On the Complexity of VAS Reachability

Sylvain Schmitz

based on joint works with D. Figueira, S. Figueira, J. Leroux, and Ph. Schnoebelen

Centre Fédéré en Vérification, February 22, 2019

Outline

well-quasi-orders (wqo):

- proving algorithm termination

a toolbox for wqo complexity

- upper bounds
- lower bounds
- complexity classes

this talk: focus on one problem

- reachability in vector addition systems

OUTLINE

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Vector Addition Systems

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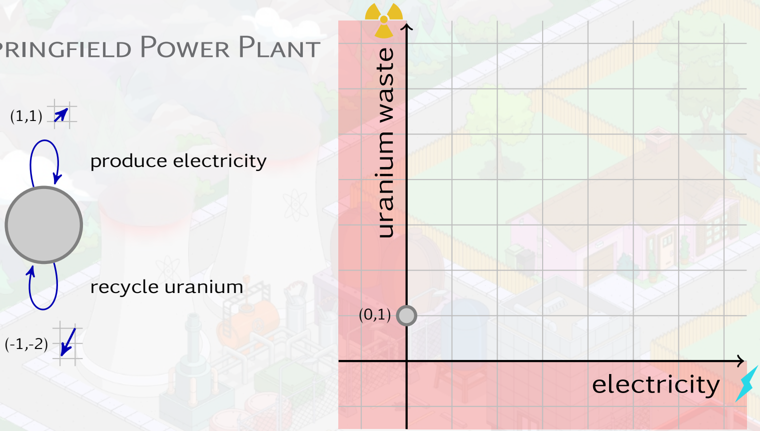
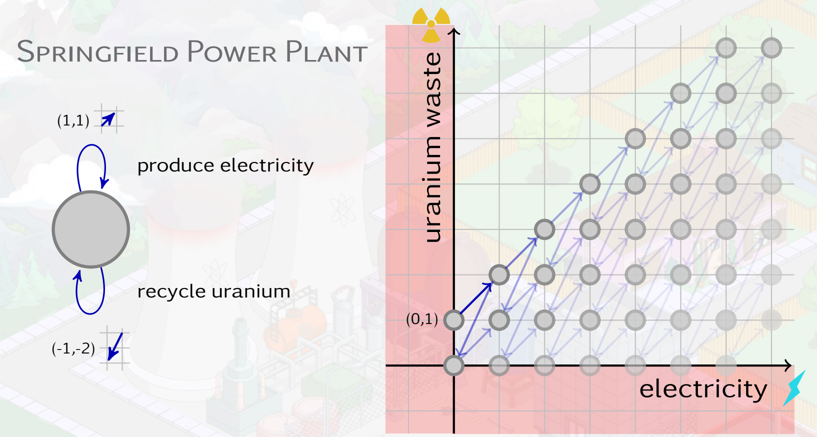
Vector Addition Systems

Springfield Power Plant

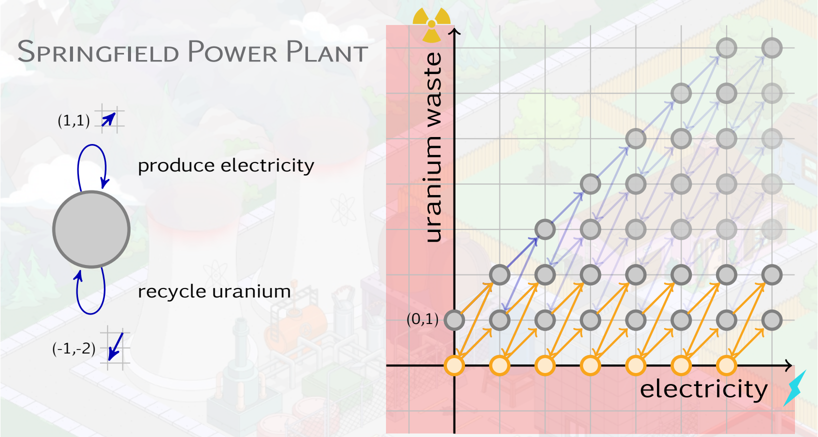


Vector Addition Systems



Can we produce unbounded electricity with no left-over uranium waste?

Vector Addition Systems



Can we produce unbounded electricity with no left-over uranium waste? Yes, $(\infty,0)$ is reachable

Importance of the Problem

Reachability Problem

input: *a vector addition system and two configurations* **source** *and* **target**

question: **source** $\rightarrow^*$ **target***?*

Importance of the Problem

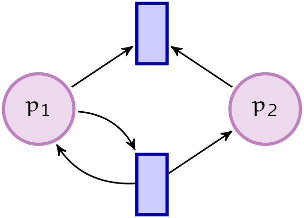
Discrete Resources

- modelling: items, money, energy, molecules, ...
- distributed computing: active threads in thread pool
- data: isomorphism types in data logics and data-centric systems

Importance of the Problem

Example: Petri Nets

Petri net

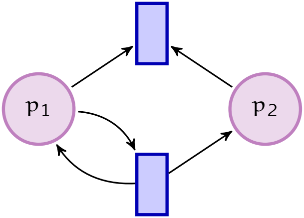


VAS

Importance of the Problem

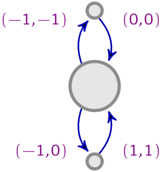
Example: Petri Nets

Petri net



$\Rightarrow$

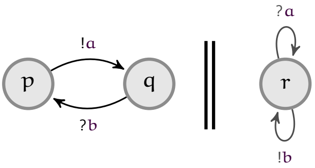
VAS



Importance of the Problem

Example: Unordered CFSM

Communicating Finite-State Machine

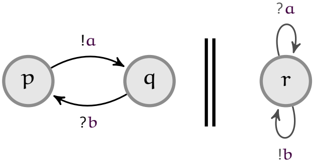


unordered channel:

Importance of the Problem

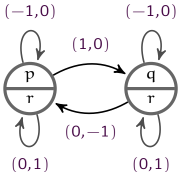
Example: Unordered CFSM

Communicating Finite-State Machine



$\Rightarrow$

VAS



unordered channel:

aabab

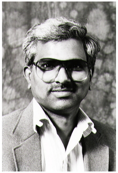
counters:

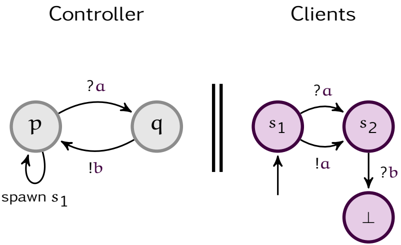
a $\boxed{3}$ $\boxed{2}$ b

Importance of the Problem

Example: Asynchronous Rendez-vous

[German & Prasad Sistla'92]





Importance of the Problem

Example: Asynchronous Rendez-vous

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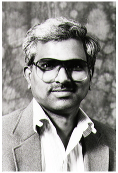
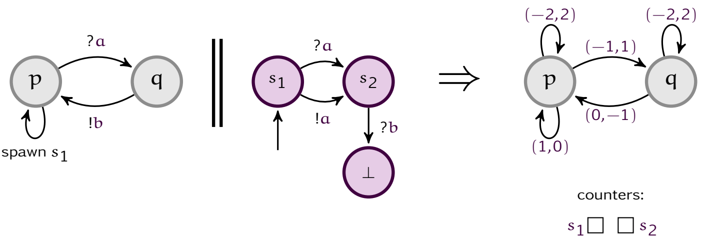


Controller Clients VAS



Importance of the Problem

Central Decision Problem [S.'16]

Large number of problems interreducible with reachability in vector addition systems

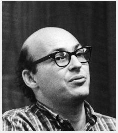
The Complexity of Reachability in Vector Addition Systems

SYLVAIN SCHMITZ,
LSV, ENS Cachan & CNRS & INRIA, Université Paris-Saclay

Importance of the Problem

Theorem (Minsky'67)

Reachability is undecidable in 2-dimensional Minsky machines (vector addition systems with zero tests).



Importance of the Problem

1962
1976
1981
1982
1992
2011
2015
2017
2018
2019


R. J. Lipton: EXPSPACE lower bound


E. W. Mayr: decidability by decomposition

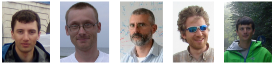

S. R. Kosaraju: decidability by decomposition


J.-L. Lambert: decidability by decomposition


J. Leroux: decidability by Presburger inductive invariants

Leroux & S.: cubic Ackermann upper bound ($\mathbf{F}_{\omega^3}$)

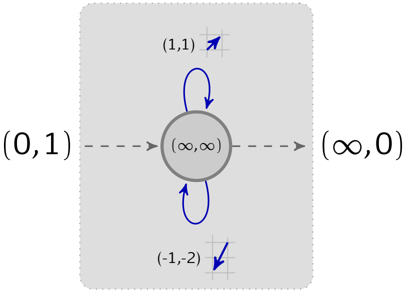
S.: quadratic Ackermann upper bound ($\mathbf{F}_{\omega^2}$)


W. Czerwinski, S. Lasota, R. Lazić, J. Leroux, F. Mazowiecki: Tower lower bound ($\mathbf{F}_3$)

J. Leroux & S.: Ackermann upper bound ($\mathbf{F}_\omega$)

"Simple Runs" (Θ Condition)

[Mayr'81, Kosaraju'82, Lambert'92]



"Simple Runs" (Θ Condition)

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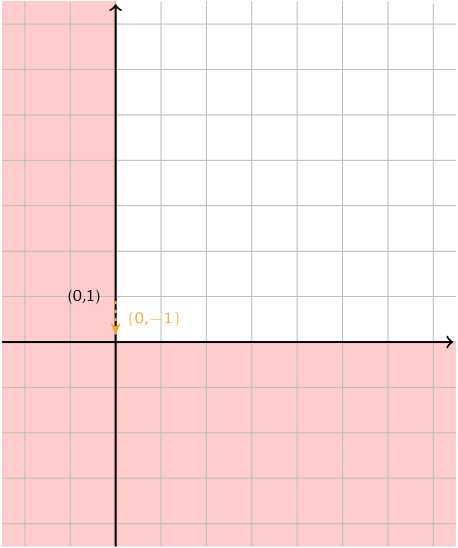

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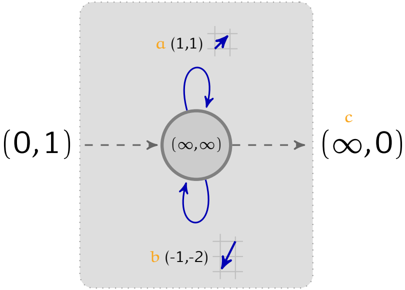
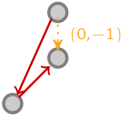


