# On the Complexity of VAS Reachability

Sylvain Schmitz
based on joint works with D. Figueira, S. Figueira, J. Leroux, and Ph. Schnoebelen

LSV, ENS Paris-Saclay & CNRS, Université Paris-Saclay

CAALM 2019

# Outline

well-quasi-orders (wqo):

- proving algorithm termination

a toolbox for wqo complexity

- upper bounds
- lower bounds
- complexity classes

this talk: focus on one problem

- reachability in vector addition systems

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# Vector Addition Systems

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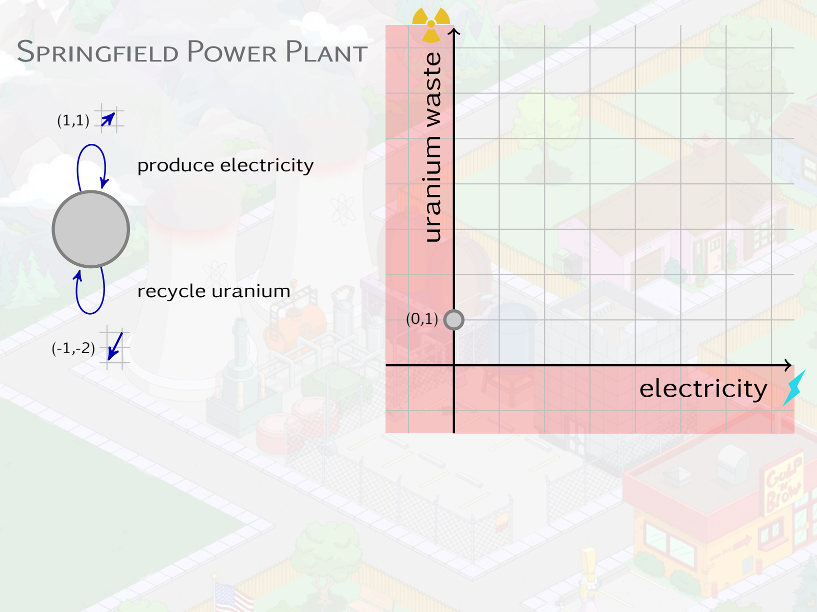
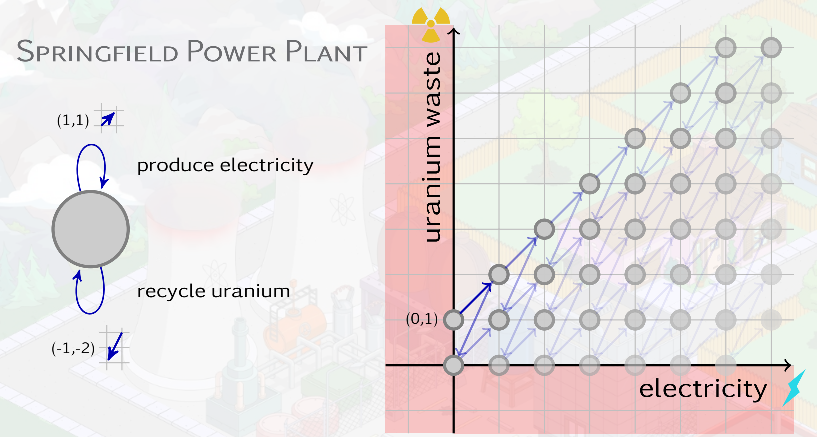
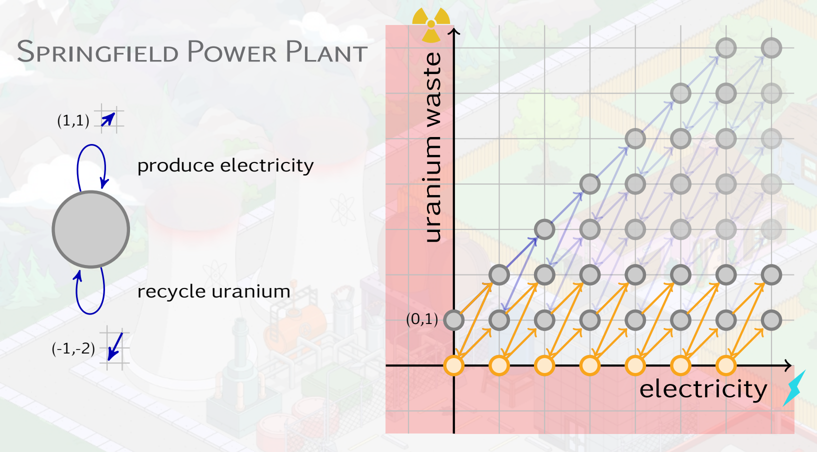


# Vector Addition Systems




Can we produce unbounded electricity with no left-over uranium waste?

# Vector Addition Systems




Can we produce unbounded electricity with no left-over uranium waste? Yes, $(\infty,0)$ is reachable

# Importance of the Problem

**Reachability Problem**

input: *a vector addition system and two configurations* **source** *and* **target**

question: **source** $\rightarrow^*$ **target***?*

# Importance of the Problem

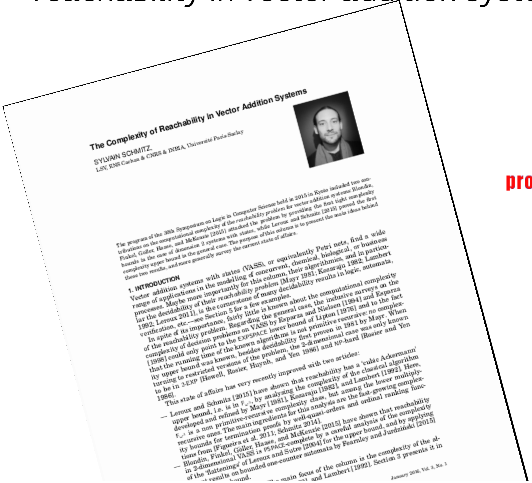
### Discrete Resources

- modelling: items, money, energy, molecules, ...
- distributed computing: active threads in thread pool
- data: isomorphism types in data logics and data-centric systems

# Importance of the Problem

## Central Decision Problem [S.'16]

Large number of problems interreducible with reachability in vector addition systems



**The Complexity of Reachability in Vector Addition Systems**

SYLVAIN SCHMITZ,
LSV, ENS Cachan & CNRS & INRIA, Université Paris-Saclay

The program of the 30th Symposium on Logic in Computer Science held in 2015 in Kyoto included two contributions on the computational complexity of the *reachability problem* for vector addition systems: Blondin, Finkel, Goller, Haase, and McKenzie [2015] attacked the problem by providing the first tight complexity bounds in the case of dimension 2 systems with states, while Leroux and Schmitz [2015] proved the first complexity upper bound in the general case. The purpose of this column is to present the main ideas behind these two results, and more generally survey the current state of affairs.

**1. INTRODUCTION**

Vector addition systems with states (VASS), or equivalently Petri nets, find a wide range of applications in the modelling of concurrent, chemical, biological, or business processes. Maybe more importantly for this column, their algorithmics, and in particular the decidability of their *reachability problem* [Mayr 1981; Kosaraju 1982; Lambert 1992; Leroux 2011], is the cornerstone of many decidability results in logic, automata, verification, etc.—see Section 5 for a few examples.

In spite of its importance, fairly little is known about the computational complexity of the reachability problem. Regarding the general case, the inclusive surveys on the complexity of decision problems on VASS by Esparza and Nielsen [1994] and Esparza [1998] could only point to the EXPSPACE lower bound of Lipton [1976] and to the fact that the running time of the known algorithms is not primitive recursive: *no* complexity upper bound was known, besides decidability first proven in 1981 by Mayr. When turning to restricted versions of the problem, the 2-dimensional case was only known to be in 2-EXP [Howell, Rosier, Huynh, and Yen 1986] and NP-hard [Rosier and Yen 1986].

This state of affairs has very recently improved with two articles:

— Leroux and Schmitz [2015] have shown that reachability has a 'cubic Ackermann' upper bound, i.e. is in $F_{\omega^3}$, by analysing the complexity of the classical algorithm developed and refined by Mayr [1981], Kosaraju [1982], and Lambert [1992]. Here, $F_{\omega^3}$ is a non primitive-recursive complexity class, but among the lower multiply-recursive ones. The main ingredients for this analysis are the fast-growing complexity bounds for termination proofs by well-quasi-orders and ordinal ranking functions from [Figueira et al. 2011; Schmitz 2014].

— Blondin, Finkel, Göller, Haase, and McKenzie [2015] have shown that reachability in 2-dimensional VASS is PSPACE-complete by a careful analysis of the complexity of the 'flattenings' of Leroux and Sutre [2004] for the upper bound, and by applying results on bounded one-counter automata by Fearnley and Jurdziński [2015] bound.

The main focus of the column is the complexity of the al- and Lambert [1992]. Section 3 presents it in

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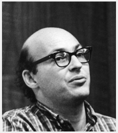
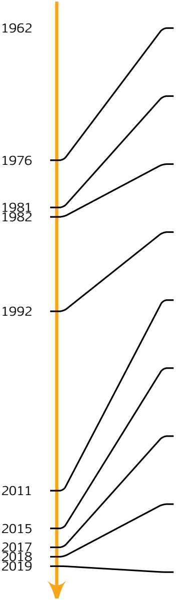
# Importance of the Problem

**Theorem** (Minsky'67)

*Reachability is undecidable in 2-dimensional Minsky machines (vector addition systems with zero tests).*



# Importance of the Problem





R. J. Lipton: EXPSPACE lower bound


E. W. Mayr: decidability by decomposition


S. R. Kosaraju: decidability by decomposition


J.-L. Lambert: decidability by decomposition


J. Leroux: decidability by Presburger inductive invariants

Leroux & S.: cubic Ackermann upper bound ($\mathbf{F}_{\omega^3}$)

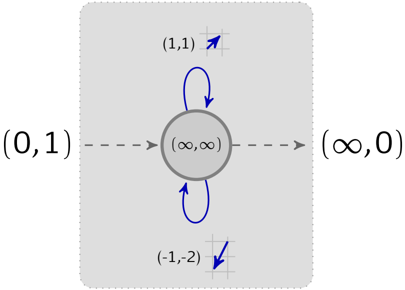
S.: quadratic Ackermann upper bound ($\mathbf{F}_{\omega^2}$)

