MPRI 2-9-1
“Algorithmic Aspects of WQO Theory”

Nov. 12th, 2020: Upper bounds for bad sequences

Recalls on WQOs

$(A, \leqslant)$ is a well-quasi-ordering (a WQO) if any infinite sequence $x_0, x_1, x_2 \ldots$ over A contains an increasing pair $x_i \leqslant x_j$ (for some $i < j$)

Ex.

1. $(\mathbb{N}, \leqslant)$ is a WQO

2. $(\prod_{i=1}^{k} A_i, \leqslant_{\text{prod}})$ is a WQO when each $(A_i, \leqslant_i)$ is (Dickson's Lemma)

   where $(x_1, \ldots, x_k) \leqslant_{\text{prod}} (y_1, \ldots, y_k) \stackrel{\text{def}}{\Leftrightarrow} \bigwedge_i x_i \leqslant_i y_i$

3. $(A^*, \leqslant_*)$ is a WQO when $(A, \leqslant)$ is (Higman's Lemma)

   where, $x = (x_1 \ldots x_n) \leqslant_* (y_1 \ldots y_m) = y$ iff $x \leqslant_{\text{prod}} y'$ for a length-n subsequence $y' = (y_{k_1} \ldots y_{k_n})$ for y (NB: $1 \leqslant k_1 < k_2 < \ldots < k_n \leqslant m$)

   E.g. over $(\mathbb{N}^2)^*$: $\left|{}^{1}_{0}\right|{}^{0}_{2} \leqslant_* \left|{}^{2}_{0}\right|{}^{1}_{1}\left|{}^{1}_{3}\right.$ while $\left|{}^{1}_{2}\right|{}^{0}_{2} \not\leqslant_* \left|{}^{2}_{0}\right|{}^{1}_{1}\left|{}^{1}_{3}\right.$

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RECALLS ON WQOs

Def. A sequence $x_0, x_1, \ldots$ over A is bad $\stackrel{\text{def}}{\Leftrightarrow}$ there is no increasing pair "$x_i \leqslant x_j$ with $i < j$"

NB. Over a WQO, a bad sequence is necessarily finite

Problem. Given A, how long can a bad sequence $x_0, x_1, ..$ over A be?

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THE LENGTH OF BAD SEQUENCES

A 1-player game over WQ $(A,\leqslant)$:

- Pick an element a_0, then some a_1, then some a_2 ..., building a sequence $a_0, a_1, a_2, a_3, \ldots$.
- Player loses when/if he creates a good sequence.

Let's play on $(\mathbb{N},\leqslant)$.

Let's play on $(\mathbb{N}^2,\leqslant_\times)$.

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Conclusions:
1. We need to restrict to sequences where x_0 and $[x_0 \ldots x_k] \mapsto x_{k+1}$ have limited complexity;
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Ordinal Indexes for Complexity Classes

The complexity analysis for WQO-based algorithms use new complexity classes: $\mathsf{F}_1, \mathsf{F}_2, \mathsf{F}_3, \ldots$

Continues with transfinite indexes: $\mathsf{F}_4, \ldots, \mathsf{F}_\omega, \mathsf{F}_{\omega+1}, \mathsf{F}_{\omega+2}, \ldots, \mathsf{F}_{\omega\cdot 2}, \mathsf{F}_{\omega\cdot 2+1}, \ldots, \mathsf{F}_{\omega\cdot 3}, \ldots, \mathsf{F}_{\omega\cdot 4}, \ldots, \mathsf{F}_{\omega^2}, \mathsf{F}_{\omega^2+1}, \ldots, \mathsf{F}_{\omega^2+\omega}, \ldots, \mathsf{F}_{\omega^2+\omega\cdot 2}, \ldots, \mathsf{F}_{\omega^2\cdot 2}, \ldots, \mathsf{F}_{\omega^3}, \ldots, \mathsf{F}_{\omega^\omega}, \ldots, \mathsf{F}_{\omega^{\omega^\omega}}, \ldots, \mathsf{F}_{\omega^{\omega^{\omega^\omega}}}, \ldots,$