Characteristic System

$$0 + 1 \cdot a - 1 \cdot b = c$$
$$1 + 1 \cdot a - 2 \cdot b = 0$$

Solution Path



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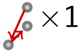
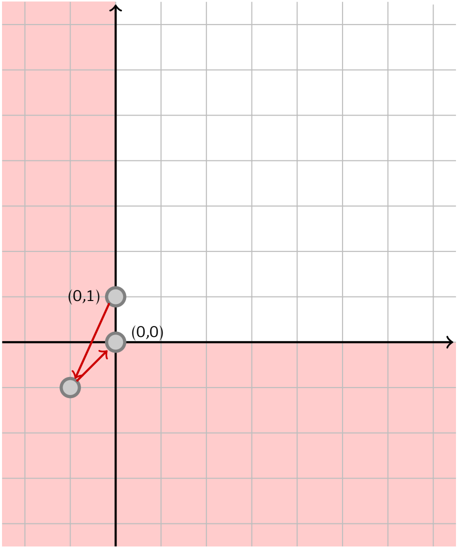


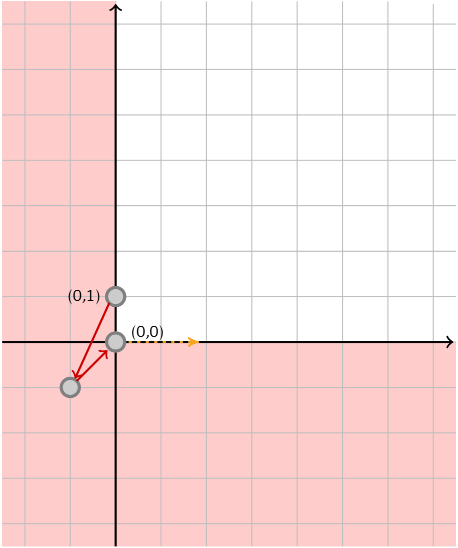


"Simple Runs" (Θ Condition)

[Mayr'81, Kosaraju'82, Lambert'92]

solution path

$\times 1$



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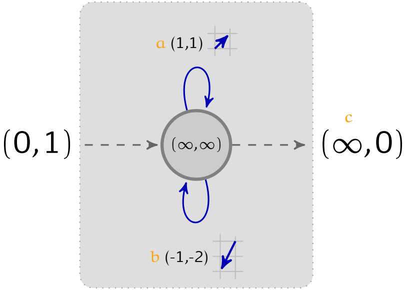
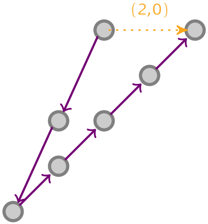


Homogeneous System

$$1\cdot a - 1\cdot b = c$$
$$1\cdot a - 2\cdot b = 0$$
$$a,b,c > 0$$

Unbounded Path



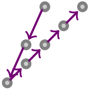
"Simple Runs" (Θ Condition)

[Mayr'81, Kosaraju'82, Lambert'92]

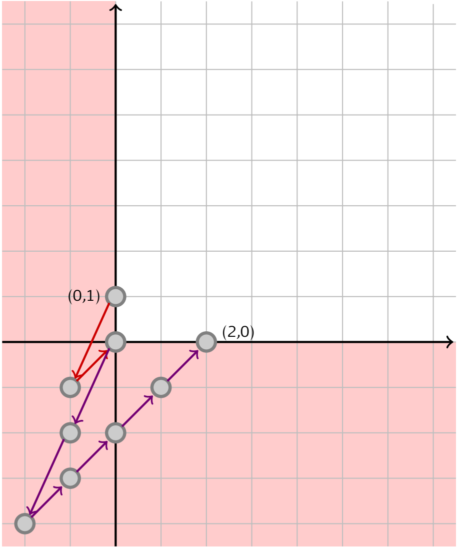
solution path

$\times 1$

unbounded path



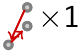
$\times 1$



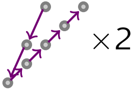
"Simple Runs" (Θ Condition)

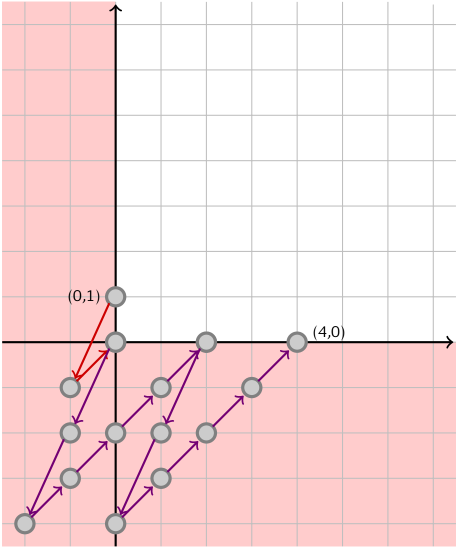
[Mayr'81, Kosaraju'82, Lambert'92]

solution path



unbounded path





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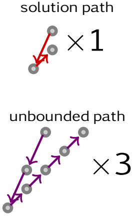
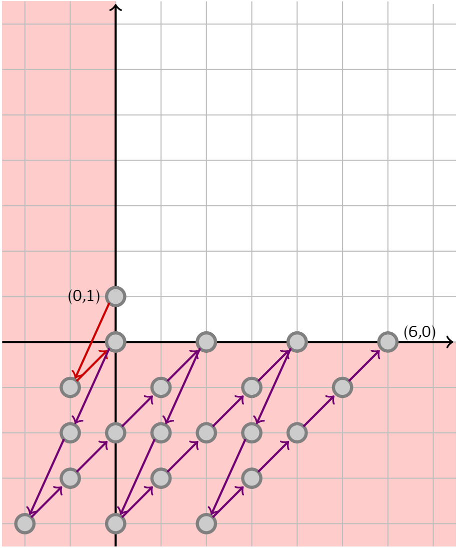
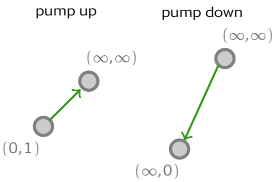




"Simple Runs" (Θ Condition)

[Mayr'81, Kosaraju'82, Lambert'92]

Pumpable Paths



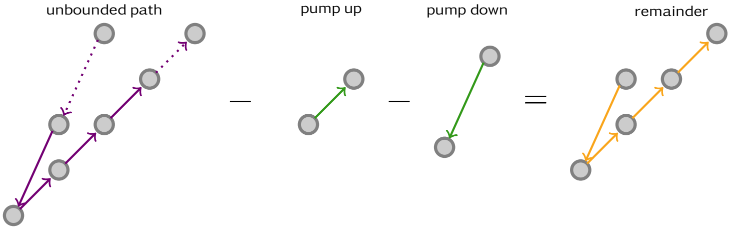
classically: uses *coverability trees* [Karp & Miller'69]

in [Leroux & S.'19] *Rackoff*-style witnesses

"Simple Runs" (Θ Condition)

[Mayr'81, Kosaraju'82, Lambert'92]

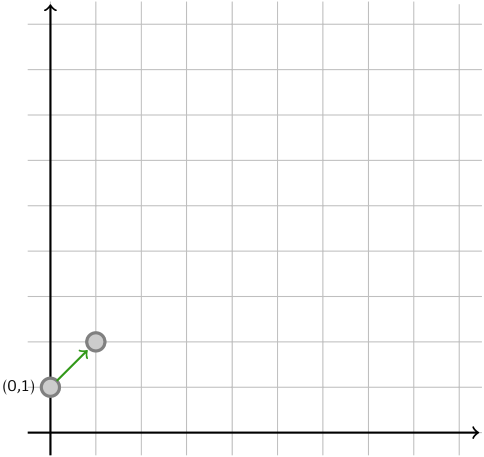
Pumpable Paths



"Simple Runs" (Θ Condition)

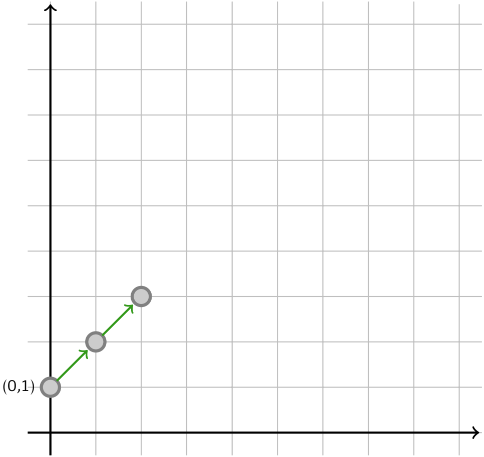
[Mayr'81, Kosaraju'82, Lambert'92]

pump up



"Simple Runs" (Θ Condition)

[Mayr'81, Kosaraju'82, Lambert'92]



"Simple Runs" (Θ Condition)

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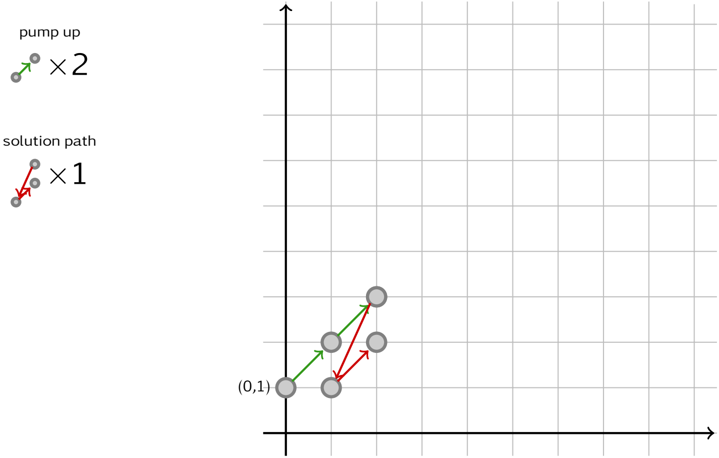

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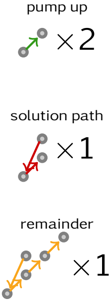
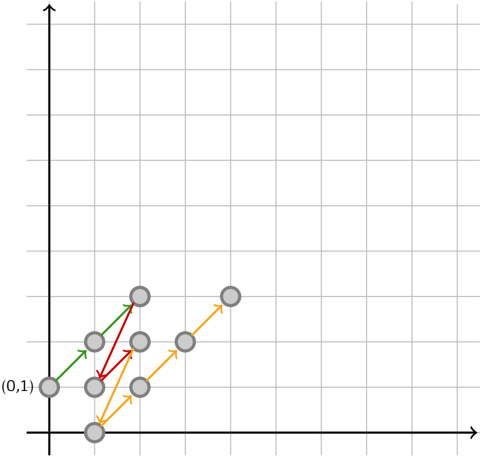




"Simple Runs" (Θ Condition)

[Mayr'81, Kosaraju'82, Lambert'92]

