Algorithmic Aspects of WQO (Well-Quasi-Ordering) Theory

Part V: Ideals of WQOs and Their Algorithms

Philippe Schnoebelen

LSV, CNRS & ENS Cachan

Chennai Mathematical Institute, Jan. 2017

Based on joint work with Sylvain Schmitz, Prateek Karandikar, K. Narayan Kumar, Alain Finkel, ..

Lecture notes & exercises available via `www.lsv.ens-cachan.fr/~phs`

IF YOU MISSED PARTS I & II

Def. $(X, \leqslant)$ is a well-quasi-ordering (a wqo) if any infinite sequence $x_0, x_1, x_2 \ldots$ over X contains an increasing pair $x_i \leqslant x_j$ (for some $i < j$)

Examples.

1. $(\mathbb{N}^k, \leqslant_\times)$ is a wqo (Dickson's Lemma)
   where, e.g., $(3,2,1) \leqslant_\times (5,2,2)$ but $(1,2,3) \not\leqslant_\times (5,2,2)$
2. $(\Sigma^*, \leqslant_*)$ is a wqo (Higman's Lemma)
   where, e.g., $abc \leqslant_* bacbc$ but $cba \not\leqslant_* bacbc$

Fact. It is possible to decide Safety, Termination, etc., for WSTS's, i.e. systems with well-quasi-ordered states and monotonic (aka compatible) steps.

Motivation for today's lecture:

WQO-based algorithms often have to handle/reason about/.. infinite upward- or downward-closed sets

- This is a non-trivial subtask
- But there exists a powerful & generic approach via ideals

IF YOU MISSED PARTS I & II

Def. $(X,\leqslant)$ is a well-quasi-ordering (a wqo) if any infinite sequence $x_0, x_1, x_2 \ldots$ over X contains an increasing pair $x_i \leqslant x_j$ (for some $i < j$)

Fact. It is possible to decide Safety, Termination, etc., for WSTS's, i.e. systems with well-quasi-ordered states and monotonic (aka compatible) steps.

Motivation for today's lecture:

WQO-based algorithms often have to handle/reason about/.. infinite upward- or downward-closed sets

- This is a non-trivial subtask
- But there exists a powerful & generic approach via ideals

Outline for Part V

- The need for data structure and algorithms for closed subsets
- Ideals and filters : basics
- Effective ideals and filters
- The Valk-Jantzen-Goubault-Larrecq algorithm
- Building complex effective wqos from simpler ones : tuples, sequences, powersets, substructures, weakening, etc.

Handling upward-closed subsets

Verifying safety for a WSTS is usually done by computing upward-closed subsets

$$B \subseteq \mathit{Pre}^{\leqslant 1}(B) \subseteq \mathit{Pre}^{\leqslant 2}(B) \subseteq \cdots \subseteq \bigcup_m \mathit{Pre}^{\leqslant m}(B) = \mathit{Pre}^*(B)$$

How is this implemented in practice?

Consider $(\mathbb{N}^2, \leqslant_\times)$ and upward-closed subsets $U, U', V, \ldots$

$U =$

There is the finite basis presentation:

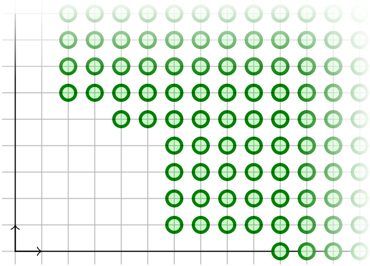
$$U = \uparrow(2,6) \cup \uparrow(4,5) \cup \uparrow(6,1) \cup \uparrow(10,0)$$

Handling upward-closed subsets

Verifying safety for a WSTS is usually done by computing upward-closed subsets

How is this implemented in practice?

Consider $(\mathbb{N}^2, \leqslant_\times)$ and upward-closed subsets $U, U', V, \ldots$

$U =$ 

There is the finite basis presentation:

$$U = \uparrow(2,6) \cup \uparrow(4,5) \cup \uparrow(6,1) \cup \uparrow(10,0)$$

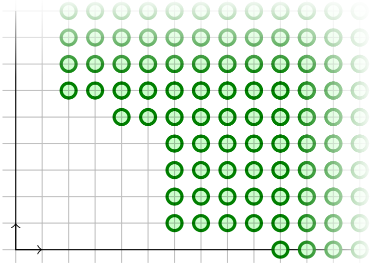
We also need algorithms for computing with this representation:

- E.g., testing whether $U \subseteq V$

Handling upward-closed subsets

How is this implemented in practice?