- We work with ordinals below ε_0 written in Cantor normal form:

$$\alpha = \omega^{\alpha_1} + \cdots + \omega^{\alpha_m} \quad \text{where } \alpha > \alpha_1 \geqslant \cdots \geqslant \alpha_m$$

NB: α is zero iff $m = 0$; it is a successor $\alpha = \beta + 1 = \beta + \omega^0$ iff $m > 0$ and $\alpha_m = 0$; otherwise it is a limit $\alpha = \lambda$

Alternative notation:

$$\alpha = \omega^{\alpha_1} \cdot c_1 + \cdots + \omega^{\alpha_m} \cdot c_m \quad \text{now with } \begin{array}{l} \alpha > \alpha_1 > \ldots > \alpha_m \\ c_1, \ldots, c_m \in \mathbb{N} \end{array}$$

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FAST-GROWING FUNCTIONS

$(F_\alpha)_{\alpha \in Ord}$: an ordinal-indexed family of functions $F_\alpha : \mathbb{N} \to \mathbb{N}$

$$F_0(x) \stackrel{\text{def}}{=} x+1 \qquad F_{\alpha+1}(x) \stackrel{\text{def}}{=} \overbrace{F_\alpha(F_\alpha(\ldots F_\alpha}^{x+1}(x)\ldots)) \qquad F_\omega(x) \stackrel{\text{def}}{=} F_{x+1}(x)$$

gives $F_1(x) = 2x+1 \approx 2x$, $F_2(x) = 2^{x+1}(x+1) - 1 \approx 2^x$,
$F_3(x) \approx \text{tower}(x)$ and $F_\omega(x) \approx \text{ACKERMANN}(x)$, the first F_α that is not primitive recursive.

Generally $F_\lambda(x) \stackrel{\text{def}}{=} F_{\lambda_x}(x)$ with $\lambda_0 < \lambda_1 < \lambda_2 < \cdots < \lambda$ a fundamental sequence for λ, given by

$$(\gamma + \omega^{\beta+1})_x \stackrel{\text{def}}{=} \gamma + \omega^\beta \cdot (x+1) \qquad (\gamma + \omega^\lambda)_x \stackrel{\text{def}}{=} \gamma + \omega^{\lambda_x}$$

E.g. $F_{\omega^2}(7) = F_{\omega\cdot 8}(7) = F_{\omega\cdot 7+8}(7) = \overbrace{F_{\omega\cdot 7+7}(F_{\omega\cdot 7+7}(\cdots(F_{\omega\cdot 7+7}}^{8}(7))\cdots))$

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THE FAST-GROWING HIERARCHY

By Schmitz (2013), after Wainer & Löb (1970), Grzegorczyk (1953)

$\mathbb{F}_\alpha \stackrel{\text{def}}{=} \bigcup_{p \in \mathcal{F}_{<\alpha}} \mathrm{FDT{\small IME}}(F_\alpha(p(n)))$, ie all functions in time $F_\alpha(\textit{negligible}(n))$

$$\mathcal{F}_{<\alpha} \stackrel{\text{def}}{=} \bigcup_{\beta<\alpha} \mathcal{F}_\beta \qquad \mathcal{F}_\alpha \stackrel{\text{def}}{=} \bigcup_{c\in\mathbb{N}} \mathbb{F}^c_\alpha \qquad \mathbb{F}^c_\alpha \stackrel{\text{def}}{=} \bigcup_{p\in\mathcal{F}_{<\alpha}} \mathrm{FDT{\small IME}}(F^c_\alpha(p(n)))$$

1. These classes admit many other characterizations and capture some well-known cases:

$\mathbb{F}_2 = \mathrm{E} = \mathrm{DT{\small IME}}(2^{O(n)}),\ \mathcal{F}_{<3} = \mathrm{FE{\small LEM}},\ \mathcal{F}_{<\omega} = \mathrm{PR},\ \mathcal{F}_{<\omega^\omega} = \mathrm{MPR}$

2. A strict hierarchy: $\mathbb{F}_\beta \subsetneq \mathbb{F}^{c+1}_\beta \subsetneq \mathbb{F}_\alpha$ for all $\beta < \alpha$ and $c > 0$.