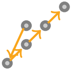
pump up

$\times 2$

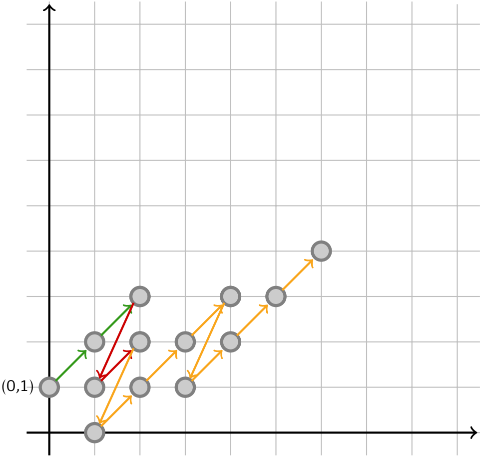
solution path

$\times 1$

remainder



$\times 2$



"Simple Runs" (Θ Condition)

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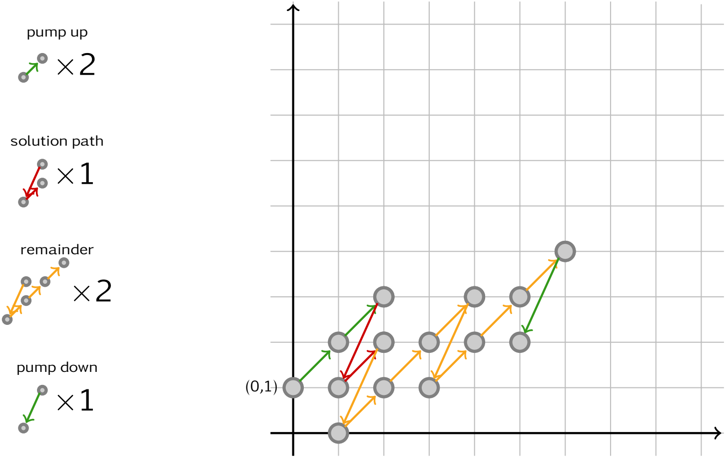

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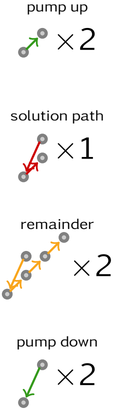

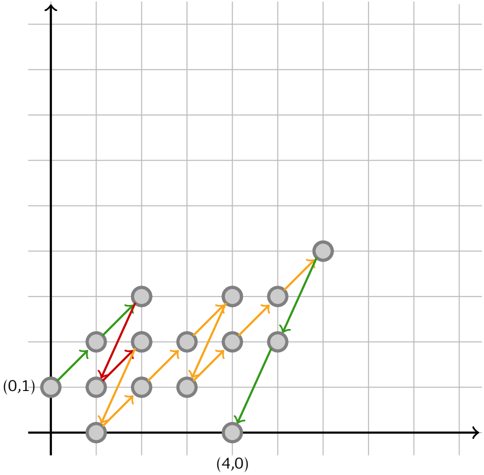

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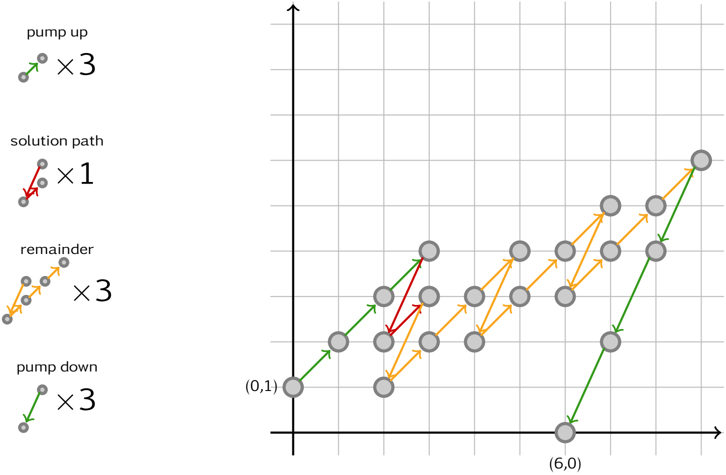
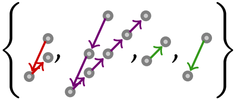


Decomposition Algorithm

[Mayr'81, Kosaraju'82, Lambert'92]

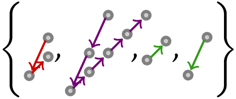
can we build a "simple run"?



Decomposition Algorithm

[Mayr'81, Kosaraju'82, Lambert'92]

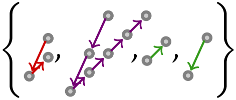
can we build a "simple run"? yes

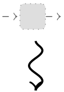


Decomposition Algorithm

[Mayr'81, Kosaraju'82, Lambert'92]

can we build a "simple run"? no





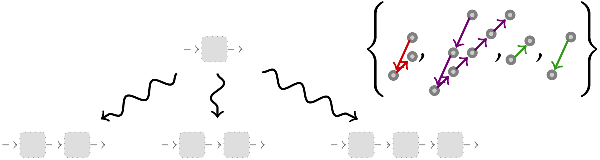
☹

decompose

Decomposition Algorithm

[Mayr'81, Kosaraju'82, Lambert'92]

can we build a "simple run"? no



decompose

Decomposition Algorithm

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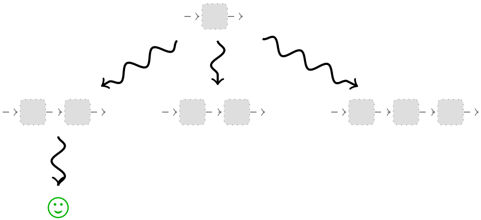

Decomposition Algorithm

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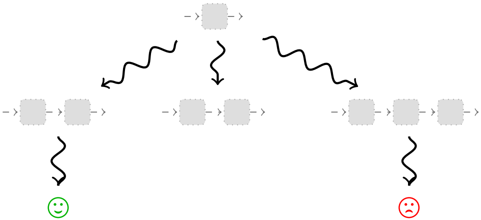

Decomposition Algorithm

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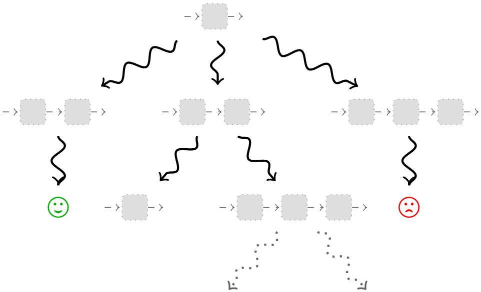


Termination

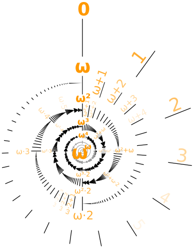
"Finally the checker has to verify that the process comes to an end. Here again he should be assisted by the programmer giving a further definite assertion to be verified. This may take the form of a quantity which is asserted to decrease continually and vanish when the machine stops."

[Turing'49]

Termination

"Finally the checker has to verify that the process comes to an end. Here again he should be assisted by the programmer giving a further definite assertion to be verified. This may take the form of a quantity which is asserted to decrease continually and vanish when the machine stops. To the pure mathematician it is natural to give an ordinal number."

[Turing'49]



Termination of the Decomposition Algorithm

[Mayr'81, Kosaraju'82, Lambert'92, Leroux & S.'19]

Ranking Function

$$\omega^{\omega} \quad (\omega^{d+1} \text{ in dim. } d)$$

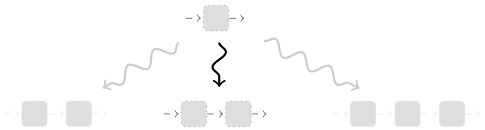
$$\vee$$

$$\alpha_0$$

Termination of the Decomposition Algorithm

[Mayr'81, Kosaraju'82, Lambert'92, Leroux & S.'19]

Ranking Function



$\omega^\omega \quad (\omega^{d+1}$ in dim. d$)$

$\vee$

α_0

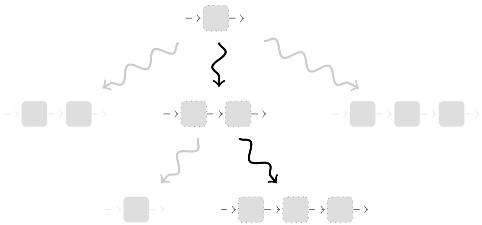
$\vee$

α_1

Termination of the Decomposition Algorithm

[Mayr'81, Kosaraju'82, Lambert'92, Leroux & S.'19]

Ranking Function

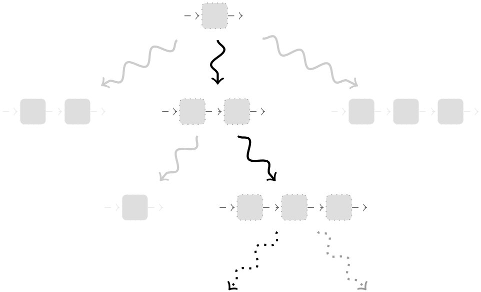


$$\omega^\omega \quad (\omega^{d+1} \text{ in dim. } d)$$
$$\vee$$
$$\alpha_0$$
$$\vee$$
$$\alpha_1$$
$$\vee$$
$$\alpha_2$$

Termination of the Decomposition Algorithm

[Mayr'81, Kosaraju'82, Lambert'92, Leroux & S.'19]

Ranking Function



$$\omega^\omega \quad (\omega^{d+1} \text{ in dim. } d)$$
$$\vee$$
$$\alpha_0$$
$$\vee$$
$$\alpha_1$$
$$\vee$$
$$\alpha_2$$
$$\vee$$
$$\vdots$$

Upper Bounds

How to bound the running time of algorithms with ordinal-based termination proofs?

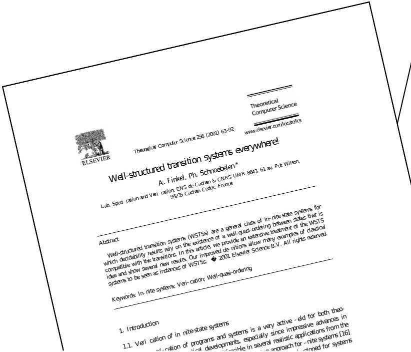
Upper Bounds

How to bound the running time of algorithms with wqo-based termination proofs?

Upper Bounds

How to bound the running time of algorithms with wqo-based termination proofs?

wqos ubiquitous in infinite-state verification


Theoretical Computer Science 256 (2001) 63–92

Theoretical Computer Science

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Well-structured transition systems everywhere!