Consider $(\mathbb{N}^2, \leqslant_\times)$ and upward-closed subsets $U, U', V, \ldots$

$U =$ 

There is the finite basis presentation:

$$U = \uparrow(2,6) \cup \uparrow(4,5) \cup \uparrow(6,1) \cup \uparrow(10,0)$$

We also need algorithms for computing with this representation:

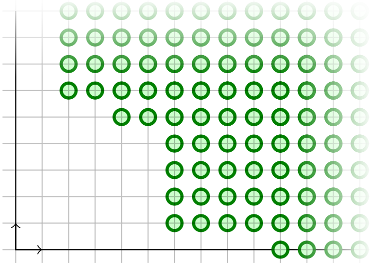
- E.g., testing whether $U \subseteq V$
- E.g., performing $U \leftarrow U \cup V$ or $U \leftarrow U \cap V$

Handling upward-closed subsets

How is this implemented in practice?

Consider $(\mathbb{N}^2, \leqslant_\times)$ and upward-closed subsets $U, U', V, \ldots$

$$U =$$



There is the finite basis presentation:

$$U = \uparrow(2,6) \cup \uparrow(4,5) \cup \uparrow(6,1) \cup \uparrow(10,0)$$

We also need algorithms for computing with this representation:

- E.g., testing whether $U \subseteq V$
- E.g., performing $U \leftarrow U \cup V$ or $U \leftarrow U \cap V$

Upward-Closed Subsets of $(\Sigma^*, \leqslant_*)$

Let us consider words with subword ordering, e.g., for lossy channel systems:

$$U = \uparrow abc \cup \cdots \cup \uparrow ddca \quad V = \uparrow bb \cup \cdots$$

How do we compare such sets?

How do we add to them ?

How do we remove from them ? E.g., how do we perform $U \leftarrow U \cap \uparrow cbab$ or $U \leftarrow U \smallsetminus \downarrow baccbab$?

Bottom line: These are feasible but not trivial !

- Can we handle $\mathbb{N}^k$ and Σ^* efficiently ?
- What about other WQOs? E.g. over $(\mathbb{N}^2)^*$: $\uparrow\left(\left|\begin{smallmatrix}\mathbf{2}\\\mathbf{0}\end{smallmatrix}\right|\begin{smallmatrix}\mathbf{0}\\\mathbf{2}\end{smallmatrix}\right) \cap \uparrow\left(\left|\begin{smallmatrix}\mathbf{1}\\\mathbf{1}\end{smallmatrix}\right|\begin{smallmatrix}\mathbf{1}\\\mathbf{0}\end{smallmatrix}\right)$

UPWARD-CLOSED SUBSETS OF $(\Sigma^*, \leqslant_*)$

Let us consider words with subword ordering, e.g., for lossy channel systems:

$$U = \uparrow abc \cup \cdots \cup \uparrow ddca \quad V = \uparrow bb \cup \cdots$$

How do we compare such sets?

How do we add to them ?

How do we remove from them ? E.g., how do we perform $U \leftarrow U \cap \uparrow cbab$ or $U \leftarrow U \smallsetminus \downarrow baccbab$?

Bottom line: These are feasible but not trivial !

- Can we handle $\mathbb{N}^k$ and Σ^* efficiently ?
- What about other WQOs? E.g. over $(\mathbb{N}^2)^*$: $\uparrow(\left|\begin{smallmatrix}2\\0\end{smallmatrix}\right|\begin{smallmatrix}0\\2\end{smallmatrix}) \cap \uparrow(\left|\begin{smallmatrix}1\\1\end{smallmatrix}\right|\begin{smallmatrix}1\\0\end{smallmatrix})$

UPWARD-CLOSED SUBSETS OF $(\Sigma^*, \leqslant_*)$

Let us consider words with subword ordering, e.g., for lossy channel systems:

$$U = \uparrow abc \cup \cdots \cup \uparrow ddca \quad V = \uparrow bb \cup \cdots$$

How do we compare such sets?

How do we add to them ?

How do we remove from them ? E.g., how do we perform $U \leftarrow U \cap \uparrow cbab$ or $U \leftarrow U \setminus \downarrow baccbab$?

Bottom line: These are feasible but not trivial !