3. There exist $\mathbb{F}_\alpha$-complete problems for each $\alpha \geqslant 2$

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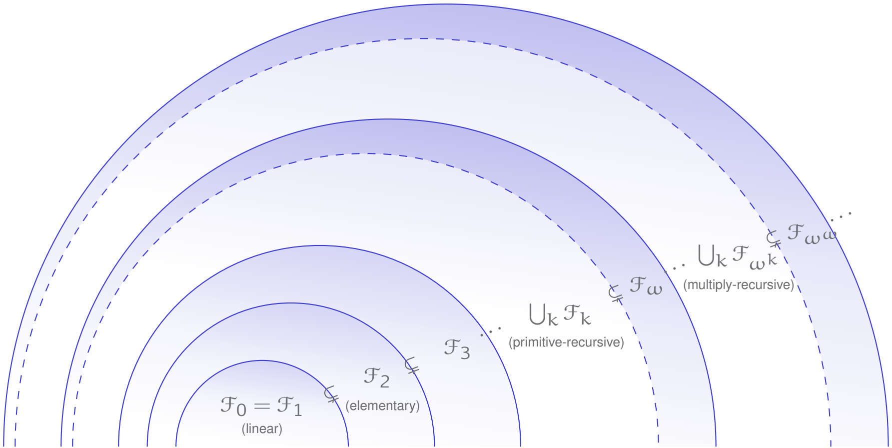

Def. $\mathcal{F}_\alpha = \bigcup_{k\in\mathbb{N}} \mathsf{FDT}\textsc{ime}\big(F_\alpha^k(n)\big) = \mathsf{FDT}\textsc{ime}\big(F_\alpha^{O(1)}(n)\big)$

The Length Function

Let $n \in \mathbb{N}$ and $g : \mathbb{N} \to \mathbb{N}$ be strictly increasing

Def. A sequence $x_0, x_1, \ldots$ is (g,n)-controlled

$\overset{\text{def}}{\Leftrightarrow} |x_i| < g^i(n) = \underbrace{g(g(\ldots g}_{i \text{ times}}(n)\ldots))$ for all $i = 0, 1, \ldots$

Def. $L(A, g, n) \overset{\text{def}}{=}$ length of longest (g,n)-controlled bad sequence $x_0, x_1, \ldots, x_l$

Ex. $L(\mathbb{N}, g, n) = n$

Fact. $L(A, g, n)$ is a well-defined integer
(if each $A_{<k} \overset{\text{def}}{=} \{x \in A \mid |x| < k\}$ is finite –the norm function is proper).
It is computable if g is recursive (and $(A, \leqslant)$ and ..)

Notation. Below we write $L_{A,g}(n)$, and even $L_A(n)$ when g is understood.

Our goal. A complexity upper bound for $L_{A,g}$

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Fact. $\mathsf{L}(A, g, n)$ is a well-defined integer
(if each $A_{<k} \overset{\text{def}}{=} \{x \in A \mid |x| < k\}$ is finite –the norm function is proper).
It is computable if g is recursive (and $(A, \leqslant)$ and ..)

Notation. Below we write $\mathsf{L}_{A,g}(n)$, and even $\mathsf{L}_A(n)$ when g is understood.

Our goal. A complexity upper bound for $\mathsf{L}_{A,g}$

Residuals

Def. For $x \in A$, $A/x \stackrel{\text{def}}{=} A - \uparrow\{x\} = \{y \in A \mid y \not\geq x\}$ is a residual of A.