A. Finkel, Ph. Schnoebelen*

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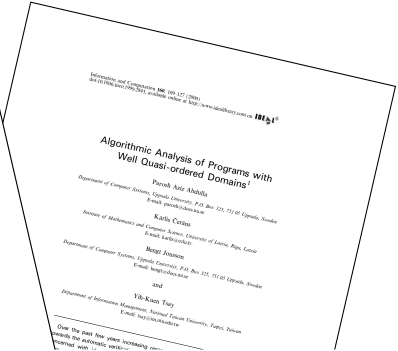
Abstract

Well-structured transition systems (WSTSs) are a general class of infinite-state systems for which decidability results rely on the existence of a well-quasi-ordering between states that is compatible with the transitions. In this article, we provide an extensive treatment of the WSTS idea and show several new results. Our improved definitions allow many examples of classical systems to be seen as instances of WSTSs. © 2001 Elsevier Science B.V. All rights reserved.

Keywords: Infinite systems; Verification; Well-quasi-ordering

1. Introduction

1.1. Verification of infinite-state systems


Information and Computation 160, 109–127 (2000)
doi:10.1006/inco.1999.2843, available online at http://www.idealibrary.com on IDEAL

Algorithmic Analysis of Programs with Well Quasi-ordered Domains[1]

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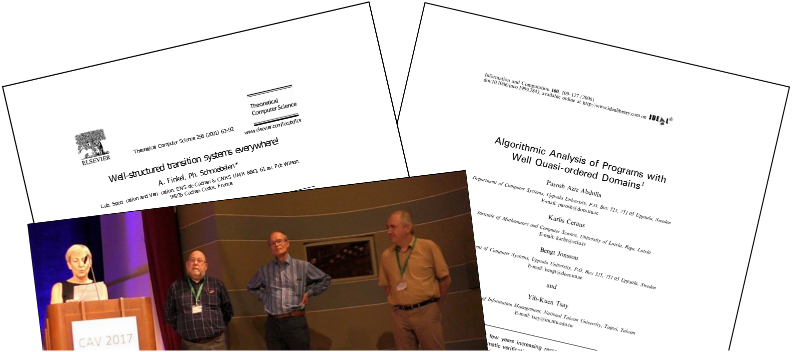
Upper Bounds

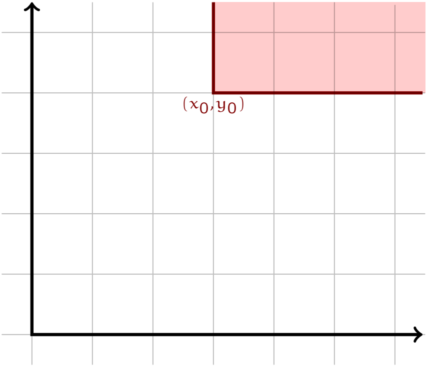
How to bound the running time of algorithms with wqo-based termination proofs?

wqos ubiquitous in infinite-state verification



A One-Player Game

- over $\mathbb{Q}_{\geqslant 0} \times \mathbb{Q}_{\geqslant 0}$
- given initially (x_0, y_0)
- Eloise plays (x_j, y_j) s.t. $\forall 0 \leqslant i < j, x_i > x_j$ or $y_i > y_j$



- Can Eloise win, i.e. play indefinitely?
- If not, how long can she last?

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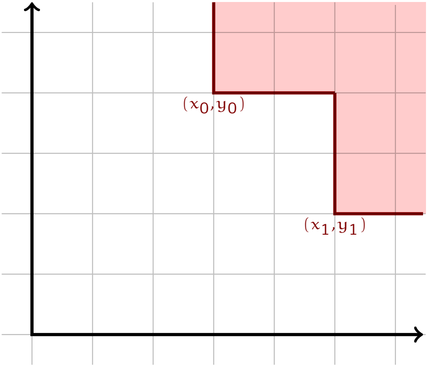

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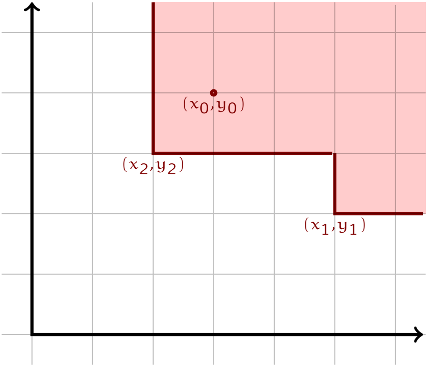

- Can Eloise win, i.e. play indefinitely?
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If $(x_0, y_0) \neq (0,0)$, then choosing $(x_j, y_j) = (\frac{x_0}{2^j}, \frac{y_0}{2^j})$ wins.

A One-Player Game

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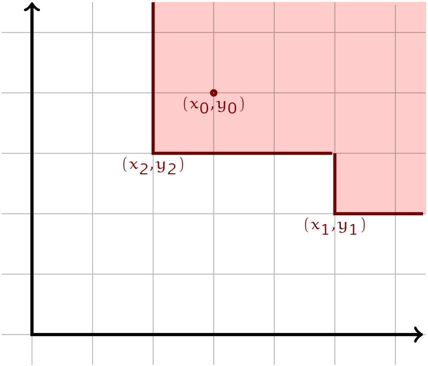

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$$
\begin{aligned}
\text{purple} \quad & \text{if } x_i > x_j \text{ but } y_i \leqslant y_j, \\
\text{red} \quad & \text{if } x_i > x_j \text{ and } y_i > y_j, \\
\text{orange} \quad & \text{if } y_i > y_j \text{ but } x_i \leqslant x_j.
\end{aligned}
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$(3,4) \qquad (5,2) \qquad (2,3) \qquad \cdots$

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$$(3,4) \qquad (5,2) \qquad (2,3) \qquad \cdots$$

By the infinite Ramsey Theorem, there exists an infinite monochromatic subset of indices.

Assume there exists an infinite sequence $(x_j, y_j)_j$ of moves over $\mathbb{N}^2$. Consider the pairs of indices $i < j$: color (i,j)

purple if $x_i > x_j$ but $y_i \leqslant y_j$,

red if $x_i > x_j$ and $y_i > y_j$,

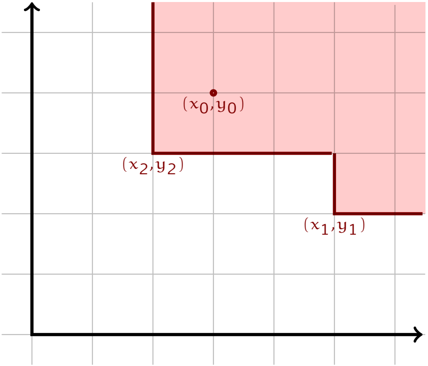
orange if $y_i > y_j$ but $x_i \leqslant x_j$.

$(3,4) \qquad (5,2) \qquad (2,3) \qquad \cdots$

By the infinite Ramsey Theorem, there exists an infinite monochromatic subset of indices. In all cases, it implies the existence of an infinite decreasing sequence in $\mathbb{N}$, a contradiction.

A One-Player Game

- over $\mathbb{N} \times \mathbb{N}$
- given initially (x_0, y_0)
- Eloise plays (x_j, y_j) s.t. $\forall 0 \leqslant i < j, x_i > x_j$ or $y_i > y_j$



- Can Eloise win, i.e. play indefinitely?
- If not, how long can she last?

Bad Sequences

Over a qo $(X, \leqslant)$

- $x_0, x_1, \ldots$ is bad if $\forall i < j \,.\, x_i \not\leqslant x_j$
- $(X, \leqslant)$ wqo iff all bad sequences are finite
-

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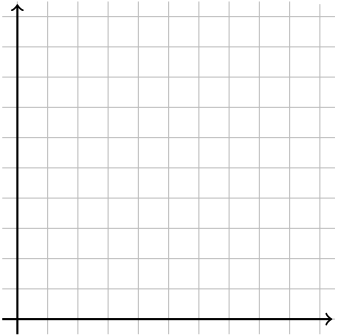


Example (over $\mathbb{N}^2$)

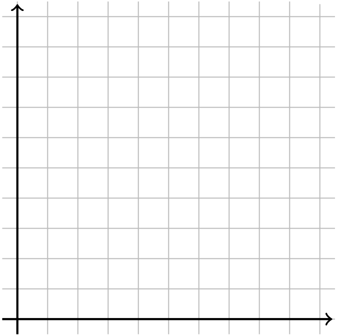
Bad Sequences

Over a qo $(X, \leqslant)$

$x_0, x_1, \ldots$ is bad if $\forall i < j \,.\, x_i \not\leqslant x_j$

$(X, \leqslant)$ wqo iff all bad sequences are finite

but can be of arbitrary length



Example (over $\mathbb{N}^2$)

Controlled Bad Sequences

Over a qo $(X, \leqslant)$ with norm $\|\cdot\|$

- $x_0, x_1, \ldots$ is bad if $\forall i < j \,.\, x_i \not\leqslant x_j$
- $(X, \leqslant)$ wqo iff all bad sequences are finite
- controlled by $g\colon \mathbb{N} \to \mathbb{N}$ monotone and inflationary and $n_0 \in \mathbb{N}$ if $\forall i \,.\, \|x_i\| \leqslant g^i(n_0)$
  [Cichoń & Tahhan Bittar'98]

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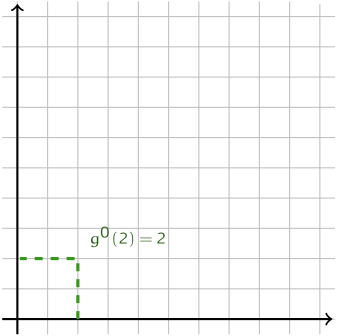

Example (over $\mathbb{N}^2$ with $n_0 = 2$ and $g(n) = n + 1$)

Controlled Bad Sequences

Over a qo $(X, \leqslant)$ with norm $\|\cdot\|$

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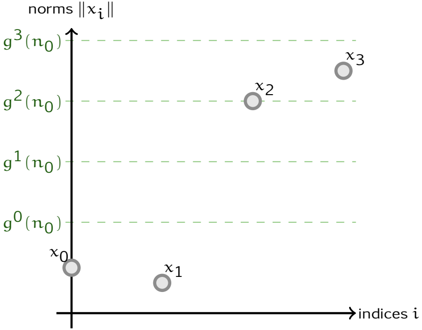
[Cichoń & Tahhan Bittar'98]

Proposition

Over $(X, \leqslant)$, assuming $\forall n \{x \in X \mid \|x\| \leqslant n\}$ finite, (g, n_0)-controlled bad sequences have a maximal length, noted $L_{g,X}(n_0)$.