- Can we handle $\mathbb{N}^k$ and Σ^* efficiently ?
- What about other WQOs? E.g. over $(\mathbb{N}^2)^*$: $\uparrow\left(\begin{smallmatrix}2\\0\end{smallmatrix}\ \begin{smallmatrix}0\\2\end{smallmatrix}\right) \cap \uparrow\left(\begin{smallmatrix}1\\1\end{smallmatrix}\ \begin{smallmatrix}1\\0\end{smallmatrix}\right)$

UPWARD-CLOSED SUBSETS OF $(\Sigma^*, \leqslant_*)$

Let us consider words with subword ordering, e.g., for lossy channel systems:

$$U = \uparrow abc \cup \cdots \cup \uparrow ddca \quad V = \uparrow bb \cup \cdots$$

How do we compare such sets?

How do we add to them ?

How do we remove from them ? E.g., how do we perform $U \leftarrow U \cap \uparrow cbab$ or $U \leftarrow U \smallsetminus \downarrow baccbab$?

Bottom line: These are feasible but not trivial !

- Can we handle $\mathbb{N}^k$ and Σ^* efficiently ?
- What about other WQOs? E.g. over $(\mathbb{N}^2)^*$: $\uparrow\left(\left|\begin{smallmatrix}\mathbf{2}\\\mathbf{0}\end{smallmatrix}\right|\begin{smallmatrix}\mathbf{0}\\\mathbf{2}\end{smallmatrix}\right) \cap \uparrow\left(\left|\begin{smallmatrix}\mathbf{1}\\\mathbf{1}\end{smallmatrix}\right|\begin{smallmatrix}\mathbf{1}\\\mathbf{0}\end{smallmatrix}\right)$

Now what about downward-closed subsets?

Problem: downward-closed D **can't always** be represented under the form $D = \downarrow x_1 \cup \cdots \cup \downarrow x_\ell$, take e.g. $D = \mathbb{N}^2$.

Recall: D can **always** be represented by excluded minors:

$$D = X \setminus \uparrow m_1 \setminus \uparrow m_2 \cdots \setminus \uparrow m_\ell$$

This amounts to $D = \neg U$ with $U = \uparrow m_1 \cup \cdots \cup \uparrow m_\ell$.

Problem: Not very convenient for simple sets:
— How do you represent $\downarrow(2,2)$ in $(\mathbb{N}^2, \leqslant_\times)$? And $\downarrow ab$ in $(\Sigma^*, \leqslant_*)$?

$$\downarrow(2,2) = \neg\big[\uparrow(0,3) \cup \uparrow(3,0)\big] \qquad \downarrow ab = \neg\big[\uparrow ba \cup \uparrow c \cup \cdots\big]$$

— How do you compute $D \cup D'$?

There is a better solution: decompose into primes!

NOW WHAT ABOUT DOWNWARD-CLOSED SUBSETS?

Problem: downward-closed D **can't always** be represented under the form $D = \downarrow x_1 \cup \cdots \cup \downarrow x_\ell$, take e.g. $D = \mathbb{N}^2$.

Recall: D can **always** be represented by excluded minors:

$$D = X \smallsetminus \uparrow m_1 \smallsetminus \uparrow m_2 \cdots \smallsetminus \uparrow m_\ell$$

This amounts to $D = \neg U$ with $U = \uparrow m_1 \cup \cdots \cup \uparrow m_\ell$.

Problem: Not very convenient for simple sets:

— How do you represent $\downarrow(2,2)$ in $(\mathbb{N}^2, \leqslant_\times)$? And $\downarrow ab$ in $(\Sigma^*, \leqslant_*)$?

$$\downarrow(2,2) = \neg\big[\uparrow(0,3) \cup \uparrow(3,0)\big] \qquad \downarrow ab = \neg\big[\uparrow ba \cup \uparrow c \cup \cdots\big]$$

— How do you compute $D \cup D'$?

There is a better solution: decompose into primes!

Now what about downward-closed subsets?

Problem: downward-closed D **can't always** be represented under the form $D = \downarrow x_1 \cup \cdots \cup \downarrow x_\ell$, take e.g. $D = \mathbb{N}^2$.

Recall: D can **always** be represented by excluded minors:

$$D = X \smallsetminus \uparrow m_1 \smallsetminus \uparrow m_2 \cdots \smallsetminus \uparrow m_\ell$$

This amounts to $D = \neg U$ with $U = \uparrow m_1 \cup \cdots \cup \uparrow m_\ell$.