Ex. $\mathbb{N}/5 = \{0,1,2,3,4\}$ and $\Gamma^*/ab = (b+c)^*(a+c)^*$ (for $\Gamma = \{a,b,c\}$)

Fact. (Descent Equation)

$$L_A(n) = \max_{x \in A_{<n}} \left\{1 + L_{A/x}(g(n))\right\} \qquad (*)$$

NB. (*) can be used as a well-founded recursive definition since taking residuals eventually deplete A completely

Indeed, in a sequence of residuals

$$A \supsetneq A/x_0 \supsetneq A/x_0/x_1 \supsetneq A/x_0/x_1/x_2 \supsetneq \cdots$$

the sequence of elements $x_0, x_1, x_2, \ldots$ is necessarily bad, hence finite

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ROADMAP

$$\mathsf{L}_A(n) = \max_{x \in A_{<n}} \left\{ 1 + \mathsf{L}_{A/x}(g(n)) \right\} \qquad (*)$$

1. Define an algebra of WQOs to manage the A argument of L_A

2. "Compute" A/x algebraically, perhaps overapproximating

3. Use ordinal arithmetic to represent/compute with the A_i's and to classify L_A in the Fast-Growing Hierarchy

AN ALGEBRA OF WQOS WITH NORMS

"WQO with norm" $\stackrel{\text{def}}{=}$ a WQO $(A, \leqslant_A)$ equipped with a norm function $|.|_A : A \to \mathbb{N}$ (and usually just written "A")

Ex. $\mathbb{N}$ with $|n|_{\mathbb{N}} \stackrel{\text{def}}{=} n$ or Γ^* with $|abba|_{\Gamma^*} \stackrel{\text{def}}{=} 4$

Simple (normed) WQOs can be combined/expanded to yield more complex (normed) WQOs

Disjoint sum. $A_1 + A_2 \stackrel{\text{def}}{=} \{1\} \times A_1 + \{2\} \times A_2$

$(i,x) \leqslant_{A_1+A_2} (j,y) \stackrel{\text{def}}{\Leftrightarrow} i = j \wedge x \leqslant_{A_i} y \qquad |(i,x)|_{A_1+A_2} \stackrel{\text{def}}{=} |x|_{A_i}$

Cartesian product.

$(x_1,x_2) \leqslant_{A_1 \times A_2} (y_1,y_2) \stackrel{\text{def}}{\Leftrightarrow} x_1 \leqslant_{A_1} y_1 \wedge x_2 \leqslant_{A_2} y_2$

$|(x_1,x_2)|_{A_1 \times A_2} \stackrel{\text{def}}{=} \max(|x_1|_{A_1}, |x_2|_{A_2})$

Finite sequences. $(x_1,\ldots,x_n) \leqslant_{A^*} (y_1,\ldots,y_m) \stackrel{\text{def}}{\Leftrightarrow} \bigwedge_{i=1}^{n} x_i \leqslant_A y_{k_i}$ for some $1 \leqslant k_1 < k_2 < \cdots < k_n \leqslant m$

$|(x_1,\ldots,x_n)|_{A^*} \stackrel{\text{def}}{=} \max(n, |x_1|_A, \ldots, |x_n|_A)$

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An algebra of WQOs with norms — continued

We consider all "elementary WQOs"

$$A ::= \emptyset \mid A + A \mid A \times A \mid A^*$$

Def. $\Gamma_p = \{a_1, \ldots, a_p\}$ is a p-letter alphabet well-ordered by Id_{Γ_p} and normed with $|a_i|_{\Gamma_p} = 0$

Fact. $\Gamma_0 \equiv \emptyset$ and $\Gamma_1 \equiv \emptyset^*$ are elementary WQOs (modulo isomorphism). $\Gamma_p \equiv \Gamma_1 + \cdots + \Gamma_1$ also is elementary

Fact. $\mathbb{N} \equiv \Gamma_1^*$ is elementary

NB. If $A \equiv B$ then $L_{A,g}(n) = L_{B,g}(n)$.

Reasoning modulo isomorphism is simplified by laws like $\emptyset \times A \equiv \emptyset$ or $A \times (B + C) \equiv A \times B + A \times C$.

