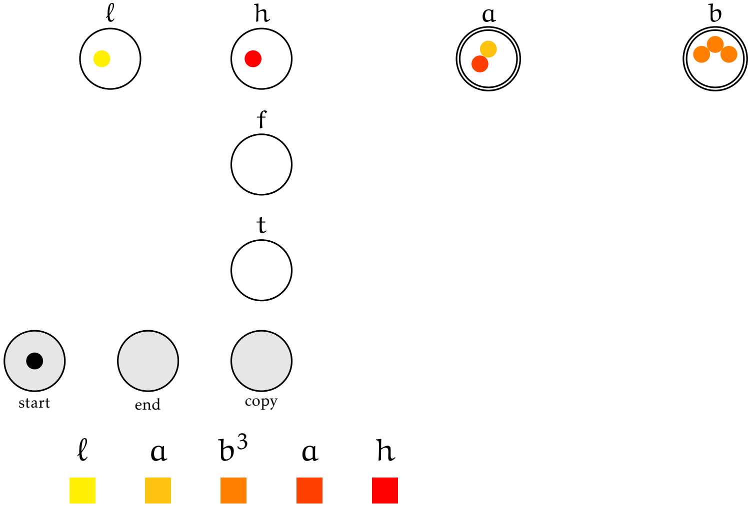
  constrained multiset rewriting systems: natural numbers

  data nets: elements of some linear-ordered dense domain

- equal expressiveness (Abdulla et al., Bonnet et al.)

Petri Data Nets (PDNs)

Example (weak copy)



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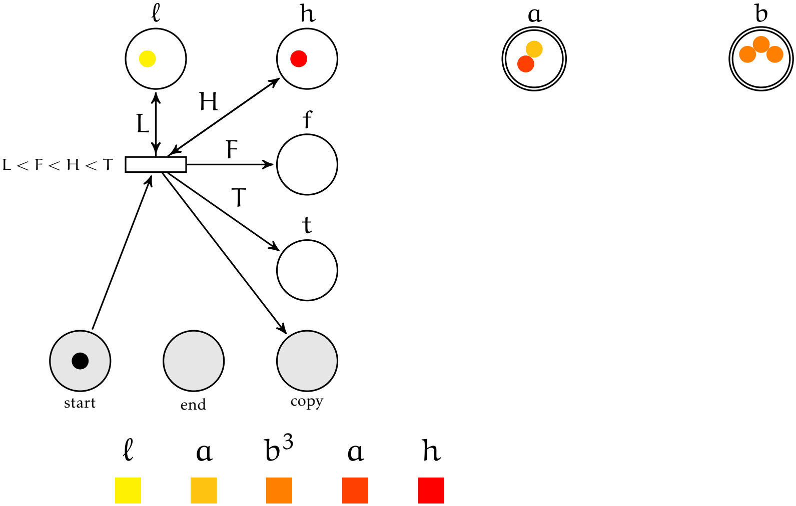

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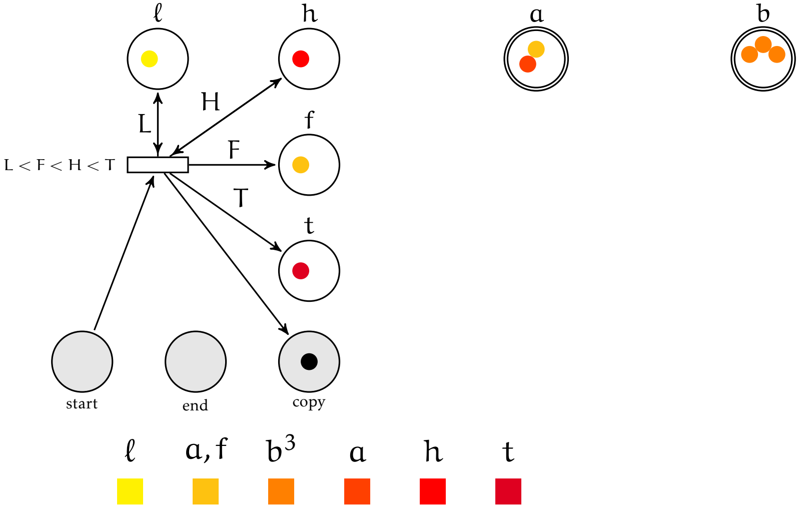

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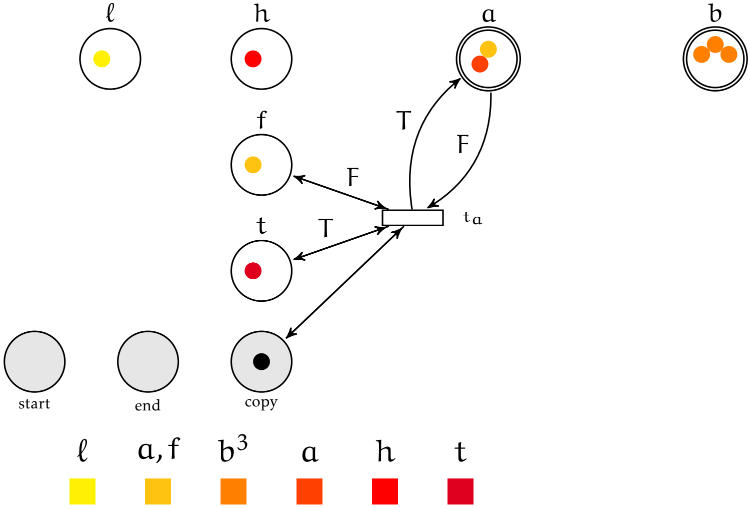