    
W. Czerwinski, S. Lasota, R. Lazić, J. Leroux
F. Mazowiecki: TOWER lower bound

J. Leroux & S.: Ackermann upper bound ($\mathbf{F}_{\omega}$)

# "Simple Runs" (Θ Condition)

[Mayr'81, Kosaraju'82, Lambert'92]



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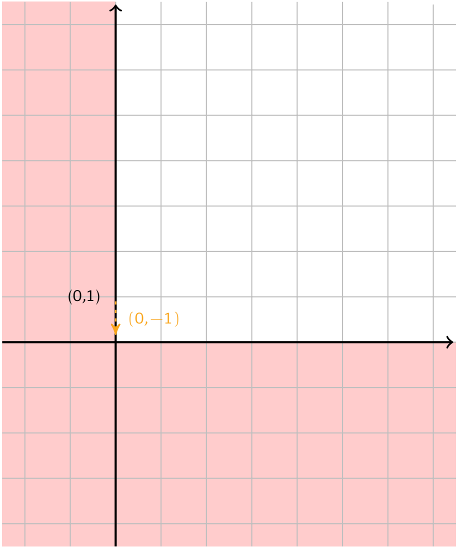

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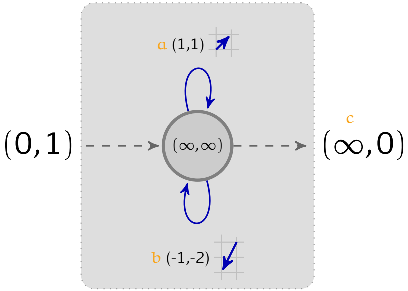
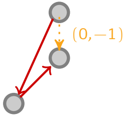



## Characteristic System

$$0 + 1 \cdot a - 1 \cdot b = c$$
$$1 + 1 \cdot a - 2 \cdot b = 0$$

## Solution Path

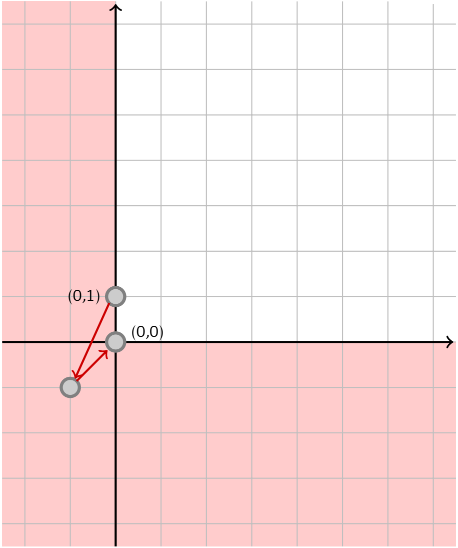


# "Simple Runs" (Θ Condition)

[Mayr'81, Kosaraju'82, Lambert'92]

solution path

$\times 1$



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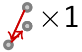


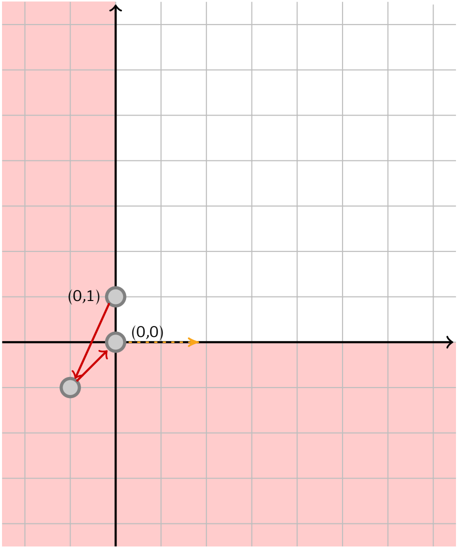

# "Simple Runs" (Θ Condition)

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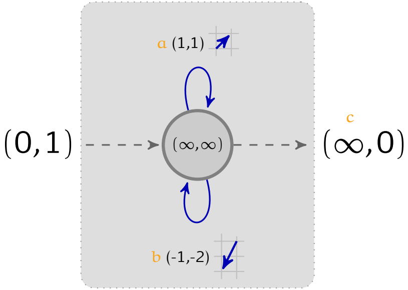
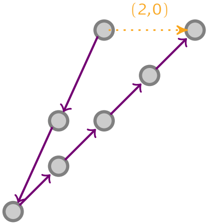



## Homogeneous System

$$1\cdot a - 1\cdot b = c$$
$$1\cdot a - 2\cdot b = 0$$
$$a,b,c > 0$$

## Unbounded Path



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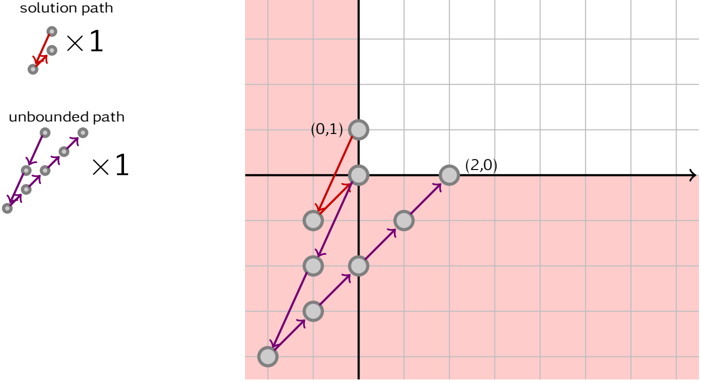
[Mayr'81, Kosaraju'82, Lambert'92]

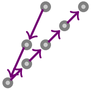


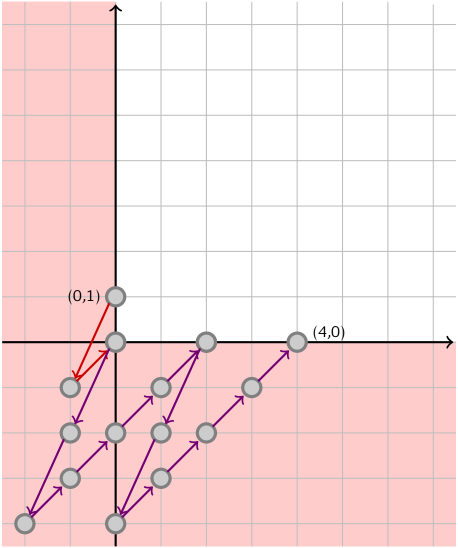
# "Simple Runs" (Θ Condition)

[Mayr'81, Kosaraju'82, Lambert'92]

solution path


unbounded path



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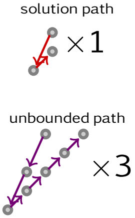
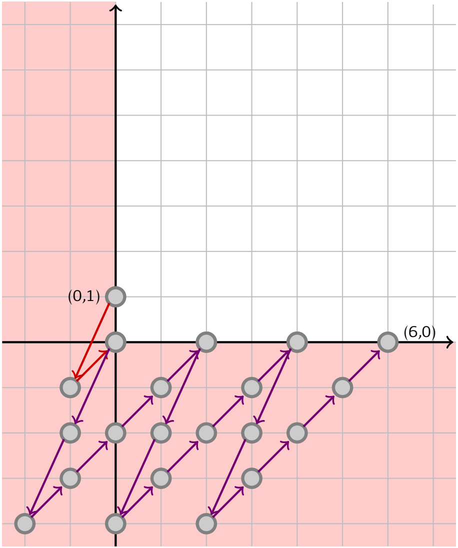
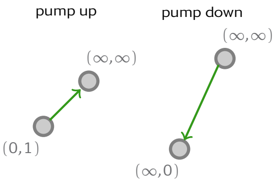


# "Simple Runs" (Θ Condition)

[Mayr'81, Kosaraju'82, Lambert'92]

## Pumpable Paths



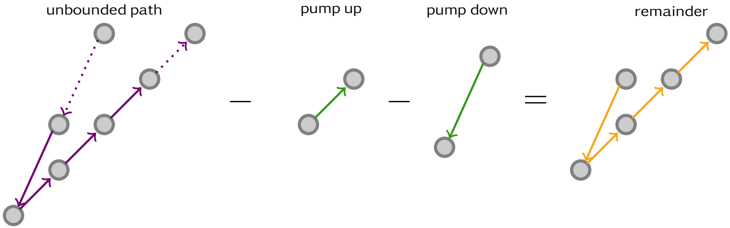

classically: uses *coverability trees* [Karp & Miller'69]

in [Leroux & S.'19] *Rackoff*-style witnesses

# "Simple Runs" (Θ Condition)

[Mayr'81, Kosaraju'82, Lambert'92]

## Pumpable Paths

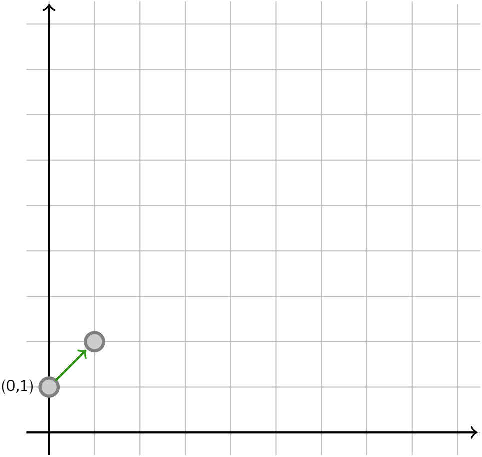


# "Simple Runs" (Θ Condition)

[Mayr'81, Kosaraju'82, Lambert'92]

pump up

$\times 1$

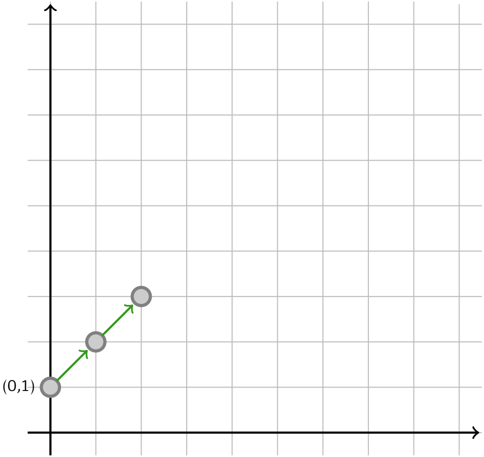


# "Simple Runs" (Θ Condition)

[Mayr'81, Kosaraju'82, Lambert'92]