We write $A.k$ for $A + \cdots + A$ (equivalently, $\Gamma_k \times A$), and A^k for $A \times \cdots \times A$

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Earlier we observed $\Gamma_3^*/ab = (b+c)^*(a+c)^*$

Can we write $\Gamma_3^*/ab \equiv \Gamma_2^* \times \Gamma_2^*$? This would (perhaps) simplify the computation of $L_{A/x}(n)$ in the Descent Equation

Answer. $\Gamma_3^*/ab \not\equiv \Gamma_2^* \times \Gamma_2^*$

However, Γ_3^*/ab can be reflected in $\Gamma_2^* \times \Gamma_2^*$

Def. $h : A \hookrightarrow B \overset{\text{def}}{\Leftrightarrow} h : A \to B$ is a mapping that satisfies $|h(x)|_B \leqslant |x|_A$ and $h(x) \leqslant_B h(y) \Rightarrow x \leqslant_A y$

For $x \in \Gamma_3^*/ab$ we let $h(x) = \langle x_1, x_2 \rangle$ where $x = x_1 x_2$ is a factorization with x_1 the longest prefix in $(b+c)^*$ (hence $x_2 \in \epsilon + a(a+c)^*$)
Check. $|h(x)| = \max(|x_1|, |x_2|) \leqslant |x|$
Check. $h(x) = \langle x_1, x_2 \rangle \leqslant_{\Gamma_2^* \times \Gamma_2^*} \langle y_1, y_2 \rangle = h(y)$ implies $x \leqslant_{\Gamma_3^*} y$

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REFLECTING RESIDUALS –2

Def. B reflects A, written $A \hookrightarrow B$, when $h : A \hookrightarrow B$ for some h.

Prop. $\mathsf{x} = x_0, x_1, ..$ bad in A implies $h(\mathsf{x})$ bad in B too. And x (g, n)-controlled implies $h(\mathsf{x})$ controlled too.

Cor. $A \hookrightarrow B$ implies $\mathsf{L}_A(n) \leqslant \mathsf{L}_B(n)$

Hence reflections can be used to overapproximate residuals

Prop. Reflections are transitive, compatible with isomorphism, and a precongruence for sum, product, and star

E.g., $A \hookrightarrow B$ implies $A^* \hookrightarrow B^*$ and $(A \times C) \hookrightarrow (B \times C)$.

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REFLECTING RESIDUALS: E.G., $\mathbb{N}^3/\langle 1,4,0\rangle$

Consider a bad sequence $\mathbf{x} = x_0, x_1, \ldots$ over $\mathbb{N}^3/\langle 1,4,0\rangle$

$$\mathbf{x} \;=\; \begin{vmatrix}\mathbf{0}\\\mathbf{6}\\\mathbf{3}\end{vmatrix} \quad \begin{vmatrix}\mathbf{2}\\\mathbf{3}\\\mathbf{3}\end{vmatrix} \quad \begin{vmatrix}\mathbf{0}\\\mathbf{1}\\\mathbf{8}\end{vmatrix} \quad \begin{vmatrix}\mathbf{1}\\\mathbf{3}\\\mathbf{6}\end{vmatrix} \quad \begin{vmatrix}\mathbf{1}\\\mathbf{0}\\\mathbf{9}\end{vmatrix} \quad \begin{vmatrix}\mathbf{2}\\\mathbf{1}\\\mathbf{3}\end{vmatrix} \quad \begin{vmatrix}\mathbf{2}\\\mathbf{0}\\\mathbf{3}\end{vmatrix} \quad \begin{vmatrix}\mathbf{1}\\\mathbf{0}\\\mathbf{0}\end{vmatrix} \quad \begin{vmatrix}\mathbf{0}\\\mathbf{0}\\\mathbf{0}\end{vmatrix}$$

REFLECTING RESIDUALS: E.G., $\mathbb{N}^3/\langle 1,4,0\rangle$

We use colors to witness that $\langle 1,4,0\rangle \not\leq x_i$ for $i = 0, \ldots$