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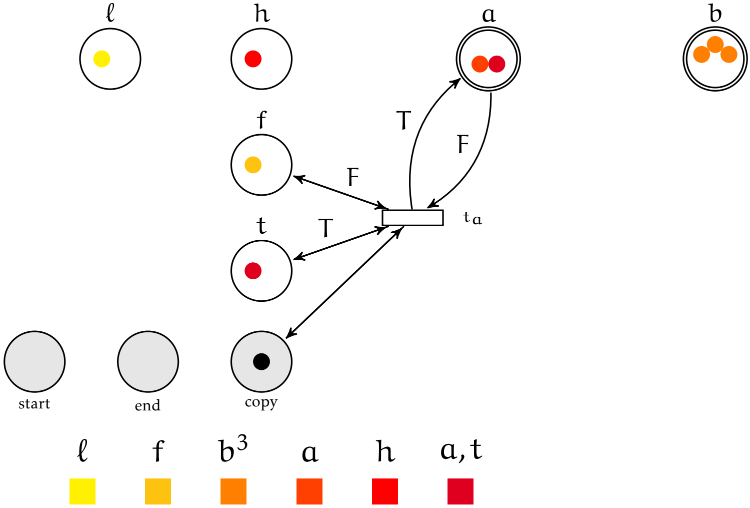

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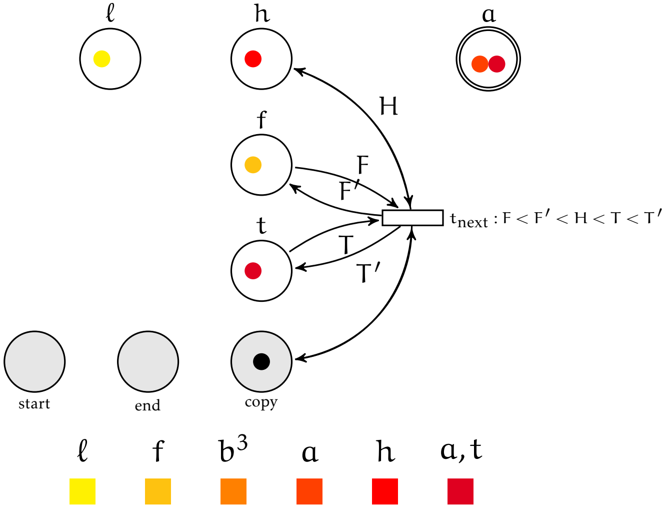

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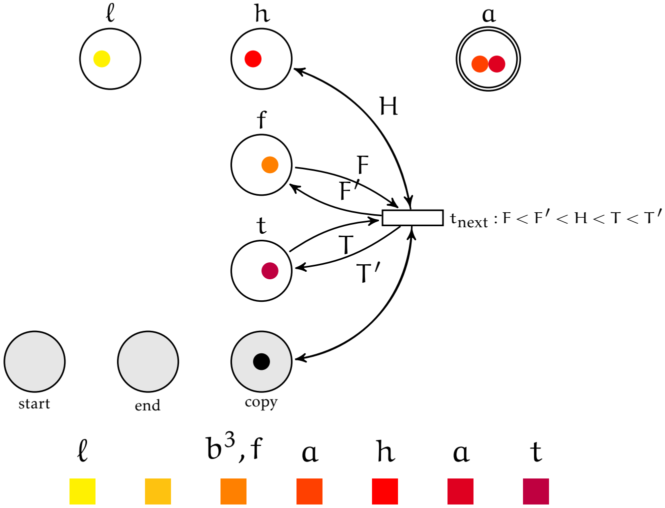

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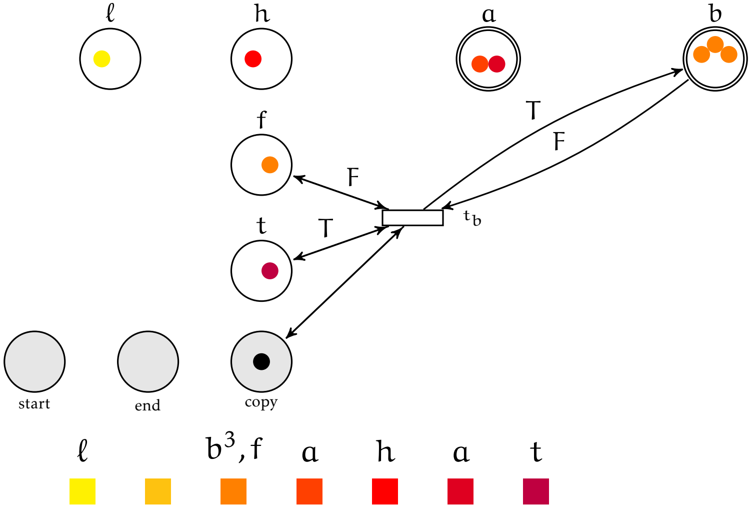