pump up


$\times 2$



# "Simple Runs" (Θ Condition)

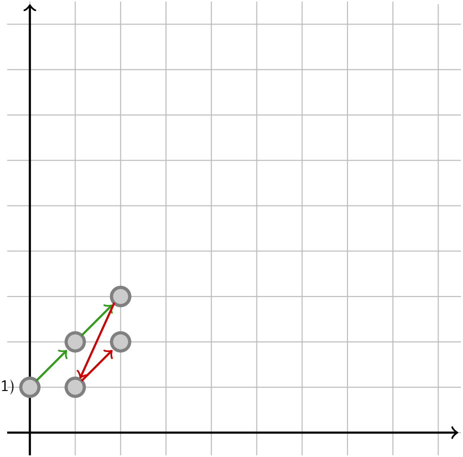
[Mayr'81, Kosaraju'82, Lambert'92]

pump up

 $\times 2$

solution path

 $\times 1$



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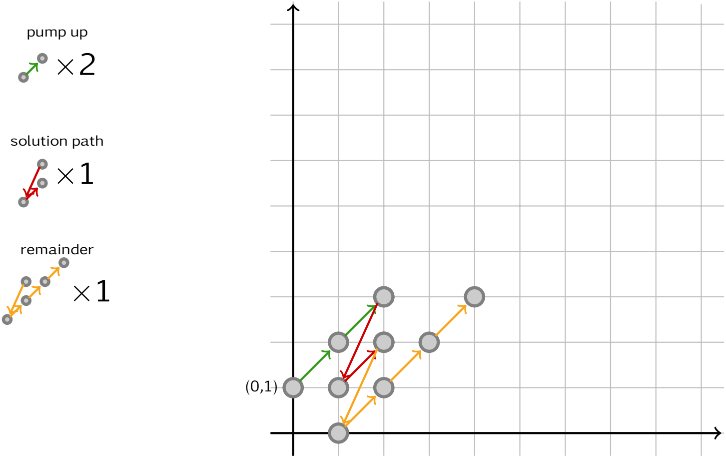

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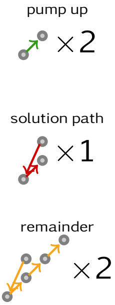


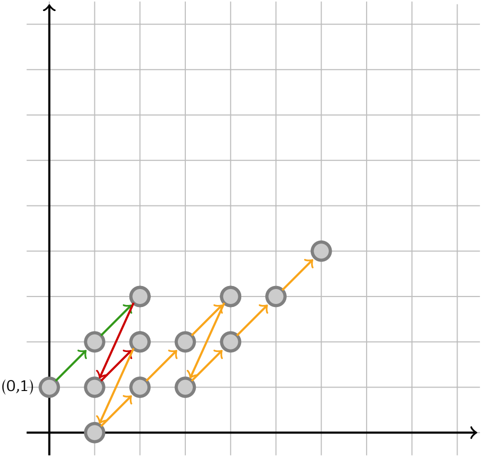

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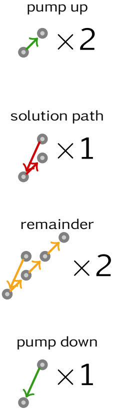


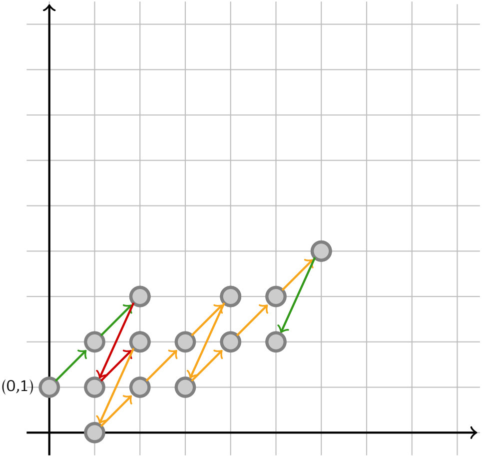

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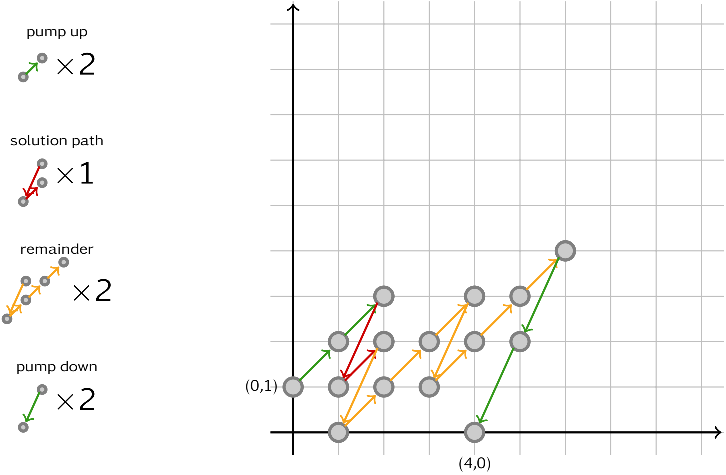

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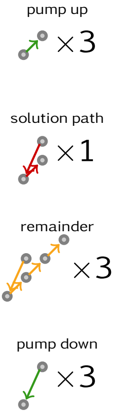
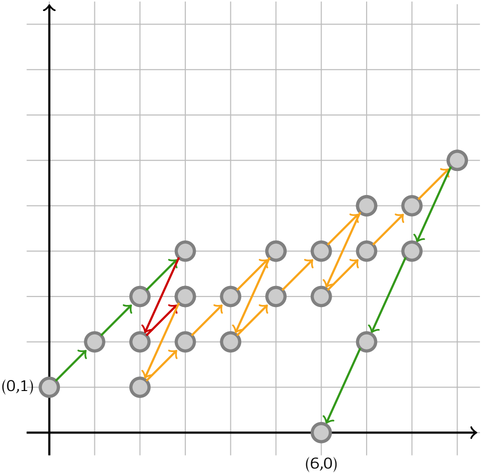
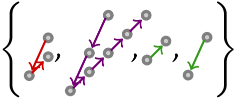


# Decomposition Algorithm

[Mayr'81, Kosaraju'82, Lambert'92]

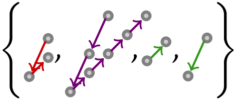
can we build a "simple run"?

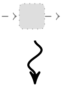


# Decomposition Algorithm

[Mayr'81, Kosaraju'82, Lambert'92]

can we build a "simple run"? yes



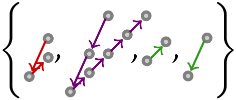


☺

# Decomposition Algorithm

[Mayr'81, Kosaraju'82, Lambert'92]

can we build a "simple run"? no



decompose

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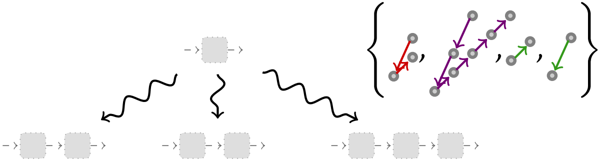

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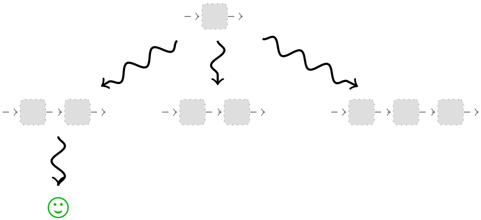

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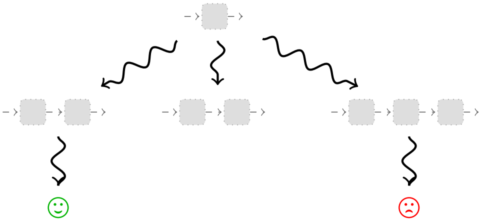

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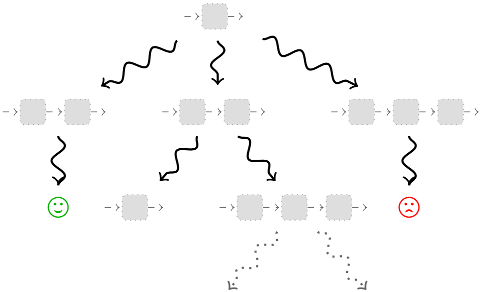


# Termination

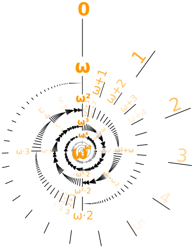
"Finally the checker has to verify that the process comes to an end. Here again he should be assisted by the programmer giving a further definite assertion to be verified. This may take the form of a quantity which is asserted to decrease continually and vanish when the machine stops."


[Turing'49]

# Termination

"Finally the checker has to verify that the process comes to an end. Here again he should be assisted by the programmer giving a further definite assertion to be verified. This may take the form of a quantity which is asserted to decrease continually and vanish when the machine stops. To the pure mathematician it is natural to give an ordinal number."

[Turing'49]

# Termination of the Decomposition Algorithm

[Mayr'81, Kosaraju'82, Lambert'92, Leroux & S.'19]

## Ranking Function

$\omega^\omega$ $\quad (\omega^d$ in dim. $d)$

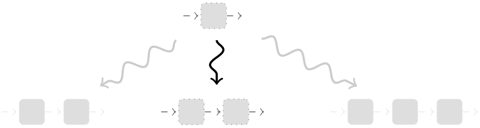
$\vee$

$\alpha_0$

# Termination of the Decomposition Algorithm

[Mayr'81, Kosaraju'82, Lambert'92, Leroux & S.'19]

## Ranking Function

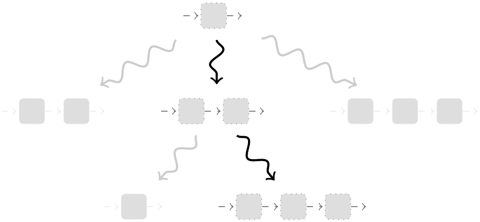


$$\omega^\omega \qquad (\omega^d \text{ in dim. } d)$$
$$\vee$$
$$\alpha_0$$
$$\vee$$
$$\alpha_1$$

# Termination of the Decomposition Algorithm

[Mayr'81, Kosaraju'82, Lambert'92, Leroux & S.'19]

## Ranking Function

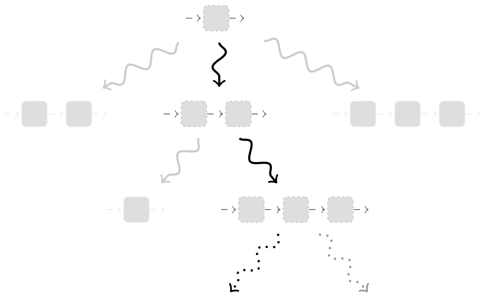


$$\omega^\omega \qquad (\omega^d \text{ in dim. } d)$$
$$\vee$$
$$\alpha_0$$
$$\vee$$
$$\alpha_1$$
$$\vee$$
$$\alpha_2$$

# Termination of the Decomposition Algorithm

[Mayr'81, Kosaraju'82, Lambert'92, Leroux & S.'19]

## Ranking Function



$$\omega^\omega \qquad (\omega^d \text{ in dim. } d)$$
$$\vee$$
$$\alpha_0$$
$$\vee$$
$$\alpha_1$$
$$\vee$$
$$\alpha_2$$
$$\vee$$
$$\vdots$$

# Upper Bounds

How to bound the running time of algorithms with ordinal-based termination proofs?