Descent Equation

(g,n_0)-controlled bad sequence $x_0,x_1,x_2,x_3,\ldots$ over a wqo $(X,\leqslant)$:



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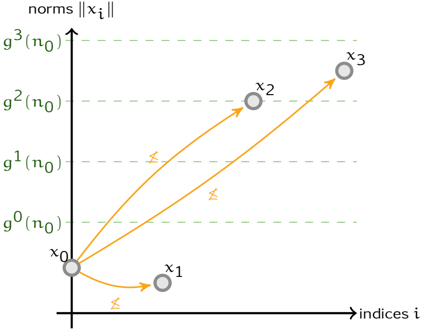

over the suffix $x_1, x_2, x_3, \ldots, \forall i > 0$,

$$x_0 \not\leq x_i$$

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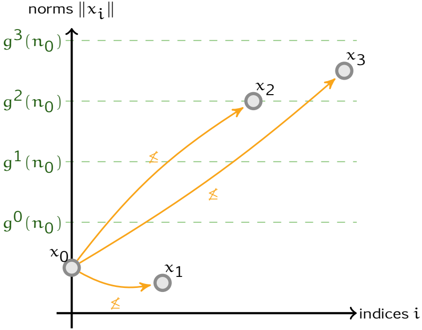

over the suffix $x_1, x_2, x_3, \ldots$, $\forall i > 0$,

$$x_i \in X \setminus \uparrow x_0 \stackrel{\text{def}}{=} \{x \in X \mid x_0 \not\leqslant x\}$$

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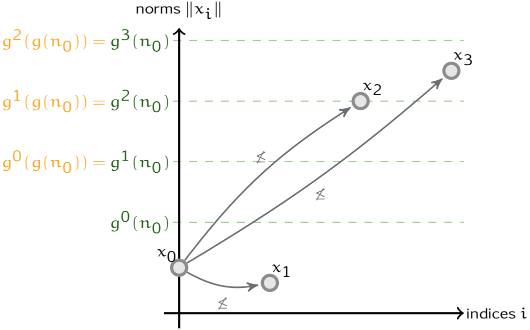

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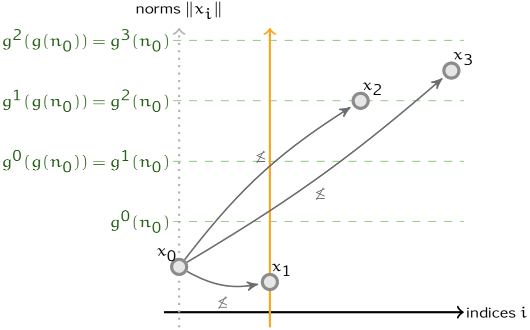

over the suffix $x_1, x_2, x_3, \ldots, \forall i > 0,$

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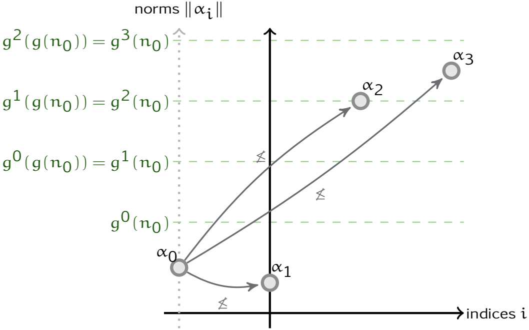
$$L_{g,X}(n_0) = \max_{x_0 \in X, \|x_0\| \leqslant n_0} 1 + L_{g, X\backslash\uparrow x_0}(g(n_0))$$

Descent Equation

(g, n_0)-controlled bad sequence $\alpha_0, \alpha_1, \alpha_2, \alpha_3, \ldots$ over an ordinal α:



over the suffix $\alpha_1, \alpha_2, \alpha_3, \ldots, \forall i > 0$,

$\alpha_i \in \alpha_0 \stackrel{\text{def}}{=} \{\beta \in \alpha \mid \beta \not\geq \alpha_0\}$

$\|\alpha_i\| \leqslant g^{i-1}(g(n_0))$

$$L_{g,\alpha}(n_0) = \max_{\alpha_0 \in \alpha, \|\alpha_0\| \leqslant n_0} 1 + L_{g,\alpha_0}(g(n_0))$$

The Case of Ordinals

[S.'14]

- Cantor Normal Form (CNF) for ordinals $\alpha < \varepsilon_0$:

$$\alpha = \omega^{\alpha_1} \cdot c_1 + \cdots + \omega^{\alpha_k} \cdot c_k$$
$$\alpha > \alpha_1 > \cdots > \alpha_k \text{ in CNF}, \qquad 0 < c_1, \ldots, c_k < \omega$$

- Norm of ordinals $\alpha < \varepsilon_0$: "maximal constant"

$$\|\alpha\| \stackrel{\text{def}}{=} \max_{1 \leqslant i \leqslant k} (\max(\|\alpha_i\|, c_i))$$

Example

$$\|\omega^{\omega^2}\| = 2$$
$$\|\omega^{\omega \cdot 5} + \omega^2 \cdot 3\| = 5$$

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Recall the descent equation:

$$L_{g,\alpha}(n_0) = \max_{\alpha_0 \in \alpha, \|\alpha_0\| \leqslant n_0} 1 + L_{g,\alpha_0}(g(n_0))$$

Proposition (variant of [Buchholtz, Cichoń & Weiermann'94])

Let $0 < \alpha < \varepsilon_0$ *and* $\|\alpha\| \leqslant n_0$. *Then*

$$L_{g,0}(n_0) = 0 \qquad L_{g,\alpha}(n_0) = 1 + L_{g,P_{n_0}(\alpha)}(g(n_0))$$

$P_x(\alpha)$ denotes the predecessor at x of $\alpha > 0$: "maximal ordinal $\beta < \alpha$ s.t. $\|\beta\| \leqslant x$"

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$$P_3(\omega^2) = \omega \cdot 3 + 3$$

$$\begin{aligned} P_3(\omega^{\omega^2}) = {} & \omega^{\omega\cdot 3+3}\cdot 3 + \omega^{\omega\cdot 3+2}\cdot 3 + \omega^{\omega\cdot 3+1}\cdot 3 + \omega^{\omega\cdot 3}\cdot 3 \\ & + \omega^{\omega\cdot 2+3}\cdot 3 + \omega^{\omega\cdot 2+2}\cdot 3 + \omega^{\omega\cdot 2+1}\cdot 3 + \omega^{\omega\cdot 2}\cdot 3 \\ & + \omega^{\omega+3}\cdot 3 + \omega^{\omega+2}\cdot 3 + \omega^{\omega+1}\cdot 3 + \omega^{\omega}\cdot 3 \\ & + \omega^{3}\cdot 3 + \omega^{2}\cdot 3 + \omega\cdot 3 + 3 \end{aligned}$$

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These functions were already known in the literature as the Cichón hierarchy

The Case of Ordinals

[S.'14]

Length Function Theorem (for Ordinals)

Let $\alpha < \varepsilon_0$ and $n_0 \geqslant \|\alpha\|$. Then the longest (g, n_0)-controlled descending sequence over α is of length $L_{g,\alpha}(n_0)$ in the Cichón hierarchy.

Relating Norm and Length

[Cichoń & Tahhan Bittar'98]

Recall the definition of the Cichoń Hierarchy:

$$L_{g,0}(x) \stackrel{\text{def}}{=} 0 \qquad Lg,\alpha(x) \stackrel{\text{def}}{=} 1 + L_{g,P_x(\alpha)}(g(x)) \text{ for } \alpha > 0$$

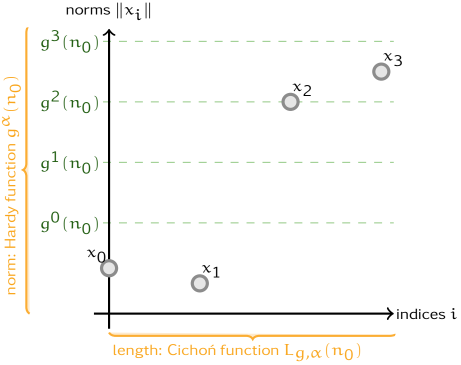
Definition (Hardy Hierarchy)

For $g : \mathbb{N} \to \mathbb{N}$, define $(g^\alpha : \mathbb{N} \to \mathbb{N})_\alpha$ by

$$g^0(x) \stackrel{\text{def}}{=} x \qquad g^\alpha(x) \stackrel{\text{def}}{=} g^{P_x(\alpha)}(g(x)) \text{ for } \alpha > 0$$

Relating Norm and Length

[Cichoń & Tahhan Bittar'98]



$$g^\alpha(x) = g^{L_{g,\alpha}(x)}(x)$$

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Relating Norm and Length

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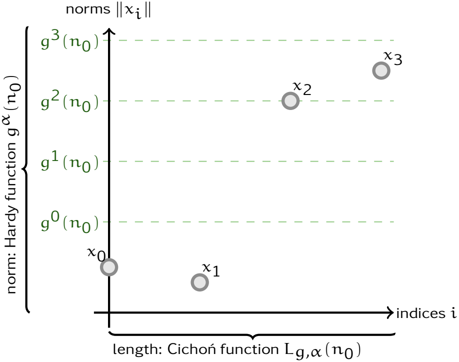

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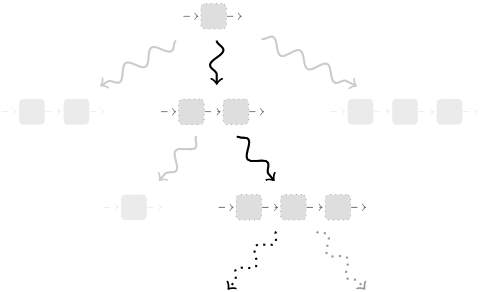
The Length of Decomposition Branches



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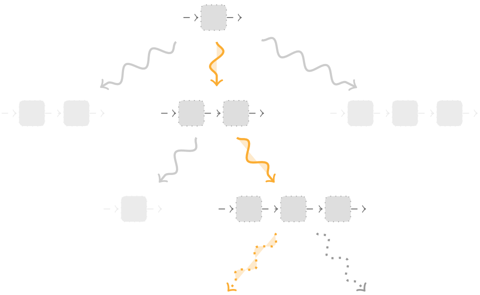
The Length of Decomposition Branches



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Consequence of (Leroux & S.'19)

An elementary control g and n the size of the reachability instance fit. Thus the decomposition algorithm runs in SPACE$(g^{\omega^\omega}(n))$, and SPACE$(g^{\omega^{d+1}}(n))$ in fixed dimension d.

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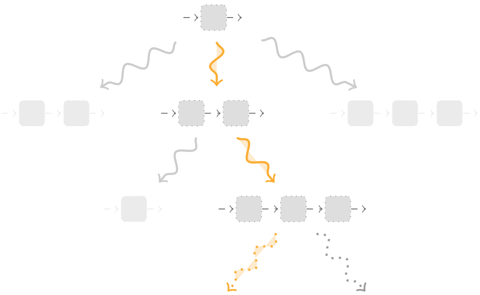

$$\begin{array}{c}\alpha_0\\ \vee\\ \alpha_1\\ \vee\\ \alpha_2\\ \vee\\ \vdots\end{array}$$

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Restating the Result

"SPACE$\left(g^{\omega^{d+1}}(n)\right)$" is unreadable!