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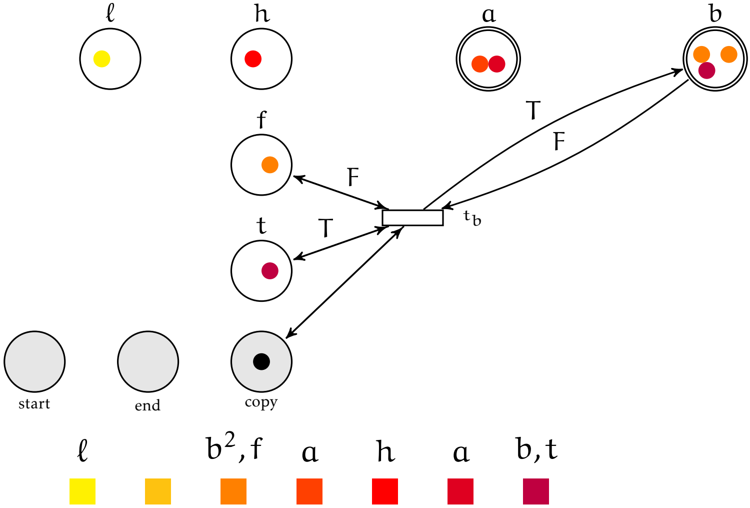

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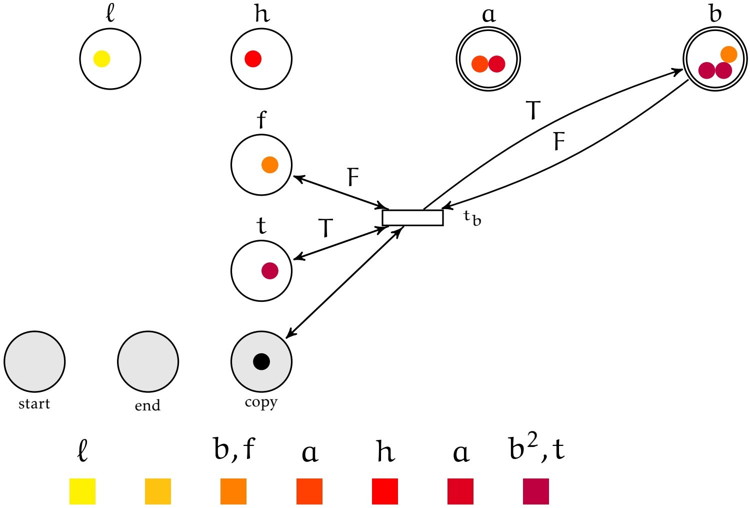

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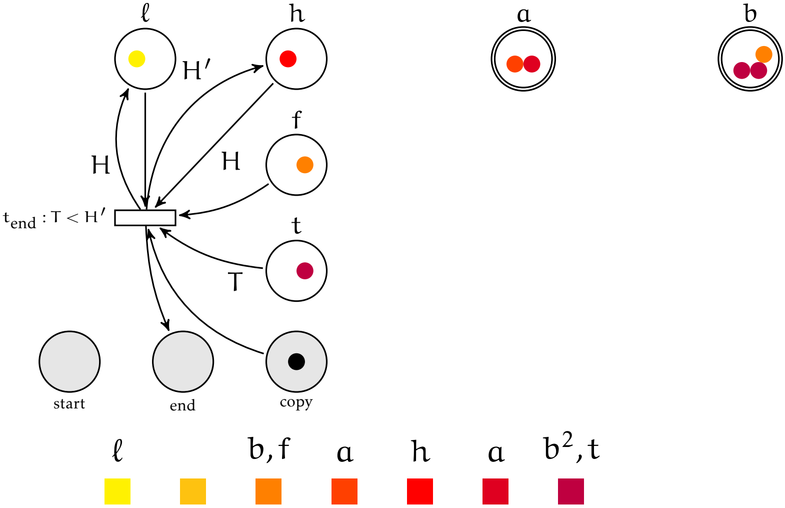