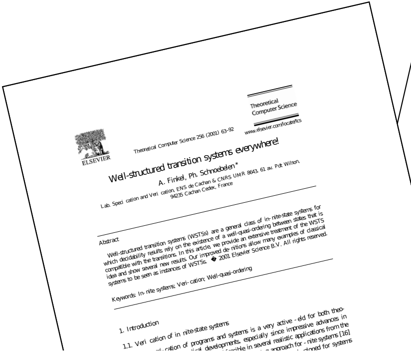
# Upper Bounds

How to bound the running time of algorithms with wqo-based termination proofs?

# Upper Bounds

How to bound the running time of algorithms with wqo-based termination proofs?

*wqos ubiquitous in infinite-state verification*


Theoretical Computer Science 256 (2001) 63–92

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Well-structured transition systems everywhere!

A. Finkel, Ph. Schnoebelen*

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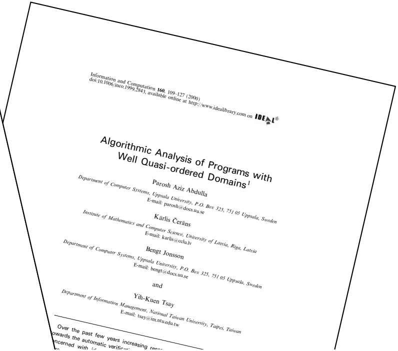
Abstract

Well-structured transition systems (WSTSs) are a general class of in nite-state systems for which decidability results rely on the existence of a well-quasi-ordering between states that is compatible with the transitions. In this article, we provide an extensive treatment of the WSTS idea and show several new results. Our improved de nitions allow many examples of classical systems to be seen as instances of WSTSs. © 2001 Elsevier Science B.V. All rights reserved.

Keywords: In nite systems; Veri cation; Well-quasi-ordering

1. Introduction

1.1. Veri cation of in nite-state systems


Information and Computation 160, 109–127 (2000)
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Algorithmic Analysis of Programs with Well Quasi-ordered Domains[1]

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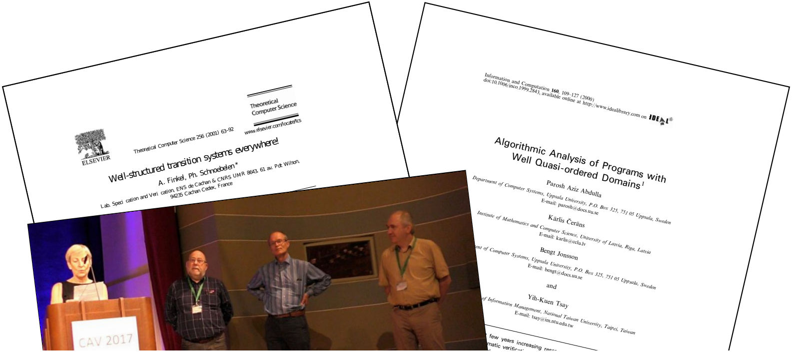
# Upper Bounds

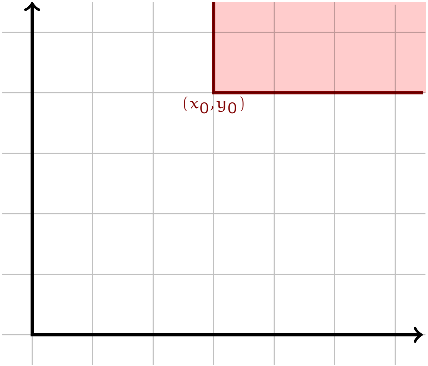
How to bound the running time of algorithms with wqo-based termination proofs?

*wqos ubiquitous in infinite-state verification*



# A One-Player Game

- over $\mathbb{Q}_{\geqslant 0} \times \mathbb{Q}_{\geqslant 0}$
- given initially $(x_0, y_0)$
- Eloise plays $(x_j, y_j)$ s.t. $\forall 0 \leqslant i < j, x_i > x_j$ or $y_i > y_j$




- Can Eloise win, i.e. play indefinitely?
- If not, how long can she last?

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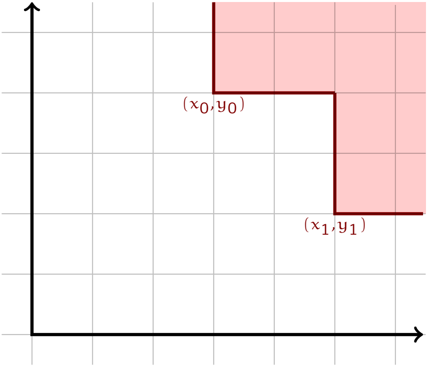


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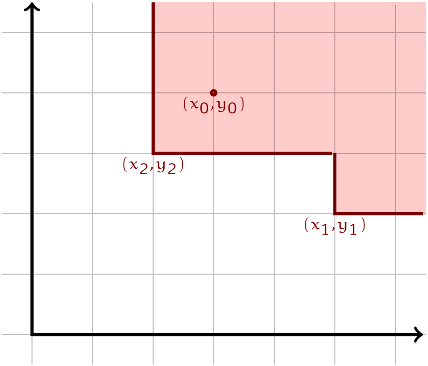


- Can Eloise win, i.e. play indefinitely?
- If not, how long can she last?

If $(x_0, y_0) \neq (0,0)$, then choosing $(x_j, y_j) = (\frac{x_0}{2^j}, \frac{y_0}{2^j})$ wins.

# A One-Player Game

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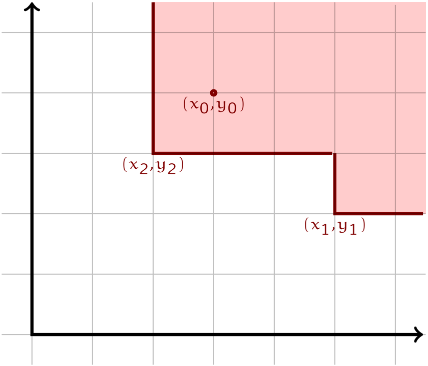


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Assume there exists an infinite sequence $(x_j, y_j)_j$ of moves over $\mathbb{N}^2$.

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purple if $x_i > x_j$ but $y_i \leqslant y_j$,

red if $x_i > x_j$ and $y_i > y_j$,

orange if $y_i > y_j$ but $x_i \leqslant x_j$.

$(3,4) \qquad (5,2) \qquad (2,3) \qquad \cdots$

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By the infinite Ramsey Theorem, there exists an infinite monochromatic subset of indices.

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$(3,4) \qquad (5,2) \qquad (2,3) \qquad \cdots$

By the infinite Ramsey Theorem, there exists an infinite monochromatic subset of indices. In all cases, it implies the existence of an infinite decreasing sequence in $\mathbb{N}$, a contradiction.

# A One-Player Game

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- given initially $(x_0, y_0)$
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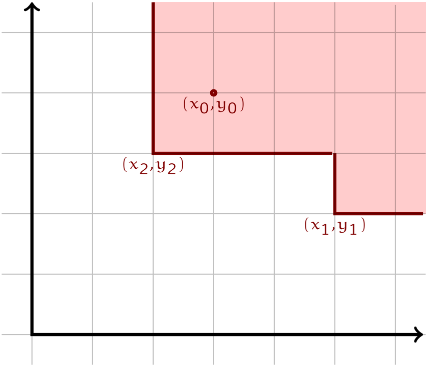


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# Bad Sequences

Over a qo $(X, \leqslant)$

- $x_0, x_1, \ldots$ is bad if $\forall i < j \,.\, x_i \not\leqslant x_j$
- $(X, \leqslant)$ wqo iff all bad sequences are finite
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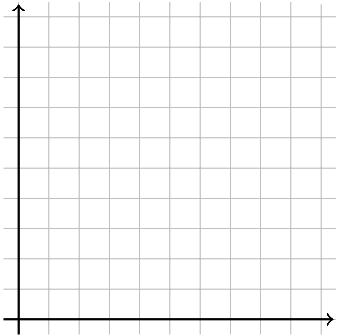


## Example (over $\mathbb{N}^2$)

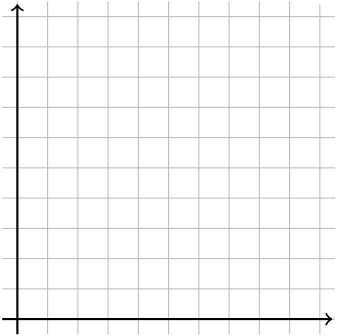
# Bad Sequences

Over a qo $(X, \leqslant)$

$x_0, x_1, \ldots$ is bad if $\forall i < j \,.\, x_i \not\leqslant x_j$

$(X, \leqslant)$ wqo iff all bad sequences are finite

but can be of arbitrary length



**Example (over $\mathbb{N}^2$)**

# Controlled Bad Sequences

Over a qo $(X, \leqslant)$ with norm $\|\cdot\|$

- $x_0, x_1, \ldots$ is bad if $\forall i < j \,.\, x_i \not\leqslant x_j$
- $(X, \leqslant)$ wqo iff all bad sequences are finite
- controlled by $g\colon \mathbb{N} \to \mathbb{N}$ monotone and inflationary and $n_0 \in \mathbb{N}$ if $\forall i \,.\, \|x_i\| \leqslant g^i(n_0)$
  [Cichoń & Tahhan Bittar'98]

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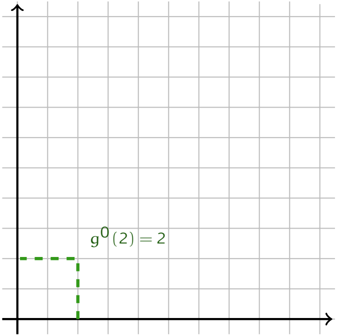