$$x = \left|\begin{matrix}\mathbf{0}\\\mathbf{6}\\\mathbf{3}\end{matrix}\right. \quad \left|\begin{matrix}\mathbf{2}\\\mathbf{3}\\\mathbf{3}\end{matrix}\right. \quad \left|\begin{matrix}\mathbf{0}\\\mathbf{1}\\\mathbf{8}\end{matrix}\right. \quad \left|\begin{matrix}\mathbf{1}\\\mathbf{3}\\\mathbf{6}\end{matrix}\right. \quad \left|\begin{matrix}\mathbf{1}\\\mathbf{0}\\\mathbf{9}\end{matrix}\right. \quad \left|\begin{matrix}\mathbf{2}\\\mathbf{1}\\\mathbf{3}\end{matrix}\right. \quad \left|\begin{matrix}\mathbf{2}\\\mathbf{0}\\\mathbf{3}\end{matrix}\right. \quad \left|\begin{matrix}\mathbf{1}\\\mathbf{0}\\\mathbf{0}\end{matrix}\right. \quad \left|\begin{matrix}\mathbf{0}\\\mathbf{0}\\\mathbf{0}\end{matrix}\right.$$

Reflecting residuals: e.g., $\mathbb{N}^3/\langle 1,4,0\rangle$

$$x = \begin{vmatrix} 0 \\ 6 \\ 3 \end{vmatrix} \begin{vmatrix} 2 \\ 3 \\ 3 \end{vmatrix} \begin{vmatrix} 0 \\ 1 \\ 8 \end{vmatrix} \begin{vmatrix} 1 \\ 3 \\ 6 \end{vmatrix} \begin{vmatrix} 1 \\ 0 \\ 9 \end{vmatrix} \begin{vmatrix} 2 \\ 1 \\ 3 \end{vmatrix} \begin{vmatrix} 2 \\ 0 \\ 3 \end{vmatrix} \begin{vmatrix} 1 \\ 0 \\ 0 \end{vmatrix} \begin{vmatrix} 0 \\ 0 \\ 0 \end{vmatrix}$$

$$\begin{vmatrix} . \\ 6 \\ 3 \end{vmatrix} \begin{vmatrix} . \\ 1 \\ 8 \end{vmatrix} \begin{vmatrix} . \\ 0 \\ 0 \end{vmatrix} \qquad x_i[1] = 0$$

$$\begin{vmatrix} 1 \\ . \\ 9 \end{vmatrix} \begin{vmatrix} 2 \\ . \\ 3 \end{vmatrix} \begin{vmatrix} 1 \\ . \\ 0 \end{vmatrix} \qquad x_i[2] = 0$$

$$\begin{vmatrix} 2 \\ . \\ 3 \end{vmatrix} \qquad x_i[2] = 1$$

$$\begin{vmatrix} 2 \\ . \\ 3 \end{vmatrix} \begin{vmatrix} 1 \\ . \\ 6 \end{vmatrix} \qquad x_i[2] = 3$$

REFLECTING RESIDUALS: E.G., $\mathbb{N}^3/\langle 1,4,0\rangle$

$$x = \begin{array}{|c|c|c|c|c|c|c|c|c} 0 & 2 & 0 & 1 & 1 & 2 & 2 & 1 & 0 \\ 6 & 3 & 1 & 3 & 0 & 1 & 0 & 0 & 0 \\ 3 & 3 & 8 & 6 & 9 & 3 & 3 & 0 & 0 \end{array}$$

$$\begin{array}{|c|c|c|c|c|c|c|c|c} \bullet & 2 & \bullet & 1 & 1 & 2 & 2 & 1 & \bullet \\ 6 & \bullet & 1 & \bullet & \bullet & \bullet & \bullet & \bullet & 0 \\ 3 & 3 & 8 & 6 & 9 & 3 & 3 & 0 & 0 \end{array}$$

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Reflecting residuals: e.g., $\mathbb{N}^3/\langle 1,4,0\rangle$

$$x = \begin{array}{|c|c|c|c|c|c|c|c|c} \mathbf{0} & \mathbf{2} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{2} & \mathbf{2} & \mathbf{1} & \mathbf{0} \\ \mathbf{6} & \mathbf{3} & \mathbf{1} & \mathbf{3} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{3} & \mathbf{3} & \mathbf{8} & \mathbf{6} & \mathbf{9} & \mathbf{3} & \mathbf{3} & \mathbf{0} & \mathbf{0} \end{array}$$

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$$\mathbb{N}^3/\langle 1,4,0\rangle \hookrightarrow \bullet \times \mathbb{N}^2 + \bullet \times \mathbb{N}^2 + \bullet \times \mathbb{N}^2 + \bullet \times \mathbb{N}^2 + \bullet \times \mathbb{N}^2 \hookrightarrow \Gamma_5 \times \mathbb{N}^2$$