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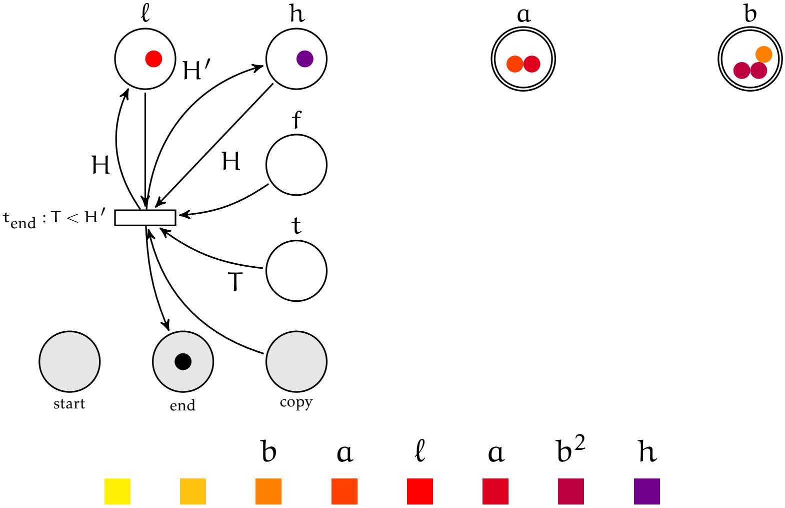
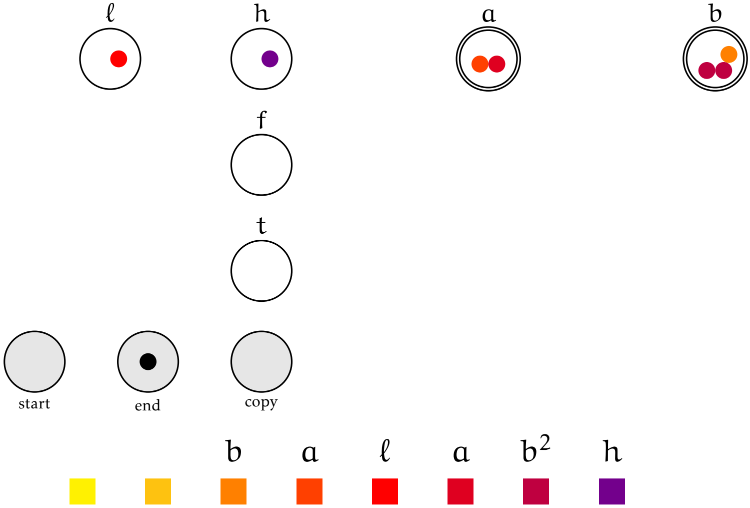


Petri Data Nets (PDNs)

Example (weak copy: $a, b^2 \sqsubseteq a, b^3, a$)



Coverability

input a Petri data net $\mathcal{N}$ and a place p of $\mathcal{N}$

question does there exist a reachable marking with at least one token in p?

Proposition (Lazić et al.)

Petri data net coverability is F_{ω^ω}*-hard.*

Proposition

Petri data net coverability is in $F_{\omega^{\omega^\omega}}(n+O(1))$.

Hardy Hierarchy

Ordinal-indexed hierarchy of functions $(H^\alpha \colon \mathbb{N} \to \mathbb{N})_{\alpha<\varepsilon_0}$:

$$H^0(x) = x \quad H^{\alpha+1}(x) = H^\alpha(x+1) \quad H^\lambda(x) = H^{\lambda_x}(x)$$

where

$$(\gamma + \omega^{\beta+1})_x = \gamma + \omega^\beta \cdot x \quad (\gamma + \omega^\lambda)_x = \gamma + \omega^{\lambda_x}$$

Intuitively: "transfinite iteration of $x \mapsto x+1$"

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Example

$$H^1(x) = x + 1 \quad H^\omega(x) = 2x \quad H^{\omega^2}(x) = 2^x x$$

H^{ω^3} non elementary

H^{ω^ω} non primitive-recursive

$H^{\omega^{\omega^\omega}}$ non multiply-recursive

$F_{\omega^{\omega^\omega}} = H^{\omega^{\omega^{\omega^\omega}}}$ this talk

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Example

Let $\Omega = \omega^{\omega^{\omega^{\omega}}}$ and $k > 0$ then

$$\Omega_k = \omega^{\omega^{\omega^{k}}} \qquad (\Omega_k)_k = \omega^{\omega^{\omega^{k-1} \cdot k}}$$

$$F_{\omega^{\omega^{\omega}}}(k) = H^{\Omega}(k) = H^{(\Omega_k)_k}(k)$$

Hardy Computations

$$H^0(x) = x \quad H^{\alpha+1}(x) = H^{\alpha}(x+1) \quad H^{\lambda}(x) = H^{\lambda_x}(x)$$

$$(\gamma + \omega^{\beta+1})_x = \gamma + \omega^{\beta} \cdot x \quad (\gamma + \omega^{\lambda})_x = \gamma + \omega^{\lambda_x}$$

Rewrite system over $\Omega_k \times \mathbb{N}$:

$$\pi + 1, n \to \pi, n+1 \tag{h_1}$$

$$\pi + \omega^{\alpha+1}, n \to \pi + \omega^{\alpha} \cdot n, n \tag{h_2}$$

$$\pi + \omega^{\alpha+\omega^{\beta+1}}, n \to \pi + \omega^{\alpha+\omega^{\beta}\cdot n}, n \tag{h_3}$$

$$\pi + \omega^{\alpha+\omega^{\beta+\omega^{m+1}}}, n \to \pi + \omega^{\alpha+\omega^{\beta+\omega^{m}\cdot n}}, n \tag{h_4}$$