Restating the Result

Hardy hierarchy with base function $H(x) \stackrel{\text{def}}{=} x+1$:

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Restating the Result

Define coarse-grained classes:

$$\mathscr{F}_{<\alpha} \stackrel{\text{def}}{=} \bigcup_{\beta<\omega^\alpha} \mathsf{FDTIME}(H^\beta(n))$$

$$\mathbf{F}_\alpha \stackrel{\text{def}}{=} \bigcup_{f\in\mathscr{F}_{<\alpha}} \mathsf{DTIME}(H^{\omega^\alpha}(f(n)))$$

Consequence of (S.'16, Thm. 4.4)

VAS Reachability is in $\mathbf{F}_\omega$, and in $\mathbf{F}_{d+4}$ in fixed dimension d.

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Complexity Classes Beyond Elementary

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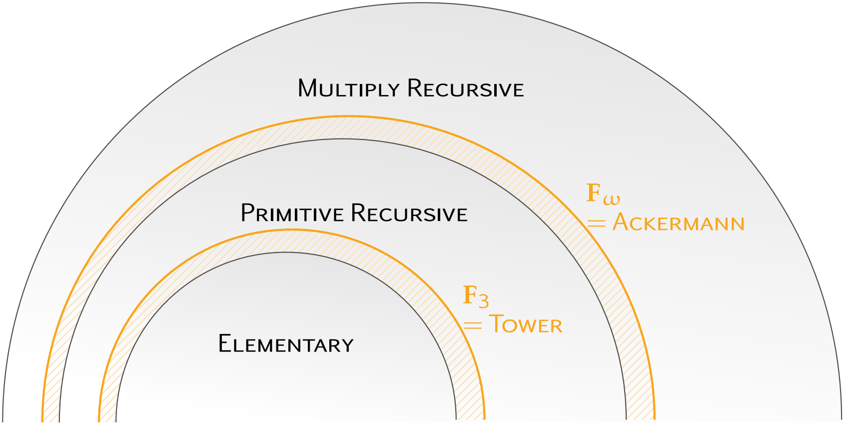

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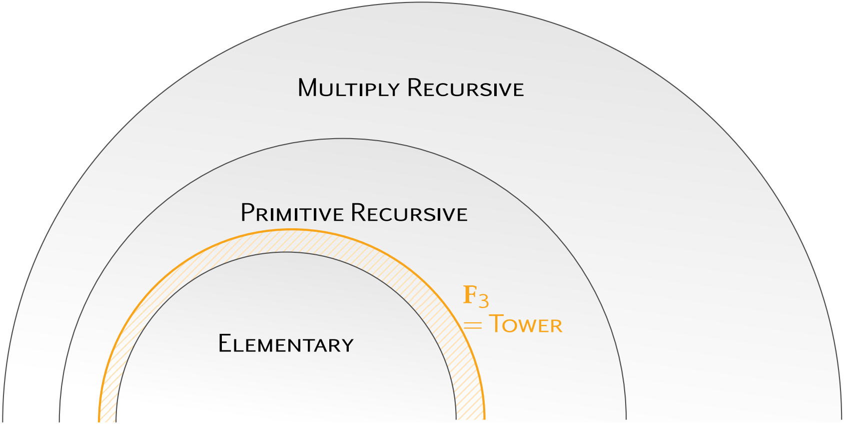

$$\mathbf{F}_3 \stackrel{\text{def}}{=} \bigcup_{e \text{ elementary}} \mathrm{DTIME}\big(\mathrm{tower}(e(n))\big)$$

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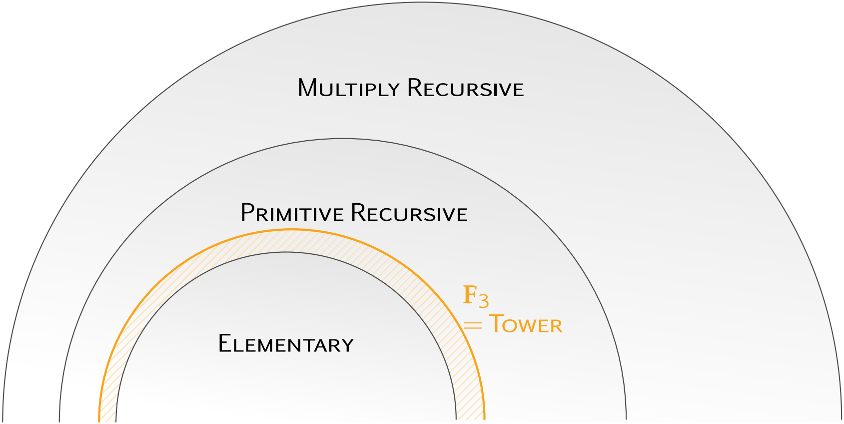

Examples of Tower-Complete Problems:

- satisfiability of first-order logic on words [Meyer'75]
- β-equivalence of simply typed λ terms [Statman'79]
- model-checking higher-order recursion schemes [Ong'06]

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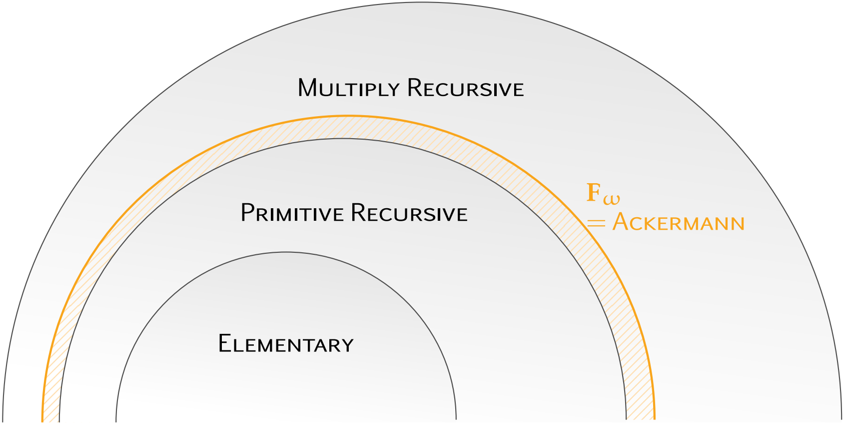

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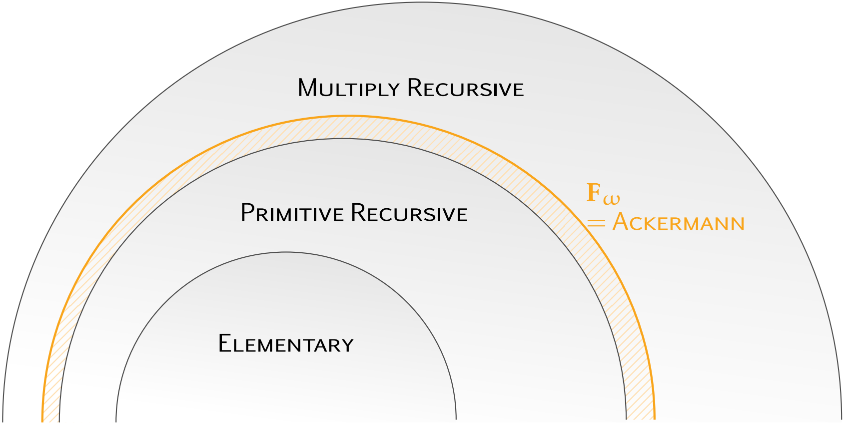

Examples of Ackermann-Complete Problems:

- reachability in lossy Minsky machines [Urquhart'98, Schnoebelen'02]
- satisfiability of safety Metric Temporal Logic [Lazić et al.'16]
- satisfiability of Vertical XPath [Figueira and Segoufin'17]

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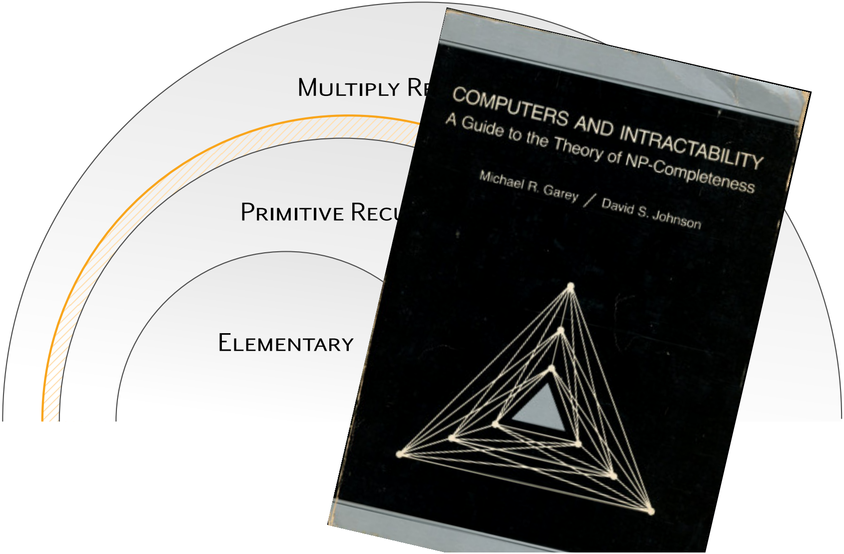

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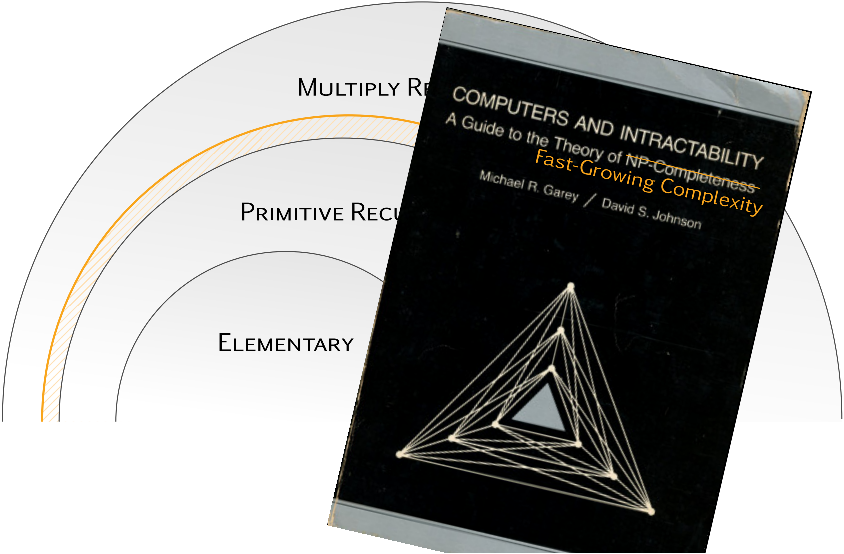

A Related Problem

labelled VAS transitions carry labels from some alphabet

$L(\mathcal{V}, \mathbf{source}, \mathbf{target})$ the language of labels in runs from **source** to **target**

$\downarrow L$ the set of scattered subwords of the words in the language L

Downwards Language Inclusion Problem

input: *two labelled VAS $\mathcal{V}$ and $\mathcal{V}'$ and configurations* $\mathbf{source}, \mathbf{target}, \mathbf{source}', \mathbf{target}'$

question: $\downarrow L(\mathcal{V}, \mathbf{source}, \mathbf{target}) \subseteq \downarrow L(\mathcal{V}', \mathbf{source}', \mathbf{target}')$?