## Example (over $\mathbb{N}^2$ with $n_0 = 2$ and $g(n) = n + 1$)

# Controlled Bad Sequences

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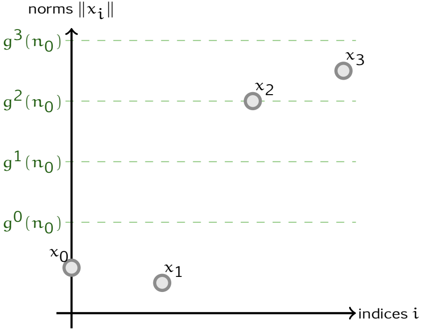
[Cichoń & Tahhan Bittar'98]

**Proposition**

*Over $(X, \leqslant)$, assuming $\forall n \, \{x \in X \mid \|x\| \leqslant n\}$ finite, $(g, n_0)$-controlled bad sequences have a maximal length, noted $L_{g,X}(n_0)$.*

# Descent Equation

$(g, n_0)$-controlled bad sequence $x_0, x_1, x_2, x_3, \ldots$ over a wqo $(X, \leqslant)$:



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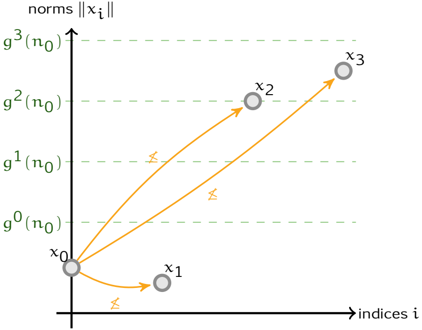


over the suffix $x_1, x_2, x_3, \ldots, \forall i > 0$,

$$x_0 \not\leq x_i$$

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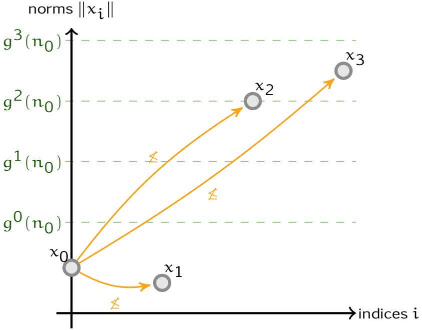


over the suffix $x_1, x_2, x_3, \ldots, \forall i > 0$,

$$x_i \in X \setminus \uparrow x_0 \stackrel{\text{def}}{=} \{x \in X \mid x_0 \not\leq x\}$$

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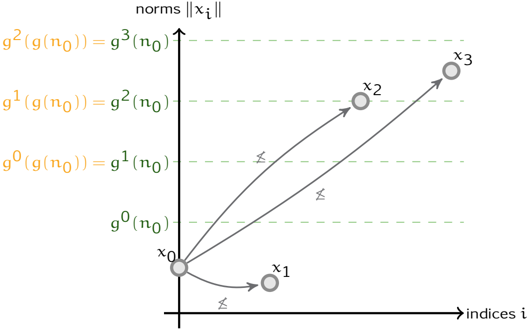


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$$x_i \in X \backslash \uparrow x_0 \stackrel{\text{def}}{=} \{x \in X \mid x_0 \not\leqslant x\}$$

$$\|x_i\| \leqslant g^{i-1}(g(n_0))$$

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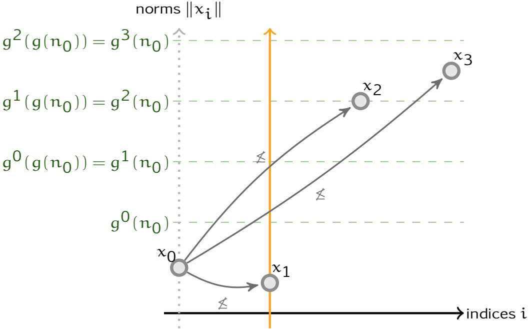


over the suffix $x_1, x_2, x_3, \ldots, \forall i > 0$,

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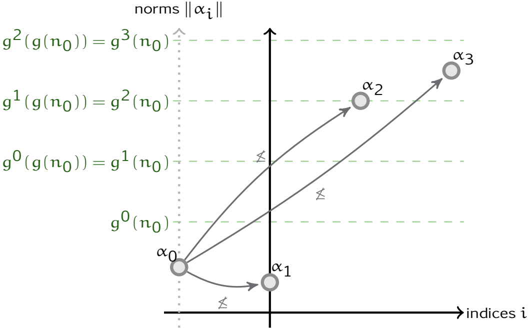
$\|x_i\| \leqslant g^{i-1}(g(n_0))$

$$L_{g,X}(n_0) = \max_{x_0 \in X, \|x_0\| \leqslant n_0} 1 + L_{g, X \backslash \uparrow x_0}(g(n_0))$$

# Descent Equation

$(g, n_0)$-controlled bad sequence $\alpha_0, \alpha_1, \alpha_2, \alpha_3, \ldots$ over an ordinal $\alpha$:




over the suffix $\alpha_1, \alpha_2, \alpha_3, \ldots, \forall i > 0$,

$$\alpha_i \in \alpha_0 \stackrel{\text{def}}{=} \{\beta \in \alpha \mid \beta \not\leq \alpha_0\}$$

$$\|\alpha_i\| \leqslant g^{i-1}(g(n_0))$$

$$L_{g,\alpha}(n_0) = \max_{\alpha_0 \in \alpha, \|\alpha_0\| \leqslant n_0} 1 + L_{g,\alpha_0}(g(n_0))$$

# The Case of Ordinals

[S.'14]

$$L_{g,\alpha}(n_0) = \max_{\alpha_0 \in \alpha, \|\alpha_0\| \leqslant n_0} 1 + L_{g,\alpha_0}(g(n_0))$$

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For a suitable norm function, there is a "maximising" ordinal $P_{n_0}(\alpha)$:

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These functions form the Cichón hierarchy.

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These functions form the Cichón hierarchy.

# Relating Norm and Length

[Cichoń & Tahhan Bittar'98]

Recall the definition of the Cichoń Hierarchy:

$$L_{g,0}(x) \stackrel{\text{def}}{=} 0 \qquad Lg,\alpha(x) \stackrel{\text{def}}{=} 1 + L_{g,P_x(\alpha)}(g(x)) \text{ for } \alpha > 0$$

**Definition** (Hardy Hierarchy)

For $g : \mathbb{N} \to \mathbb{N}$, define $(g^\alpha : \mathbb{N} \to \mathbb{N})_\alpha$ by

$$g^0(x) \stackrel{\text{def}}{=} x \qquad g^\alpha(x) \stackrel{\text{def}}{=} g^{P_x(\alpha)}(g(x)) \text{ for } \alpha > 0$$

# Relating Norm and Length

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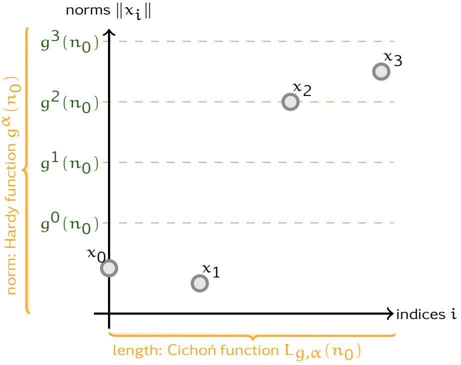


$$g^\alpha(x) = g^{L_{g,\alpha}(x)}(x)$$

$$g^\alpha(x) \geqslant L_{g,\alpha}(x) + x$$

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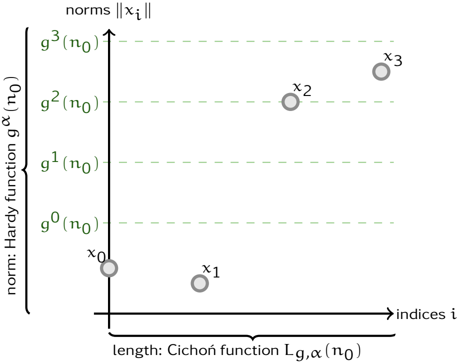


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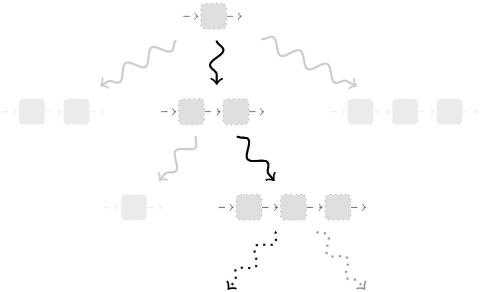
# The Length of Decomposition Branches



$$\alpha_0 > \alpha_1 > \alpha_2 > \vdots$$

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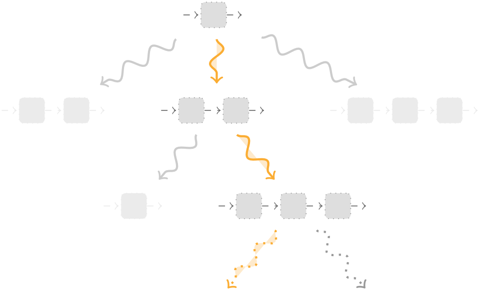

$$\begin{array}{c}\alpha_0\\ \vee\\ \alpha_1\\ \vee\\ \alpha_2\\ \vee\\ \vdots\end{array}$$

**Consequence of (Leroux & S.'19)**

*An elementary control $\mathfrak{g}$ and $\mathfrak{n}$ the size of the reachability instance fit. Thus the decomposition algorithm runs in SPACE$(\mathfrak{g}^{\omega^\omega}(\mathfrak{n}))$, and SPACE$(\mathfrak{g}^{\omega^d}(\mathfrak{n}))$ in fixed dimension $d$.*

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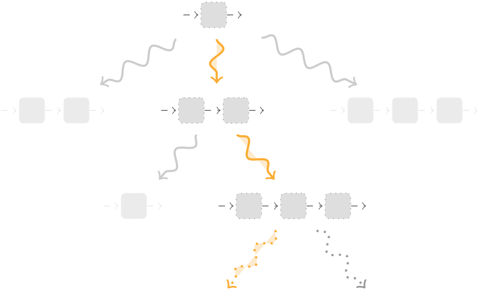

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**Consequence of (Leroux & S.'19)**

*An elementary control $g$ and $n$ the size of the reachability instance fit. Thus the decomposition algorithm runs in SPACE$(g^{\omega^\omega}(n))$, and SPACE$(g^{\omega^d}(n))$ in fixed dimension $d$.*

# Restating the Result

"$\mathsf{SPACE}\big(g^{\omega^d}(n)\big)$" is unreadable!