Hardy Computations

Rewrite system over $\Omega_k \times \mathbb{N}$:

$$\pi + 1, n \to \pi, n + 1 \qquad (h_1)$$

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$$\pi + \omega^{\alpha+\omega^{\beta+\omega^{m+1}}}, n \to \pi + \omega^{\alpha+\omega^{\beta+\omega^{m}\cdot n}}, n \qquad (h_4)$$

A Hardy computation:

$$\pi_0, n_0 \to \pi_1, n_1 \to \cdots \to 0, H^{\pi_0}(n_0)$$

Hardness of Coverability in PDNs

- Reduction: from the halting problem of a Minsky machine M with sum of counters bounded by $H^{\Omega}(|M| + O(1)) = H^{\Omega}(k)$
- weak computation of $H^{\Omega}(k)$: some smaller value could be reached
- also perform a weak computation of the inverse $(H^{\Omega})^{-1}$

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- Reduction: from the halting problem of a Minsky machine M with sum of counters bounded by $H^{\Omega}(|M| + O(1)) = H^{\Omega}(k)$: $\mathcal{N}_M$
- weak computation of $H^{\Omega}(k)$: some smaller value could be reached: $\mathcal{N}_H[k]$
- also perform a weak computation of the inverse $(H^{\Omega})^{-1}$: $\mathcal{N}_{H^{-1}}[k]$

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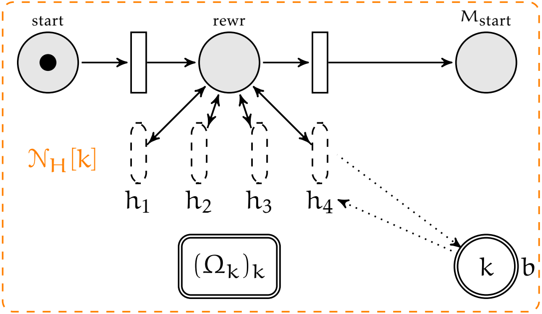

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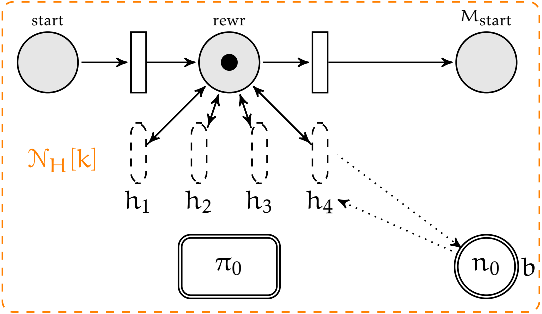

$(\Omega_k)_k, k = \pi_0, n_0$

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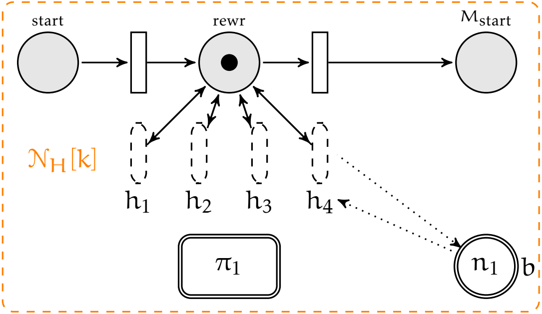

$(\Omega_k)_k, k = \pi_0, n_0 \xrightarrow{h} \pi_1, n_1$

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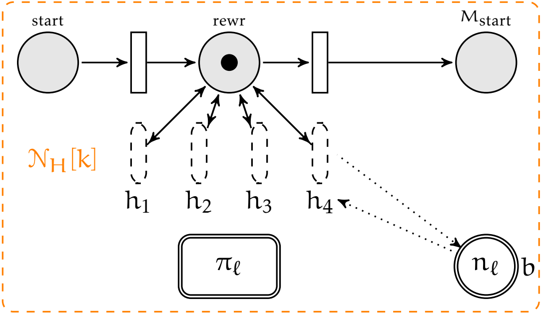

$$(\Omega_k)_k, k = \pi_0, n_0 \xrightarrow{h} \pi_1, n_1 \xrightarrow{h} \cdots \xrightarrow{h} \pi_\ell, n_\ell$$

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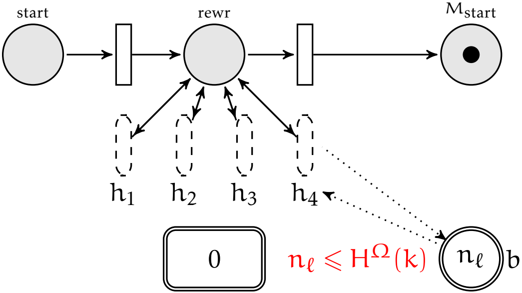