Problem: Not very convenient for simple sets:

— How do you represent $\downarrow(2,2)$ in $(\mathbb{N}^2, \leqslant_\times)$? And $\downarrow ab$ in $(\Sigma^*, \leqslant_*)$?

$$\downarrow(2,2) = \neg\big[\uparrow(0,3) \cup \uparrow(3,0)\big] \qquad \downarrow ab = \neg\big[\uparrow ba \cup \uparrow c \cup \cdots\big]$$

— How do you compute $D \cup D'$?

There is a better solution: decompose into primes!

Now what about downward-closed subsets?

Problem: downward-closed D **can't always** be represented under the form $D = {\downarrow}x_1 \cup \cdots \cup {\downarrow}x_\ell$, take e.g. $D = \mathbb{N}^2$.

Recall: D can **always** be represented by excluded minors:

$$D = X \smallsetminus {\uparrow}m_1 \smallsetminus {\uparrow}m_2 \cdots \smallsetminus {\uparrow}m_\ell$$

This amounts to $D = \neg U$ with $U = {\uparrow}m_1 \cup \cdots \cup {\uparrow}m_\ell$.

Problem: Not very convenient for simple sets:

— How do you represent ${\downarrow}(2,2)$ in $(\mathbb{N}^2, \leqslant_\times)$? And ${\downarrow}ab$ in $(\Sigma^*, \leqslant_*)$?

$${\downarrow}(2,2) = \neg\big[{\uparrow}(0,3) \cup {\uparrow}(3,0)\big] \qquad {\downarrow}ab = \neg\big[{\uparrow}ba \cup {\uparrow}c \cup \cdots\big]$$

— How do you compute $D \cup D'$?

There is a better solution: decompose into primes!

Primes, Up and Down

Fix $(X, \leqslant)$ WQO and consider $Up(X) = \{U, U', \ldots\}$ and $Down(X) = \{D, D', \ldots\}$

Def. 1. U ($\neq \emptyset$) is (up-) prime $\overset{\text{def}}{\Leftrightarrow}$ $U \subseteq (U_1 \cup U_2)$ implies $U \subseteq U_1$ or $U \subseteq U_2$.
2. D ($\neq \emptyset$) is (down-) prime $\overset{\text{def}}{\Leftrightarrow}$ $D \subseteq (D_1 \cup D_2)$ implies $D \subseteq D_1$ or $D \subseteq D_2$.

Examples: for any $x \in X$, $\uparrow x$ is up-prime and $\downarrow x$ is down-prime

Lem. (Irreducibility)
1. U is prime iff $U = U_1 \cup \cdots \cup U_n$ implies $U = U_i$ for some i
2. D is prime iff $D = D_1 \cup \cdots \cup D_n$ implies $D = D_i$ for some i

Lem. (Existence of Prime Decompositions, aka Completeness)
1. Every $U \in Up$ is a finite union of up-primes
2. Every $D \in Down$ is a finite union of down-primes

Minimal Prime Decompositions

Def. A prime decomposition U (or D) $= P_1 \cup \cdots \cup P_n$ is minimal $\overset{\text{def}}{\Leftrightarrow} \forall i,j : P_i \subseteq P_j$ implies $i = j$.

Thm. Every U (or D) has a unique minimal prime decomposition. It is called its canonical decomposition

Thm. (Primes are Filters/Ideals)
1. The up-primes of X are exactly the $\uparrow x$ for $x \in X$ (the principal filters)
2. The down-primes of X are exactly the ideals of X (see below)

Def. An ideal I of X is a non-empty directed downward-closed subset

Recall: I directed $\overset{\text{def}}{\Leftrightarrow} x,y \in I \implies \exists z \in I : x \leqslant z \geqslant y$

Example: any $\downarrow x$ is an ideal (called a principal ideal)

Example: If $x_1 < x_2 < x_3 \ldots$ is an increasing sequence then $\bigcup_i \downarrow x_i$ is an ideal

Exercise: Let us look at $\neg U$ for our earlier $U \subseteq \mathbb{N}^2$

Minimal Prime Decompositions

Def. A prime decomposition U (or D) $= P_1 \cup \cdots \cup P_n$ is minimal $\overset{\text{def}}{\Leftrightarrow} \forall i,j : P_i \subseteq P_j$ implies $i = j$.