REFLECTING RESIDUALS: E.G., $\mathbb{N}^3/\langle 1,4,0\rangle$

$$\mathbb{N}^3/\langle 1,4,0\rangle \hookrightarrow \bullet\times\mathbb{N}^2+\bullet\times\mathbb{N}^2+\bullet\times\mathbb{N}^2+\bullet\times\mathbb{N}^2+\bullet\times\mathbb{N}^2 \hookrightarrow \Gamma_5\times\mathbb{N}^2$$

$\mathbb{N}^k/\langle n_1,\ldots,n_k\rangle \hookrightarrow \Gamma_P\times\mathbb{N}^{k-1}$ for $P \stackrel{\text{def}}{=} \sum_{i=1}^{k}(n_i-1)$

REFLECTING RESIDUALS: E.G., Γ_3^*/abb

Consider a bad sequence $\mathbf{x} = x_0, x_1, \ldots$ over Γ_3^*/abb

$\mathbf{x} =$

$aaa, \quad caba, \quad caac, \quad bbcb, \quad abcc, \quad ba, \quad acacb, \quad cbc, \quad a, \quad \epsilon$

Reflecting residuals: e.g., Γ_3^*/abb

We use colors to witness that $abb \not\leq_* x_i$ for $i = 0, \ldots$

$x =$
$aaa, \quad caba, \quad caac, \quad bbcb, \quad abcc, \quad ba, \quad acacb, \quad cbc, \quad a, \quad \epsilon$

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$$x = aaa, \quad caba, \quad caac, \quad bbcb, \quad abcc, \quad ba, \quad acacb, \quad cbc, \quad a, \quad \epsilon$$

$$\langle\epsilon, aa\rangle \quad \langle c,\epsilon,a\rangle \quad \langle c, ac\rangle \quad bbcb \quad \langle\epsilon,\epsilon,cc\rangle \quad \langle b,\epsilon\rangle \quad \langle\epsilon,cac,\epsilon\rangle \quad cbc \quad \langle\epsilon,\epsilon\rangle \quad \epsilon$$

u in 1st line belongs to $\{b,c\}^*$
$\langle u,v\rangle$ in 2nd line belongs to $\{b,c\}^* \times \{a,c\}^*$
$\langle u,v,w\rangle$ in 3rd line belongs to $\{b,c\}^* \times \{a,c\}^* \times \{a,c\}^*$

REFLECTING RESIDUALS: E.G., Γ_3^*/abb

$$x = aaa, \quad caba, \quad caac, \quad bbcb, \quad abcc, \quad ba, \quad acacb, \quad cbc, \quad a, \quad \epsilon$$

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Thus $\quad \Gamma_3^*/abb \hookrightarrow \Gamma_2^* + (\Gamma_2^*)^2 + (\Gamma_2^*)^3$

Reflecting residuals: e.g., Γ_3^*/abb

$x = aaa, \quad caba, \quad caac, \quad bbcb, \quad abcc, \quad ba, \quad acacb, \quad cbc, \quad a, \quad \epsilon$

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Thus $\quad \Gamma_3^*/abb \hookrightarrow \Gamma_2^* + (\Gamma_2^*)^2 + (\Gamma_2^*)^3$

More generally $\quad \Gamma_{p+1}^*/x \hookrightarrow \sum_{i=1}^{n=|x|} (\Gamma_p^*)^i \hookrightarrow \Gamma_n \times (\Gamma_p^*)^n$

Getting rid of residuals by reflections

$$(A+B)/(1,x) \equiv (A/x)+B \qquad (A+B)/(2,x) \equiv A+(B/x)$$

$$(A\times B)/\langle x,y\rangle \hookrightarrow [(A/x)\times B]+[A\times(B/y)]$$

$$\Gamma^*_{p+1}/(x_1\ldots x_n) \hookrightarrow \Gamma_n\times(\Gamma^*_p)^n$$

More generally;