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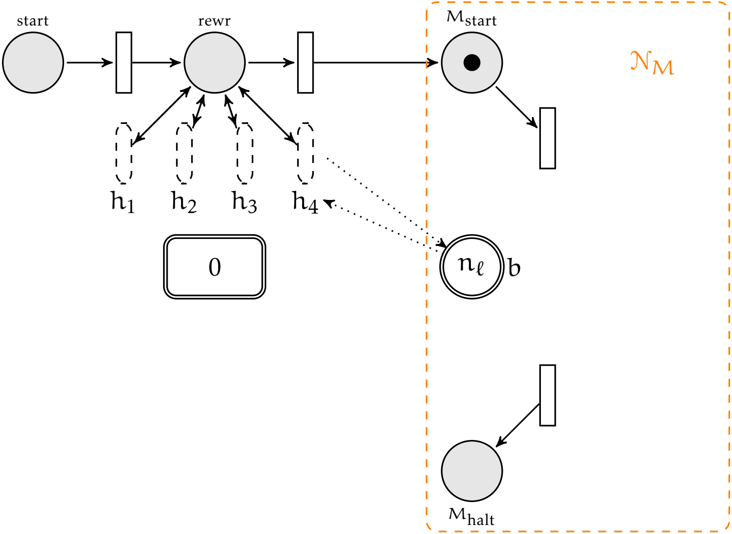

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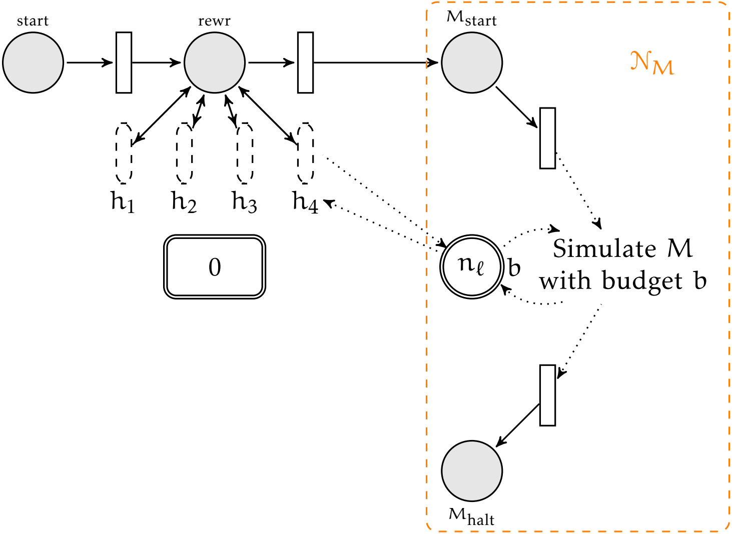

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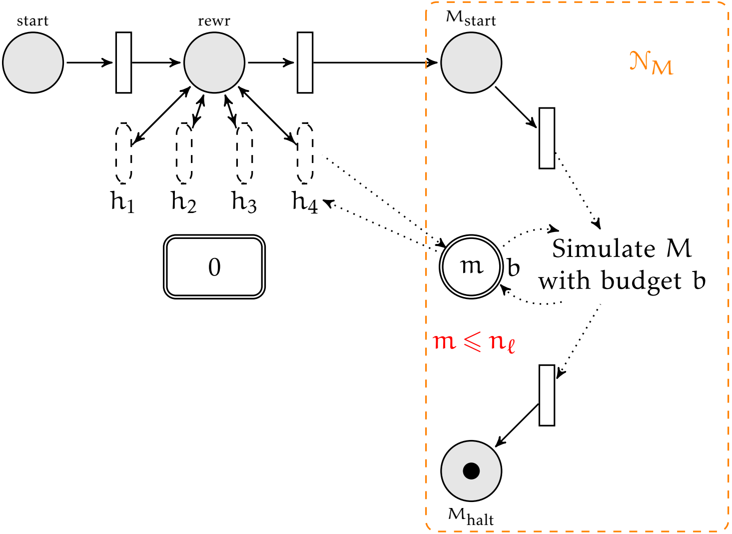

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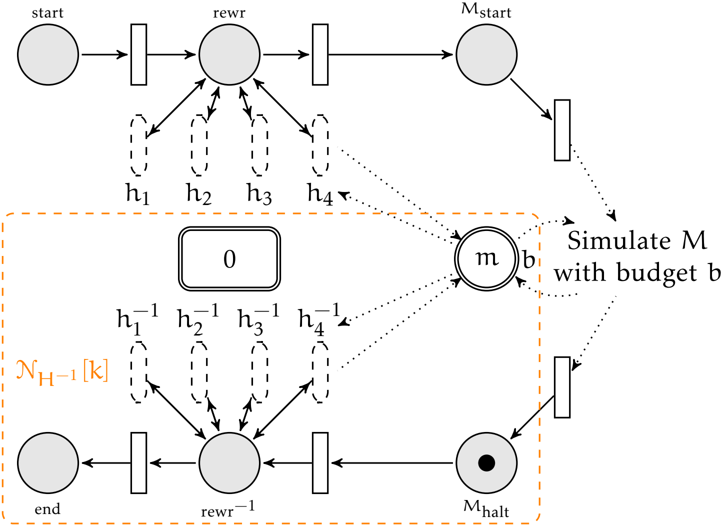