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Theorem (Habermehl, Meyer & Wimmel'10)

Given a labelled VAS $\mathcal{V}$ and configurations **source** *and* **target** *and its decomposition, one can construct a finite automaton for $\downarrow L(\mathcal{V}, \mathbf{source}, \mathbf{target})$ in polynomial time.*

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The Downwards Language Inclusion is in Ackermann.

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Theorem (Zetzsche'16)

The Downwards Language Inclusion is Ackermann*-hard.*

SUMMARY

well-quasi-orders (wqo):

- proving algorithm termination

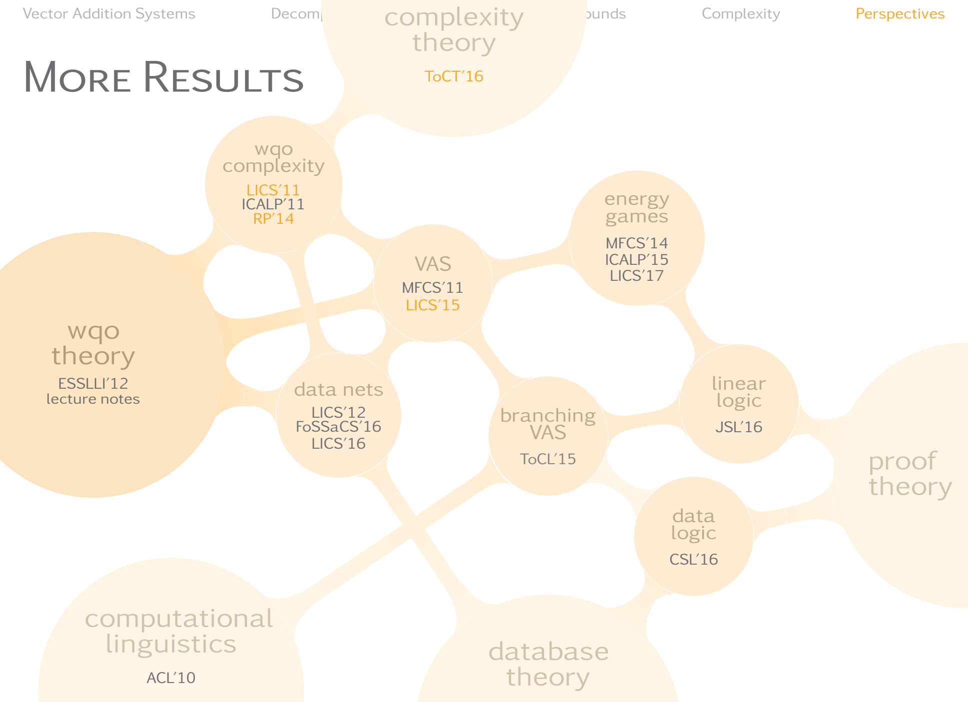
a toolbox for wqo-based complexity

- upper bounds: length function theorems (for ordinals, Dickson's Lemma, Higman's Lemma, and combinations)
- lower bounds
- complexity classes: $(\mathbf{F}_\alpha)_\alpha$

this talk: focus on one problem

- reachability in vector addition systems in $\mathbf{F}_\omega$

More Results



Perspectives

1. complexity gap for VAS reachability
   - Tower-hard [Czerwinski et al.'18]
   - decomposition algorithm: requires $\mathbf{F}_\omega$ = Ackermann time, because downward language inclusion is $\mathbf{F}_\omega$-hard [Zetzsche'16]
2. reachability in VAS extensions
   - decidable in VAS with hierarchical zero tests [Reinhardt'08]
   - what about
     - branching VAS
     - unordered data Petri nets
     - pushdown VAS

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Demystifying Reachability in Vector Addition Systems

[Leroux & S.'15]

Ideal Decomposition Theorem

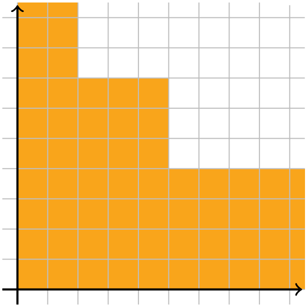
The Decomposition Algorithm computes the ideal decomposition of the set of runs from source to target.

Upper Bound Theorem

Reachability in vector addition systems is in cubic Ackermann.

Ideals of Well-Quasi-Orders $(X, \leqslant)$

- Canonical decompositions
  [Bonnet'75]
  if $D \subseteq X$ is $\downarrow$-closed, then
  $$D = I_1 \cup \cdots \cup I_n$$
  for (maximal) ideals $I_1, \ldots, I_n$



Example (over $\mathbb{N}^2$)

$$D = (\{0,\ldots,2\} \times \mathbb{N}) \cup (\{0,\ldots,5\} \times \{0,\ldots,7\}) \cup (\mathbb{N} \times \{0,\ldots,4\})$$

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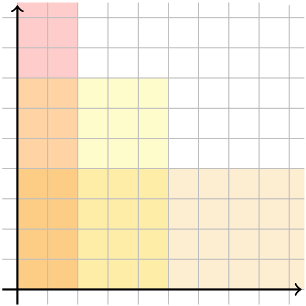

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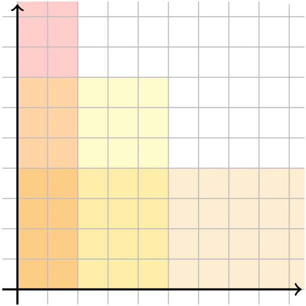
- Effective representations
  [Goubault-Larrecq et al.'17]



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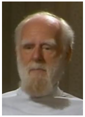
Decomposition Theorem

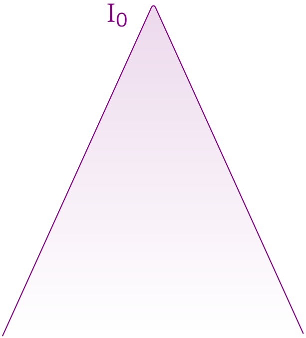
Well-Quasi-Order on Runs

combination of Dickson's and Higman's lemmata



Syntax

Semantics



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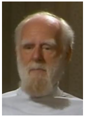

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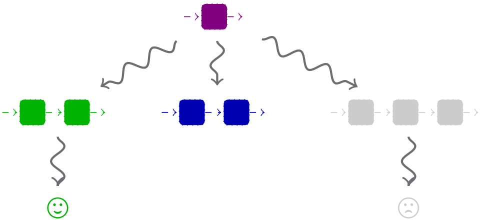

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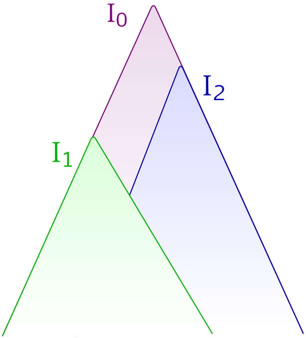

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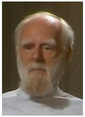

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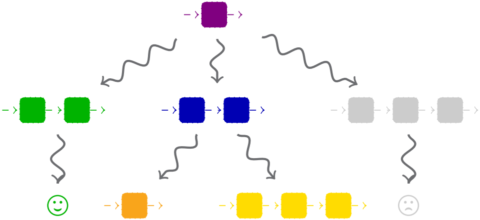

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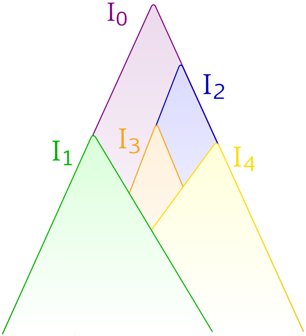

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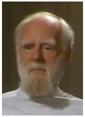

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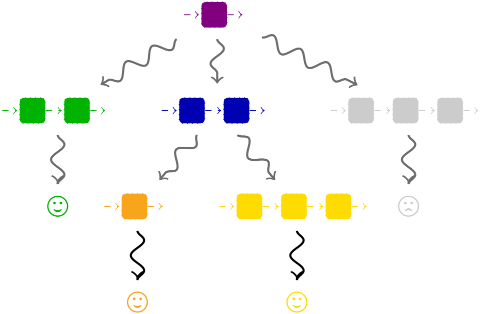

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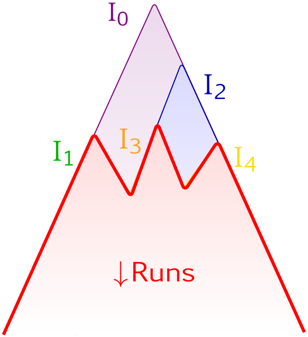
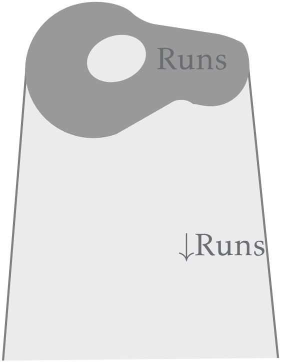


Adherence Membership

- I is adherent to Runs if $I \subseteq {\downarrow}(I \cap \mathrm{Runs})$
- semantic equivalent to Θ condition
- undecidable for arbitrary ideals
- decidable for the ideals arising in the decomposition algorithm



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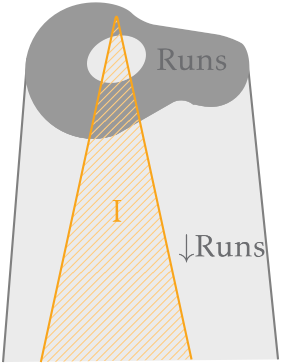

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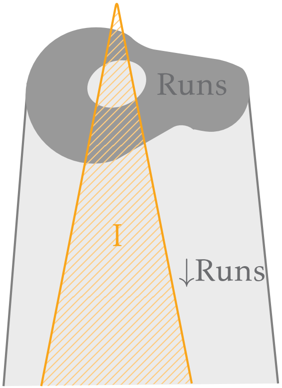

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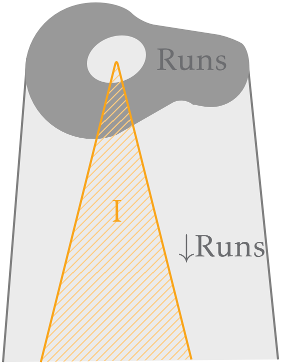