# Restating the Result

Hardy hierarchy with base function $H(x) \stackrel{\text{def}}{=} x+1$:

$$H^0(x) = x$$

$$H^k(x) = \overbrace{H \circ \cdots \circ H}^{k \text{ times}}(x) \qquad = x+k$$

$$H^\omega(x) = H^{x+1}(x) = \overbrace{H \circ \cdots \circ H}^{x+1 \text{ times}}(x) \qquad = 2x+1$$

$$H^{\omega^2}(x) = H^{\omega\cdot(x+1)} = \overbrace{H^\omega \circ \cdots \circ H^\omega}^{x+1 \text{ times}}(x) \qquad \approx 2^x$$

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$$\vdots$$

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# Restating the Result

Define coarse-grained classes:

$$\mathscr{F}_{<\alpha} \stackrel{\text{def}}{=} \bigcup_{\beta<\omega^\alpha} \mathsf{FDTIME}(H^\beta(n))$$

$$\mathbf{F}_\alpha \stackrel{\text{def}}{=} \bigcup_{f\in\mathscr{F}_{<\alpha}} \mathsf{DTIME}(H^{\omega^\alpha}(f(n)))$$

**Consequence of (S.'16, Thm. 4.4)**

*VAS Reachability is in $\mathbf{F}_\omega$, and in $\mathbf{F}_{d+3}$ in fixed dimension $d$.*

# RESTATING THE RESULT

Define coarse-grained classes:

$$\mathscr{F}_{<\alpha} \stackrel{\text{def}}{=} \bigcup_{\beta<\omega^\alpha} \mathsf{FDTIME}(H^\beta(n))$$

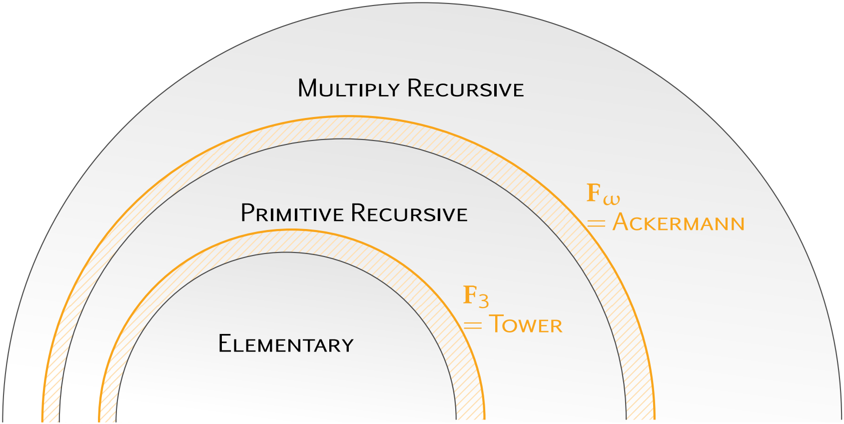
$$\mathbf{F}_\alpha \stackrel{\text{def}}{=} \bigcup_{f\in\mathscr{F}_{<\alpha}} \mathsf{DTIME}(H^{\omega^\alpha}(f(n)))$$

**CONSEQUENCE OF (S.'16, THM. 4.4)**

*VAS Reachability is in* $\mathbf{F}_\omega$*, and in* $\mathbf{F}_{d+3}$ *in fixed dimension* $d$*.*

# Complexity Classes Beyond Elementary

[S.'16]



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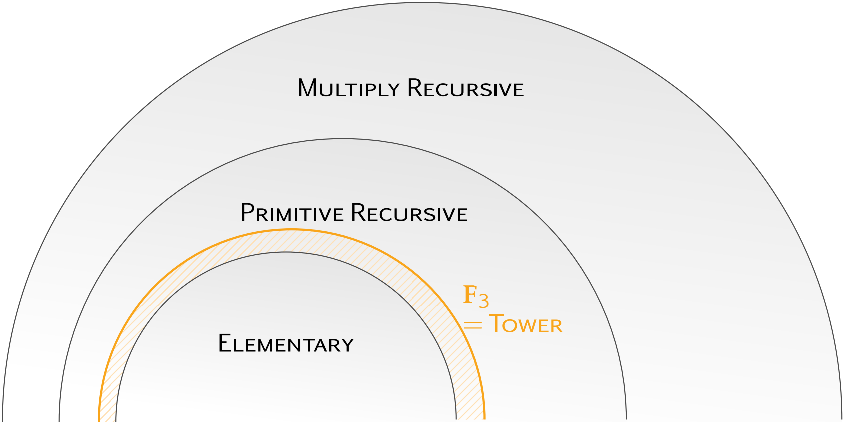


$$\mathbf{F}_3 \stackrel{\text{def}}{=} \bigcup_{e \text{ elementary}} \mathrm{DTime}\left(\mathrm{tower}(e(n))\right)$$

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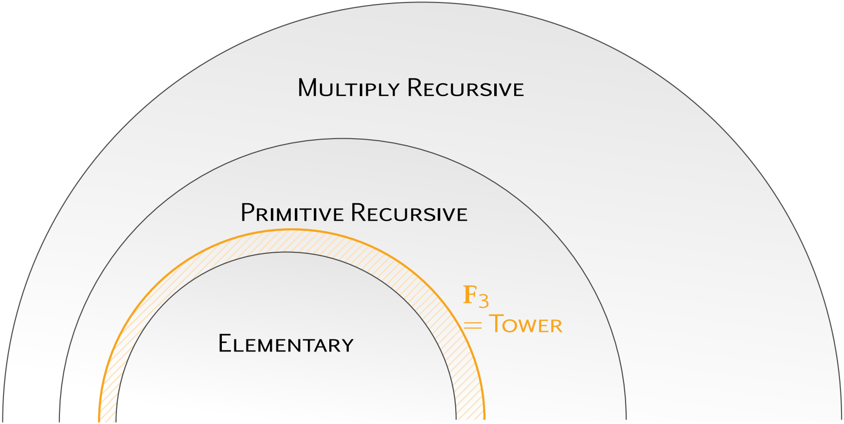


Examples of Tower-Complete Problems:

- satisfiability of first-order logic on words [Meyer'75]
- $\beta$-equivalence of simply typed $\lambda$ terms [Statman'79]
- model-checking higher-order recursion schemes [Ong'06]

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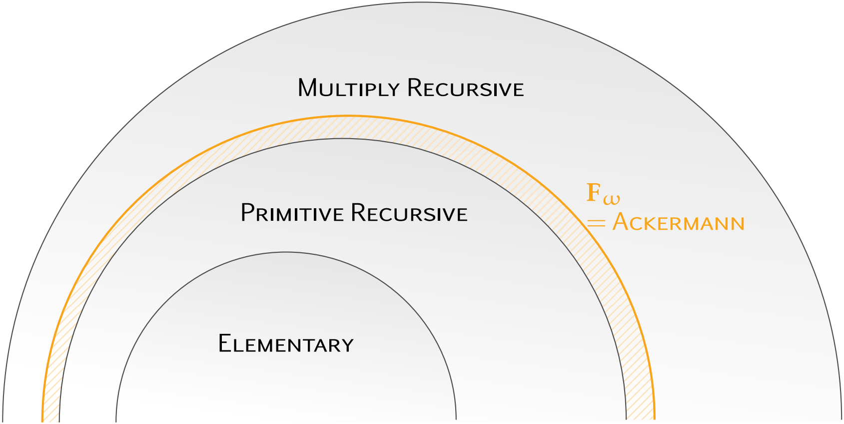


$$\mathbf{F}_\omega \stackrel{\text{def}}{=} \bigcup_{p \text{ primitive recursive}} \text{DTime}\big(\text{ack}(p(n))\big)$$

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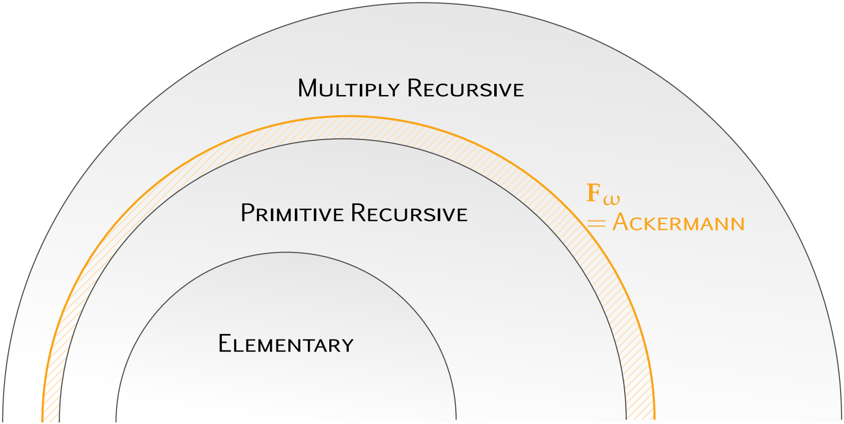


Examples of Ackermann-Complete Problems:

- reachability in lossy Minsky machines [Urquhart'98, Schnoebelen'02]
- satisfiability of safety Metric Temporal Logic [Lazić et al.'16]
- satisfiability of Vertical XPath [Figueira and Segoufin'17]

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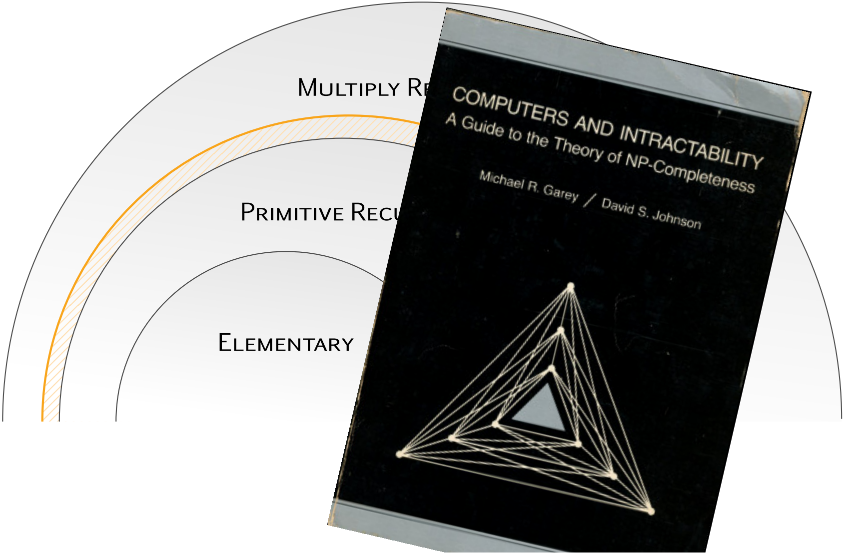