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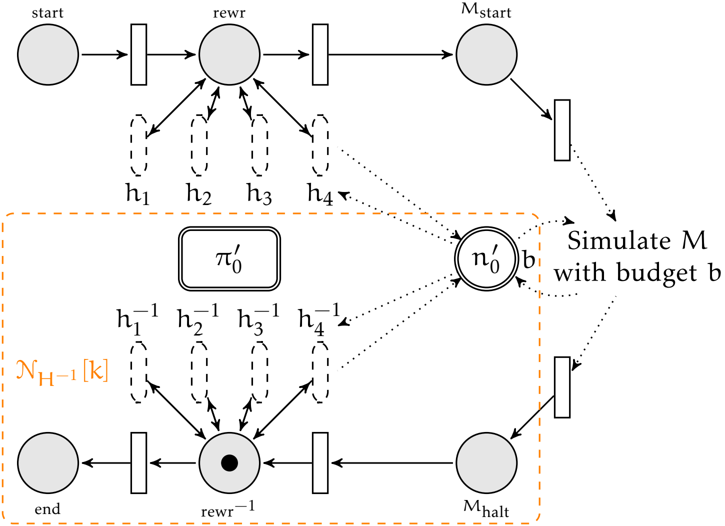

$$0, m_0 = \pi'_0, n'_0$$

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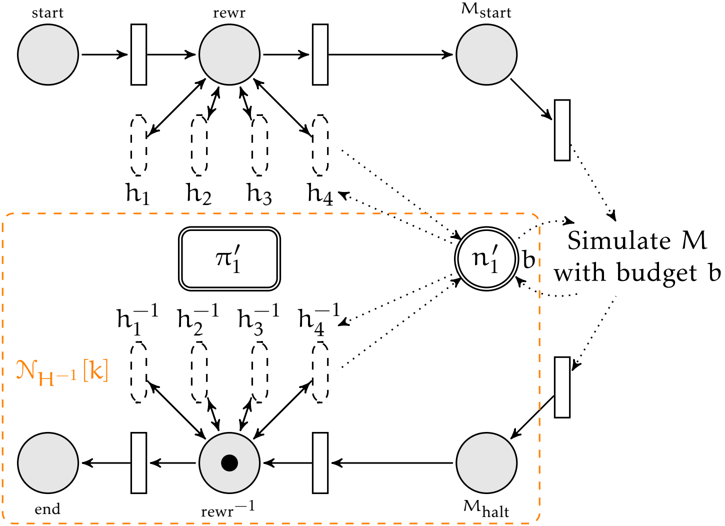

$$0, m_0 = \pi'_0, n'_0 \xrightarrow{h^{-1}} \pi'_1, n'_1$$

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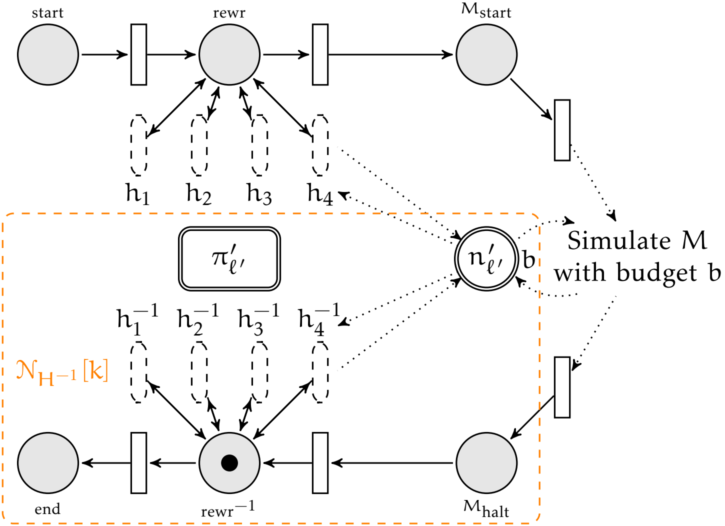

$$0, m_0 = \pi'_0, n'_0 \xrightarrow{h^{-1}} \pi'_1, n'_1 \xrightarrow{h^{-1}} \cdots \xrightarrow{h^{-1}} \pi'_{\ell'}, n'_{\ell'}$$

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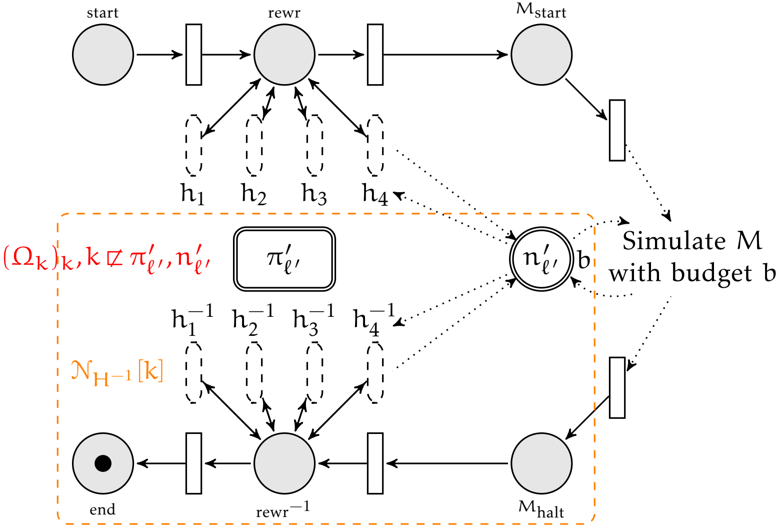