Thm. Every U (or D) has a unique minimal prime decomposition. It is called its canonical decomposition

Thm. (Primes are Filters/Ideals)
1. The up-primes of X are exactly the $\uparrow x$ for $x \in X$ (the principal filters)
2. The down-primes of X are exactly the ideals of X (see below)

Def. An ideal I of X is a non-empty directed downward-closed subset

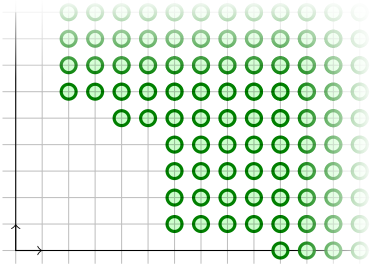
Recall: I directed $\overset{\text{def}}{\Leftrightarrow} x, y \in I \implies \exists z \in I : x \leqslant z \geqslant y$

Example: any $\downarrow x$ is an ideal (called a principal ideal)

Example: If $x_1 < x_2 < x_3 \ldots$ is an increasing sequence then $\bigcup_i \downarrow x_i$ is an ideal

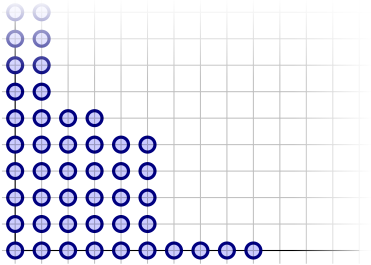
Exercise: Let us look at $\neg U$ for our earlier $U \subseteq \mathbb{N}^2$

A DOWNWARD-CLOSED SUBSET OF $\mathbb{N}^2$

$U =$ 

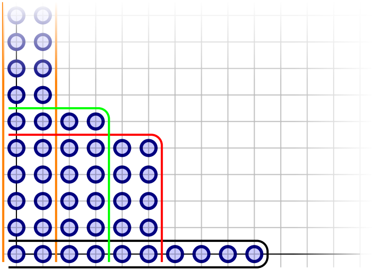
A DOWNWARD-CLOSED SUBSET OF $\mathbb{N}^2$

$$D = \neg U =$$



A DOWNWARD-CLOSED SUBSET OF $\mathbb{N}^2$

$$D = I_1 \cup \cdots \cup I_4$$



NAILING DOWN THE IDEALS

The ideals of $(\mathbb{N},\leqslant)$ are exactly all $\downarrow n$ together with $\mathbb{N}$ itself

Hence $(\mathit{Idl}(\mathbb{N}),\subseteq) \equiv (\mathbb{N}\cup\{\omega\},\leqslant)$, denoted $\mathbb{N}_\omega$ ($\equiv \omega+1$)

Thm. The ideals of $(X_1\times X_2,\leqslant_\times)$ are exactly the $J_1\times J_2$ for J_i an ideal of X_i ($i=1,2$)

Hence $(\mathit{Idl}(X_1\times X_2),\subseteq) \equiv \mathit{Idl}(X_1,\subseteq)\times\mathit{Idl}(X_2,\subseteq)$ Very nice !!!!

Coro. The ideals of $(\mathbb{N}^k,\leqslant_\times)$ are handled like $\mathbb{N}^k_\omega$

Example: Assume $U=\uparrow(2,2)$ and $D=\downarrow(4,\omega)\cup\downarrow(6,3)$.
What is $U\smallsetminus D$ and $D\smallsetminus U$?

NAILING DOWN THE IDEALS

The ideals of $(\mathbb{N}, \leqslant)$ are exactly all $\downarrow n$ together with $\mathbb{N}$ itself

Hence $(\mathit{Idl}(\mathbb{N}), \subseteq) \equiv (\mathbb{N} \cup \{\omega\}, \leqslant)$, denoted $\mathbb{N}_\omega$ ($\equiv \omega + 1$)

Thm. The ideals of $(X_1 \times X_2, \leqslant_\times)$ are exactly the $J_1 \times J_2$ for J_i an ideal of X_i ($i = 1, 2$)

Hence $(\mathit{Idl}(X_1 \times X_2), \subseteq) \equiv \mathit{Idl}(X_1, \subseteq) \times \mathit{Idl}(X_2, \subseteq)$ Very nice !!!!

Coro. The ideals of $(\mathbb{N}^k, \leqslant_\times)$ are handled like $\mathbb{N}_\omega^k$

Example: Assume $U = \uparrow(2,2)$ and $D = \downarrow(4, \omega) \cup \downarrow(6,3)$.
What is $U \setminus D$ and $D \setminus U$?