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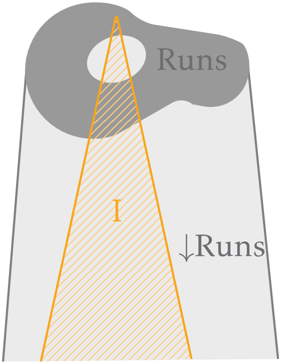

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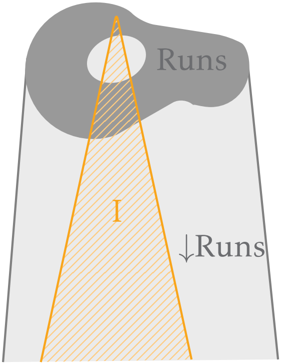

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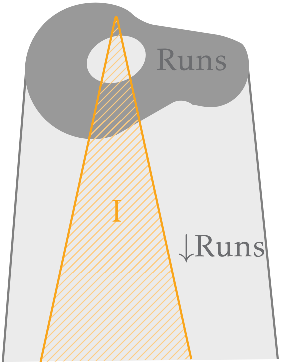

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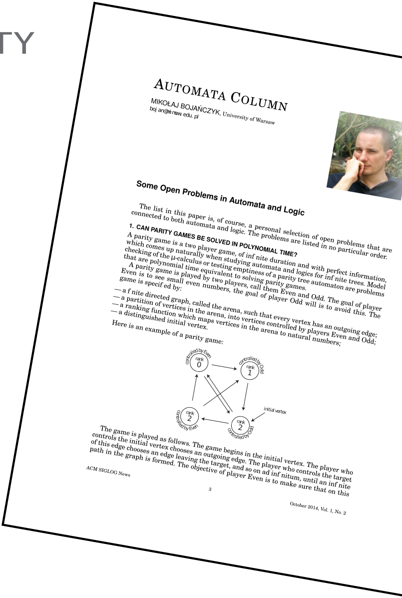
Branching VAS Reachability

- important open problem [Bojańczyk'14]
- incorrect decidability proof in [Bimbó'15]
- application domains:



AUTOMATA COLUMN

MIKOŁAJ BOJAŃCZYK, University of Warsaw
bojan@mimuw.edu.pl

Some Open Problems in Automata and Logic

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1. CAN PARITY GAMES BE SOLVED IN POLYNOMIAL TIME?

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— a f nite directed graph, called the arena, such that every vertex has an outgoing edge;
— a partition of vertices in the arena, into vertices controlled by players Even and Odd;
— a ranking function which maps vertices in the arena to natural numbers;
— a distinguished initial vertex.

Here is an example of a parity game:

The game is played as follows. The game begins in the initial vertex. The player who controls the initial vertex chooses an outgoing edge. The player who controls the target of this edge chooses an edge leaving the target, and so on ad inf nitum, until an inf nite path in the graph is formed. The objective of player Even is to make sure that on this

ACM SIGLOG News 3 October 2014, Vol. 1, No. 2

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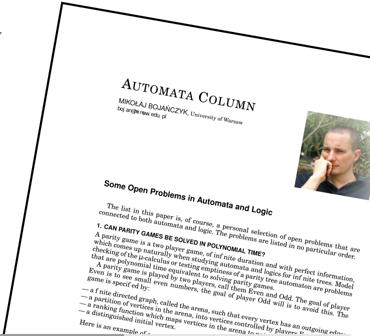
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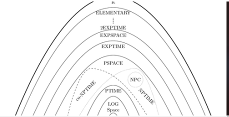		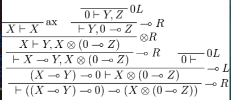	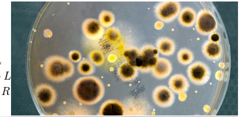
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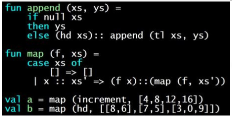			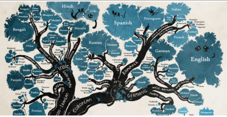

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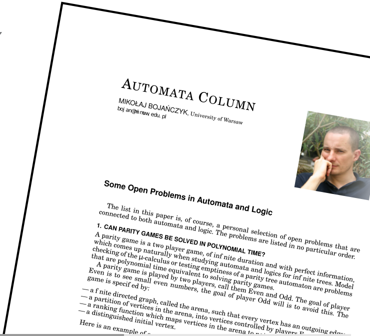
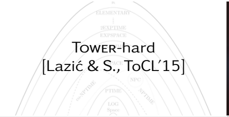


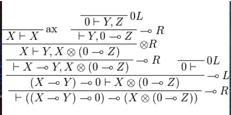
Complexity Theory



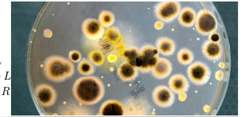
Tower-hard
[Lazić & S., ToCL'15]

Distributed Computing

Proof Theory



Computational Biology



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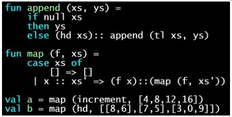

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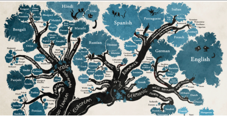
Security

Computational Linguistics



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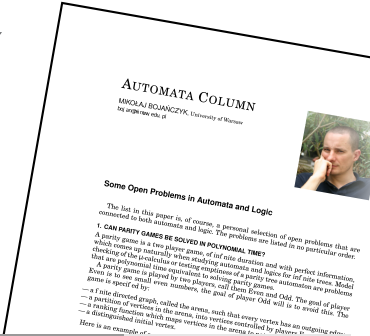
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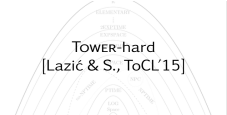
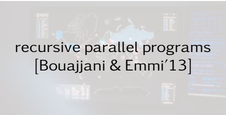
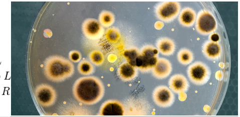
 Tower-hard [Lazić & S., ToCL'15]	 recursive parallel programs [Bouajjani & Emmi'13]	see proof below	

$$\dfrac{\dfrac{\dfrac{\dfrac{}{X \vdash X}\,\mathrm{ax} \quad \dfrac{\dfrac{}{0 \vdash Y, Z}\,0L}{\vdash Y, 0 \multimap Z}\multimap R}{X \vdash Y, X \otimes (0 \multimap Z)}\otimes R}{\vdash X \multimap Y, X \otimes (0 \multimap Z)}\multimap R \quad \dfrac{}{0 \vdash}\,0L}{\dfrac{(X \multimap Y) \multimap 0 \vdash X \otimes (0 \multimap Z)}{\vdash ((X \multimap Y) \multimap 0) \multimap (X \otimes (0 \multimap Z))}\multimap R}\multimap L$$

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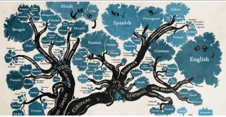
see code below			

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fun map (f, xs) =
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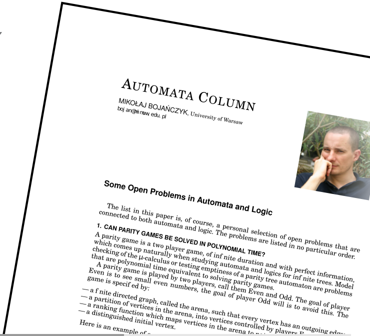
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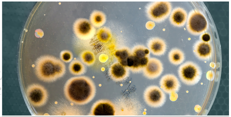
Tower-hard [Lazić & S., ToCL'15]	recursive parallel programs [Bouajjani & Emmi'13]	linear and relevance logics [de Groote et al.'04 Lazić & S., ToCL'15 S., JSL'16]	
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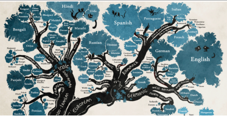
(code below)			

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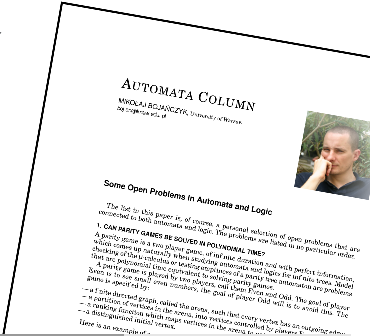
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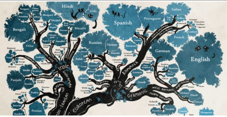

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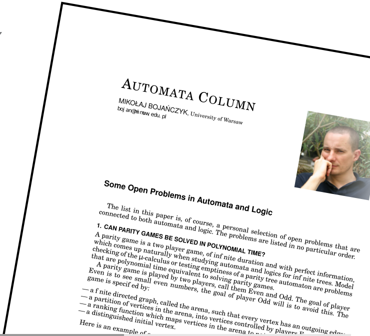
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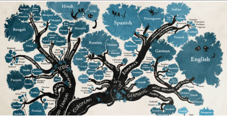
observational equivalence [Cotton-Barratt et al.'17]			

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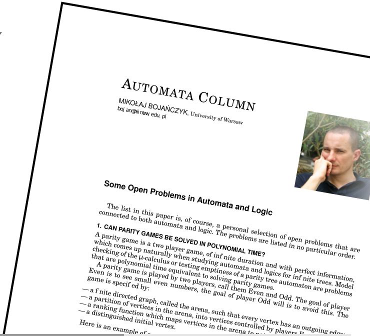
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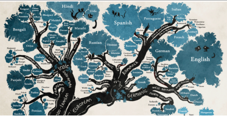
observational equivalence [Cotton-Barratt et al.'17]	data logics [Bojańczyk et al.'09, Abriola et al.'17]		

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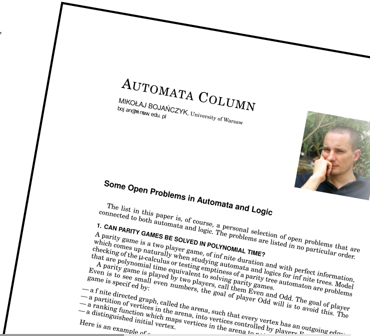

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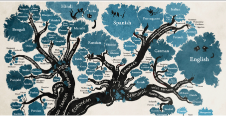
observational equivalence [Cotton-Barratt et al.'17]	data logics [Bojańczyk et al.'09, Abriola et al.'17]	security protocols [Verma & Goubault-Larrecq'05]	

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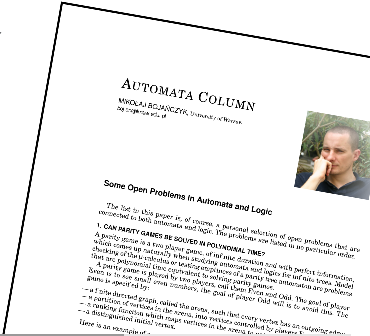
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