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E.g., $(\mathbb{N}^3)^*/x \hookrightarrow \Gamma_n\times(\mathbb{N}^3)^n\times((\Gamma_P\times\mathbb{N}^2)^*)^n$ for $P=n^2$

Getting rid of residuals by reflections

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Reflecting residuals in ordinal arithmetic

Def. "Exponential WQO" $\overset{\text{def}}{\Leftrightarrow}$ a WQO built with Γ_p^*'s, sums and products

There is a "bijective" correspondence between ordinals below ω^{ω^ω} and exponential WQOs

$$o(\Gamma_p) \overset{\text{def}}{=} p \qquad o(\Gamma_{p+1}^*) \overset{\text{def}}{=} \omega^{\omega^p}$$

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Computing residuals with ordinal arithmetic

Def. (omitted) ∂_n is a well-founded relation over ω^{ω^ω} such that

$$x \in A_{<n} \text{ and } o(A) = \alpha \quad \text{imply} \quad A/x \hookrightarrow C(\beta) \text{ for some } \beta \in \partial_n \alpha$$

Example

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\Gamma_2^* \arrow[r, "\bigcup_{|x|<4}[\cdot/x \hookrightarrow \cdot]"] \arrow[d, "o"'] & \Gamma_3 \times (\Gamma_1^*)^3 \arrow[d, "o"] \\
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CLASSIFYING L IN THE FAST-GROWING HIERARCHY

Def. $M_{\alpha,g}(n) \stackrel{\text{def}}{=} \max_{\alpha' \in \partial_n \alpha} \{1 + M_{\alpha',g}(g(n))\}$

(This is a well-founded definition)

Prop. $L_{A,g}(n) \leqslant M_{o(A),g}(n)$

Def of $(M_\alpha)_{\alpha < \omega^{\omega^\omega}}$ is similar to a standard hierarchy $(h_\alpha)_{\alpha < \cdots}$

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that satisfies $h_{\omega^\alpha}(x) \leqslant F_\alpha(x) - x$ for $(F_\alpha)_{\alpha < \cdots}$ built on h

Two problems remain:
– can one relate $\alpha' \in \partial_n \alpha$ with $\alpha_n - 1$?
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Main result

Length Function Theorems for $(\mathbb{N}^k, \leqslant_\times)$:

- If g is in $\mathcal{F}_\gamma$ for $\gamma > 0$ then $L_{g,\mathbb{N}^k}$ is in $\mathcal{F}_{\gamma+k}$
- If g is in $g \in \mathcal{F}_1$ then $L_{g,Q\times\mathbb{N}^k}$ is in $\mathbb{F}_k^{|Q|}$

Fact. The runs explored by the Termination algorithm are controlled with $|s_{\text{init}}|$ and $\mathit{Succ} : \mathbb{N} \to \mathbb{N}$.

$\Rightarrow$ Time/space bound in $\mathbb{F}_k$ for Lossy Counter Machines with k counters, and in $\mathbb{F}_\omega$ when k is not fixed.

Fact. The minimal pseudo-runs explored by the backward-chaining Coverability algorithm are controlled by $|s_{\text{target}}|$ and Succ.

$\Rightarrow \cdots$ *same upper bounds* $\cdots$

MAIN RESULT

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MORE LENGTH FUNCTION THEOREMS

For finite words with $\leqslant_*$, L_{A^*} is in $\mathbb{F}_{\omega^{|A|-1}}$, and in $\mathbb{F}_{\omega^\omega}$ when alphabet is not fixed. Applies e.g. to lossy channel systems.

For sequences over $\mathbb{N}^k$ with embedding, $L_{(\mathbb{N}^k)^*}$ is in $\mathbb{F}_{\omega^{\omega^k}}$, and in $\mathbb{F}_{\omega^{\omega^\omega}}$ when k is not fixed. Applies e.g. to timed-arc Petri nets.

For finite words with priority ordering, L_{A^*} is in $\mathbb{F}_{\varepsilon_0}$ Applies e.g. to priority channel systems and higher-order LCS.

Bottom line: we have definite complexity upper bounds for WQO-based algorithms

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