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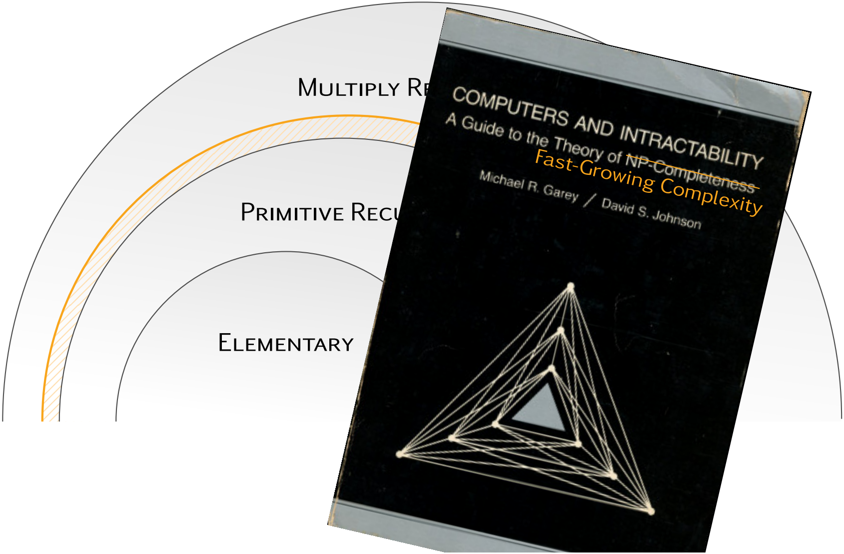

# SUMMARY

well-quasi-orders (wqo):

- proving algorithm termination

a toolbox for wqo-based complexity

- upper bounds: length function theorems (for ordinals, Dickson's Lemma, Higman's Lemma, and combinations)
- lower bounds
- complexity classes: $(\mathbf{F}_\alpha)_\alpha$

this talk: focus on one problem

- reachability in vector addition systems in $\mathbf{F}_\omega$

# Perspectives

1. complexity gap for VAS reachability
   - TOWER-hard [Czerwinski et al.'18]
     better lower bounds?
   - decomposition algorithm: requires $\mathbf{F}_\omega$ (Ackermannian) time, because downward language inclusion is $\mathbf{F}_\omega$-hard [Zetzsche'16]
2. reachability in VAS extensions
   - decidable in VAS with hierarchical zero tests [Reinhardt'08]
   - what about
     - branching VAS
     - unordered data Petri nets
     - pushdown VAS

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# Demystifying Reachability in Vector Addition Systems

[Leroux & S.'15]

**Ideal Decomposition Theorem**

*The Decomposition Algorithm computes the ideal decomposition of the set of runs from source to target.*

**Upper Bound Theorem**

*Reachability in vector addition systems is in cubic Ackermann.*

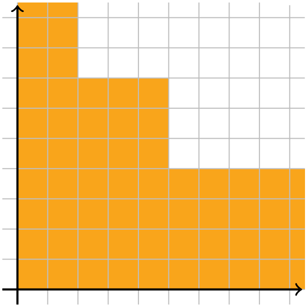
# Ideals of Well-Quasi-Orders $(X, \leqslant)$

- Canonical decompositions [Bonnet'75]

  if $D \subseteq X$ is $\downarrow$-closed, then

  $$D = I_1 \cup \cdots \cup I_n$$

  for (maximal) ideals $I_1, \ldots, I_n$



**Example (over $\mathbb{N}^2$)**

$$D = (\{0,\ldots,2\} \times \mathbb{N}) \cup (\{0,\ldots,5\} \times \{0,\ldots,7\}) \cup (\mathbb{N} \times \{0,\ldots,4\})$$

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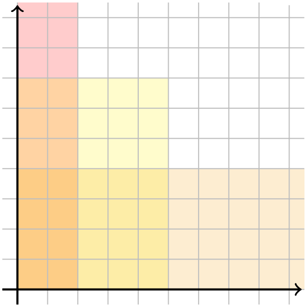

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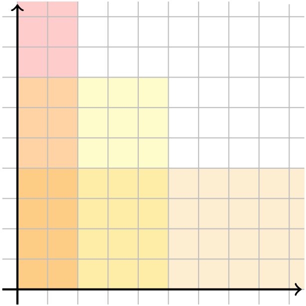
- Effective representations
  [Goubault-Larrecq et al.'17]



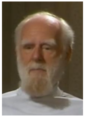
**EXAMPLE (OVER $\mathbb{N}^2$)**

$$D = [\![(2, \infty)]\!] \cup [\![(5, 7)]\!] \cup [\![(\infty, 4)]\!]$$

# Decomposition Theorem

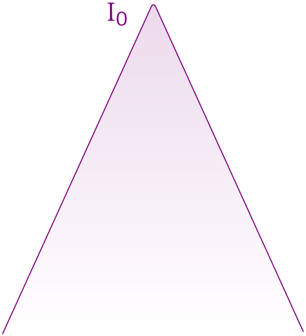
## Well-Quasi-Order on Runs

combination of Dickson's and Higman's lemmata



Syntax

Semantics



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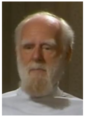

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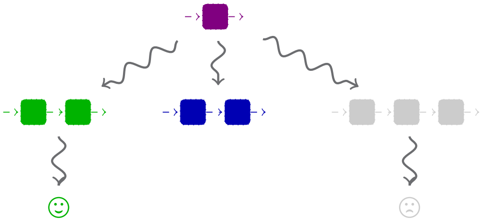

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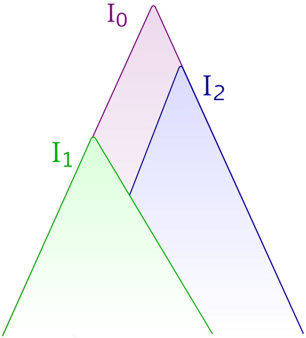

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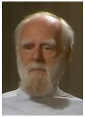

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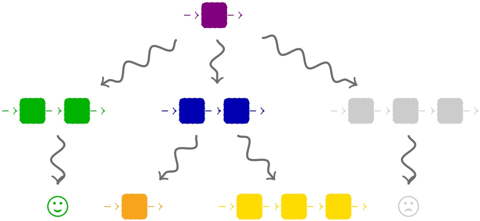

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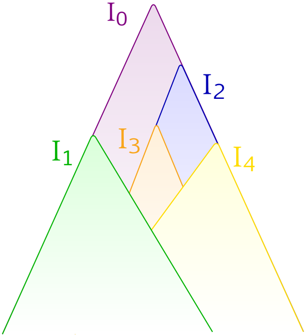

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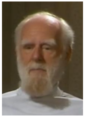

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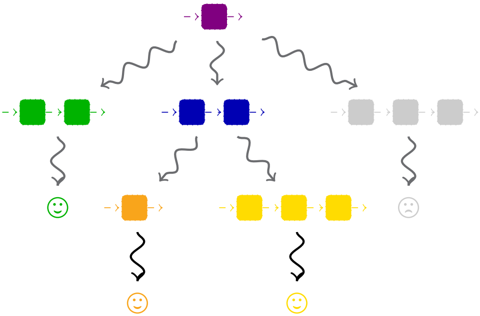

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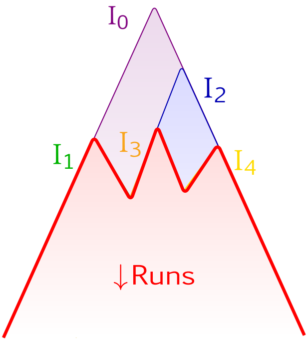
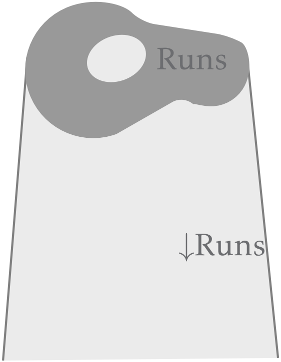


# Adherence Membership

- $I$ is adherent to Runs if $I \subseteq {\downarrow}(I \cap \text{Runs})$
- semantic equivalent to $\Theta$ condition
- undecidable for arbitrary ideals
- decidable for the ideals arising in the decomposition algorithm



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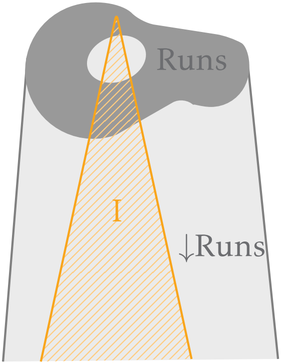


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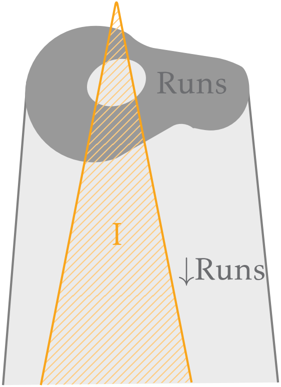


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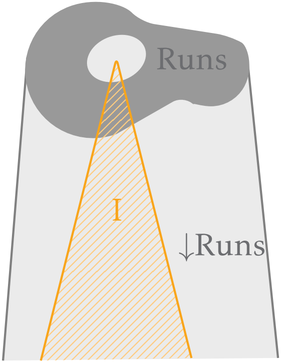


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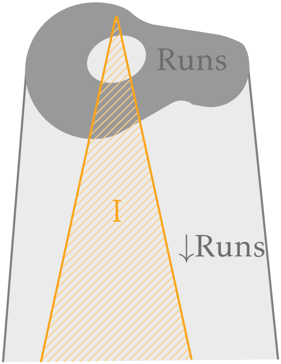


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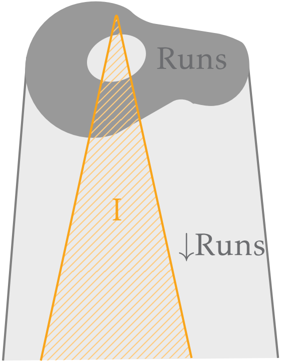


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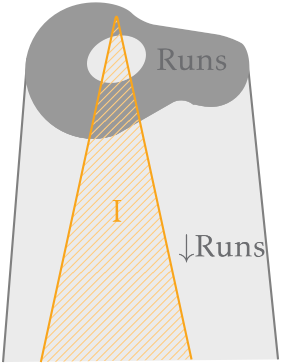


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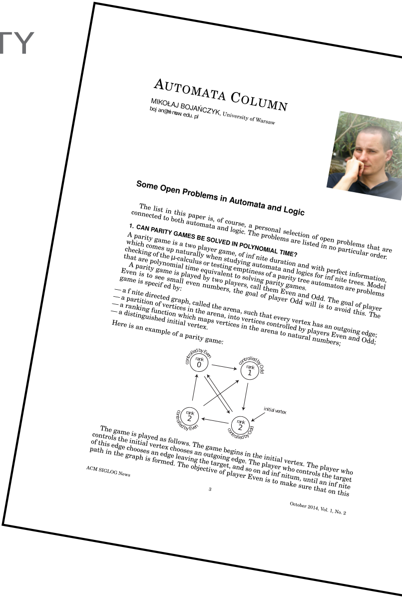

AUTOMATA COLUMN

MIKOŁAJ BOJAŃCZYK, University of Warsaw
boj an@mimuw.edu.pl

## Some Open Problems in Automata and Logic

The list in this paper is, of course, a personal selection of open problems that are connected to both automata and logic. The problems are listed in no particular order.