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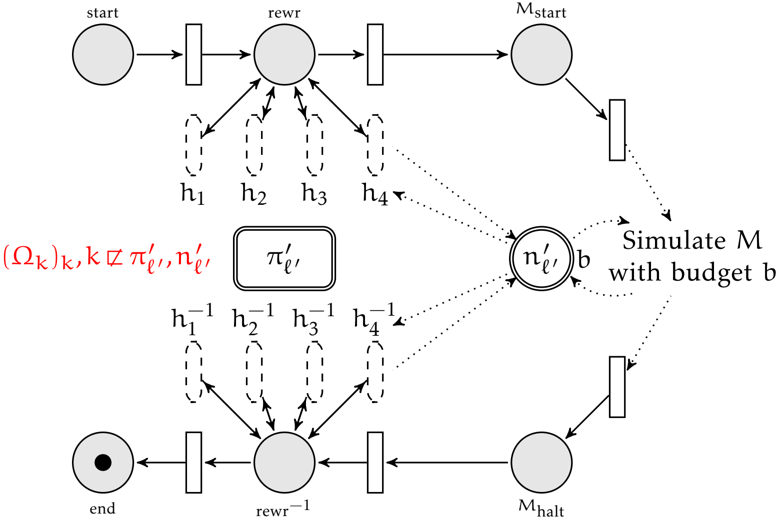

Cumulative Ordinal Encoding

- ordinals $< \omega^k$ as vectors v in $\mathbb{N}^k$:

$$\beta(v) = \omega^{k-1} \cdot v[k-1] + \cdots + \omega^0 \cdot v[0]$$

- ordinals $< \omega^{\omega^k}$ as vector sequences $\mathbf{V} = v_1 \cdots v_p$:

$$\alpha(\mathbf{V}) = \omega^{\beta(v_1)} + \cdots + \omega^{\beta(v_p)}$$

- ordinals $< \Omega_k$ as #-separated vector sequences $x = \mathbf{V}_1 \# \mathbf{V}_2 \# \cdots \# \mathbf{V}_m \#$ (aka codes):

$$\pi(x) = \omega^{\alpha(\mathbf{V}_1 \cdots \mathbf{V}_m)} + \omega^{\alpha(\mathbf{V}_1 \cdots \mathbf{V}_{m-1})} + \cdots + \omega^{\alpha(\mathbf{V}_1)}$$

Encoded Hardy Computations

Work on codes instead of ordinals:

Example

$$\pi + 1, n \to \pi, n + 1 \tag{h_1}$$

is encoded as

$$\#x, n \to x, n + 1 \tag{r_1}$$

Encoded Hardy Computations

Example

Let $k > 1$; the initial step of the encoded Hardy computation for $H^{\Omega}(k)$ is

$$x_0, k \xrightarrow{h_4} x_1, k$$

with codes

$$x_0 = (\mathbf{1}_{k-1})^k \# \, ; \qquad \pi(x_0) = (\Omega_k)_k = \omega^{\omega^{\omega^{k-1} \cdot k}}$$

$$x_1 = (\mathbf{1}_{k-1})^{k-1} (\mathbf{1}_{k-2})^k \# \, ; \quad \pi(x_1) = \omega^{\omega^{\omega^{k-1} \cdot (k-1) + \omega^{k-2} \cdot k}}$$

Properties

Proposition (Correctness)

$x, n \to y, m$ and x pure imply $H^{\pi(x)}(n) = H^{\pi(y)}(m)$.

Proposition (Robustness)

Let x, x' be pure codes and $n' > 0$. If x' is n'-trim and $x, n \sqsubset x', n'$, then $H^{\pi(x)}(n) < H^{\pi(x')}(n')$.

Why Cumulative Encodings?

Non-Robustness of Cantor Normal Form

Encoding as "$\dotplus$"-separated sequences $p = \mathbf{V}_1 \dotplus \cdots \dotplus \mathbf{V}_m$ with

$$\chi(p) = \omega^{\alpha(\mathbf{V}_1)} + \cdots + \omega^{\alpha(\mathbf{V}_m)}$$

Example

For $k = 1$:

$$\chi(1 \dotplus 0) = \omega^{\omega} + \omega \qquad \chi(10) = \omega^{\omega+1}$$

$$\begin{aligned} H^{\chi(1\dotplus 0)}(n) &= H^{\omega^{\omega}+\omega}(n) \\ &< H^{\omega^{\omega\cdot(n-1)+\omega^n}}(n) = H^{\chi(10)}(n) \end{aligned}$$

Concluding Remarks

- answers the open complexity questions on enriched nets
- first "natural" problem complete for $F_{\omega^{\omega^{\omega}}}$
- also expressiveness corollaries (see paper)

Abdulla, P.A., Delzanno, G., and Van Begin, L., 2007. Comparing the expressive power of well-structured transition systems. In Duparc, J. and Henzinger, T.A., editors, *CSL 2007*, volume 4646 of *LNCS*, pages 99–114. doi:10.1007/978-3-540-74915-8_11.

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