**1. CAN PARITY GAMES BE SOLVED IN POLYNOMIAL TIME?**

A parity game is a two player game, of inf nite duration and with perfect information, which comes up naturally when studying automata and logics for inf nite trees. Model checking of the μ-calculus or testing emptiness of a parity tree automaton are problems that are polynomial time equivalent to solving parity games.

A parity game is played by two players, call them Even and Odd. The goal of player Even is to see small even numbers, the goal of player Odd will is to avoid this. The game is specif ed by:

— a f nite directed graph, called the arena, such that every vertex has an outgoing edge;
— a partition of vertices in the arena, into vertices controlled by players Even and Odd;
— a ranking function which maps vertices in the arena to natural numbers;
— a distinguished initial vertex.

Here is an example of a parity game:



The game is played as follows. The game begins in the initial vertex. The player who controls the initial vertex chooses an outgoing edge. The player who controls the target of this edge chooses an edge leaving the target, and so on ad inf nitum, until an inf nite path in the graph is formed. The objective of player Even is to make sure that on this

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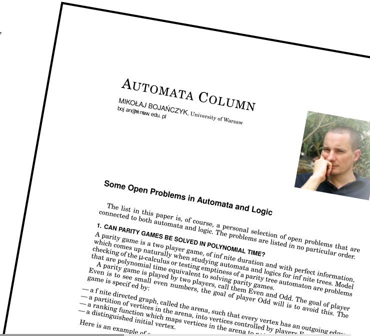
Automata Column
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Here is an example of…

| Complexity Theory | Distributed Computing | Proof Theory | Computational Biology |
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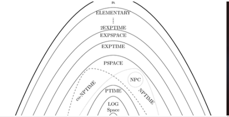
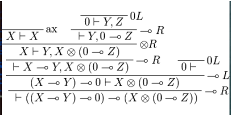
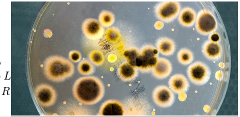
|  |  |  |  |
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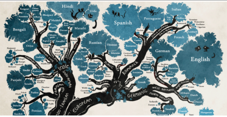
| (code below) |  |  |  |



```
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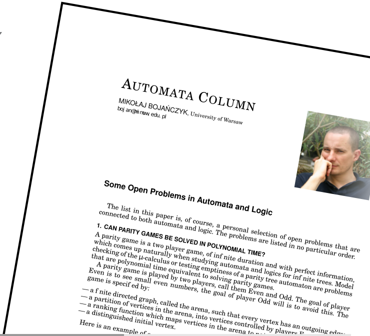

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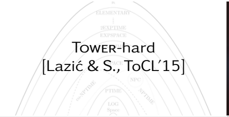
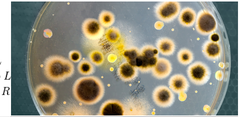


Tower-hard
[Lazić & S., ToCL'15]

$$\dfrac{\dfrac{\dfrac{\dfrac{}{X \vdash X}\,\mathrm{ax} \qquad \dfrac{\dfrac{}{0 \vdash Y, Z}\,0L}{\vdash Y, 0 \multimap Z}\multimap R}{X \vdash Y, X \otimes (0 \multimap Z)}\otimes R}{\vdash X \multimap Y, X \otimes (0 \multimap Z)}\multimap R \qquad \dfrac{}{0 \vdash}\,0L}{(X \multimap Y) \multimap 0 \vdash X \otimes (0 \multimap Z)}\multimap L}{\vdash ((X \multimap Y) \multimap 0) \multimap (X \otimes (0 \multimap Z))}\multimap R$$



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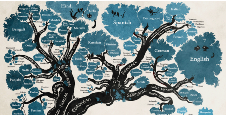

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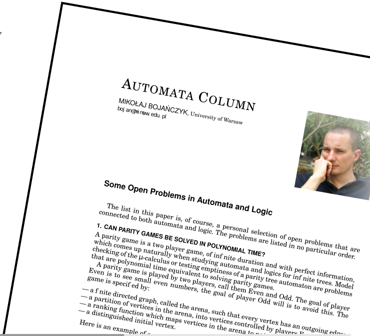
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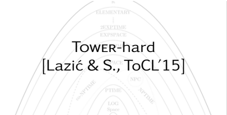


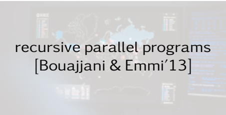

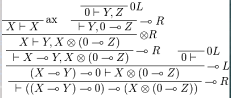


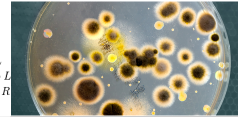

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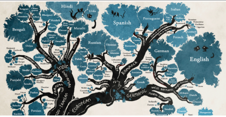

# BRANCHING VAS REACHABILITY

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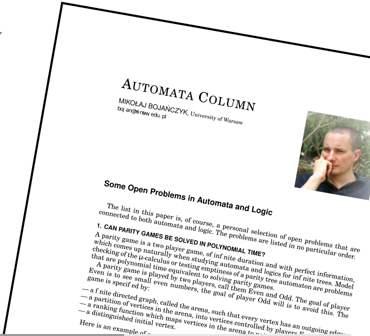
- application domains:


AUTOMATA COLUMN

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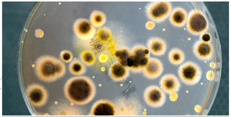
| TOWER-hard [Lazić & S., ToCL'15] | recursive parallel programs [Bouajjani & Emmi'13] | linear and relevance logics [de Groote et al.'04 Lazić & S., ToCL'15 S., JSL'16] |  |

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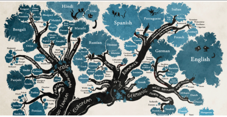
|---|---|---|---|
| see code below |  |  |  |

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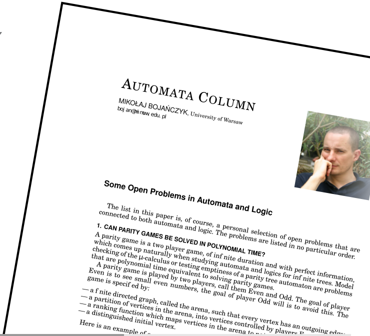
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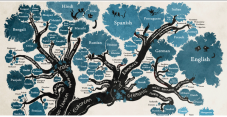

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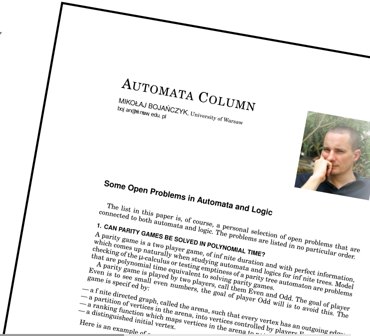
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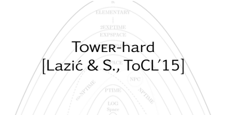
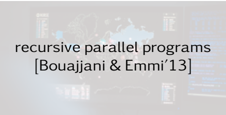

Tower-hard
[Lazić & S., ToCL'15]

**Distributed Computing**


recursive parallel programs
[Bouajjani & Emmi'13]

**Proof Theory**

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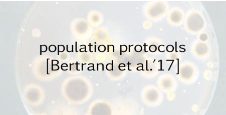
population protocols
[Bertrand et al.'17]

**Programming Languages**

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**Database Theory**

**Security**

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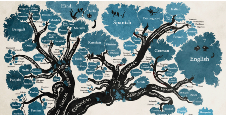

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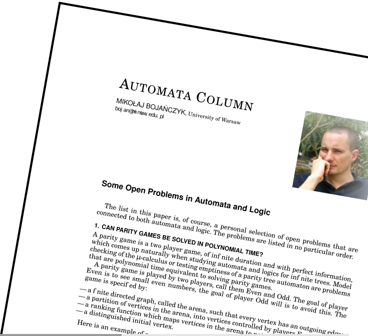
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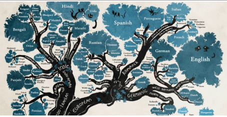
| observational equivalence [Cotton-Barratt et al.'17] | data logics [Bojańczyk et al.'09, Abriola et al.'17] |  |  |

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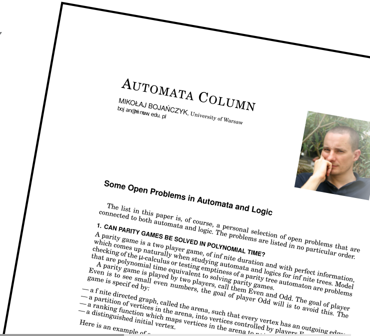

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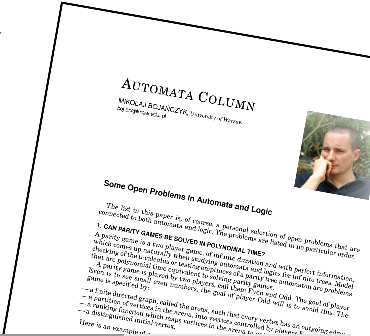

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