Nailing down the ideals

The ideals of $(\mathbb{N},\leqslant)$ are exactly all $\downarrow n$ together with $\mathbb{N}$ itself

Hence $(\mathit{Idl}(\mathbb{N}),\subseteq) \equiv (\mathbb{N} \cup \{\omega\},\leqslant)$, denoted $\mathbb{N}_\omega$ $(\equiv \omega + 1)$

Thm. The ideals of $(X_1 \times X_2,\leqslant_\times)$ are exactly the $J_1 \times J_2$ for J_i an ideal of X_i $(i = 1,2)$

Hence $(\mathit{Idl}(X_1 \times X_2),\subseteq) \equiv \mathit{Idl}(X_1,\subseteq) \times \mathit{Idl}(X_2,\subseteq)$ Very nice !!!!

Coro. The ideals of $(\mathbb{N}^k,\leqslant_\times)$ are handled like $\mathbb{N}_\omega^k$

Example: Assume $U = \uparrow(2,2)$ and $D = \downarrow(4,\omega) \cup \downarrow(6,3)$.
What is $U \setminus D$ and $D \setminus U$?

Ideals for $(\Sigma^*, \leqslant_*)$?

Recall: $\downarrow w$ is an ideal for any $w \in \Sigma^*$.

E.g. $\downarrow abc = \{abc, ab, ac, bc, a, b, c, \varepsilon\}$

What else?

- Σ^* ?
- $(ab)^* = \{\varepsilon, ab, abab, ababab, \ldots\}$?
- $a^* + b^* = \{\varepsilon, a, aa, aaa, \ldots, b, bb, bbb, \ldots\}$?
- $(a+b)^*$?

Lem. $I \cdot J \in \mathit{Idl}(\Sigma^*)$ for all $I, J \in \mathit{Idl}(\Sigma^*)$

Thm. The ideals of Σ^* are exactly the concatenation products $P = A_1 \cdot A_2 \cdots A_n$ for atoms of the form $A = \downarrow a = \{a, \varepsilon\}$ with $a \in \Sigma$ or $A = \Gamma^*$ with $\Gamma \subseteq \Sigma$.

Exercise. Use this to compute $\Sigma^* \setminus \uparrow bad$

Ideals for $(\Sigma^*, \leqslant_*)$?

Recall: $\downarrow w$ is an ideal for any $w \in \Sigma^*$.

E.g. $\downarrow abc = \{abc, ab, ac, bc, a, b, c, \varepsilon\}$

What else?

- Σ^* ?
- $(ab)^* = \{\varepsilon, ab, abab, ababab, \ldots\}$?
- $a^* + b^* = \{\varepsilon, a, aa, aaa, \ldots, b, bb, bbb, \ldots\}$?
- $(a+b)^*$?

Lem. $I \cdot J \in Idl(\Sigma^*)$ for all $I, J \in Idl(\Sigma^*)$

Thm. The ideals of Σ^* are exactly the concatenation products $P = A_1 \cdot A_2 \cdots A_n$ for atoms of the form $A = \downarrow a = \{a, \varepsilon\}$ with $a \in \Sigma$ or $A = \Gamma^*$ with $\Gamma \subseteq \Sigma$.

Exercise. Use this to compute $\Sigma^* \smallsetminus \uparrow bad$

IDEALS FOR $(\Sigma^*, \leqslant_*)$?

Recall: $\downarrow w$ is an ideal for any $w \in \Sigma^*$.

E.g. $\downarrow abc = \{abc, ab, ac, bc, a, b, c, \varepsilon\}$

What else?

- Σ^* ?
- $(ab)^* = \{\varepsilon, ab, abab, ababab, \ldots\}$?
- $a^* + b^* = \{\varepsilon, a, aa, aaa, \ldots, b, bb, bbb, \ldots\}$?
- $(a + b)^*$?

Lem. $I \cdot J \in \mathit{Idl}(\Sigma^*)$ for all $I, J \in \mathit{Idl}(\Sigma^*)$

Thm. The ideals of Σ^* are exactly the concatenation products $P = A_1 \cdot A_2 \cdots A_n$ for atoms of the form $A = \downarrow a = \{a, \varepsilon\}$ with $a \in \Sigma$ or $A = \Gamma^*$ with $\Gamma \subseteq \Sigma$.

Exercise. Use this to compute $\Sigma^* \smallsetminus \uparrow bad$

IDEALS FOR $(\Sigma^*, \leqslant_*)$?

Recall: $\downarrow w$ is an ideal for any $w \in \Sigma^*$.

E.g. $\downarrow abc = \{abc, ab, ac, bc, a, b, c, \varepsilon\}$

What else?

- Σ^* ?
- $(ab)^* = \{\varepsilon, ab, abab, ababab, \ldots\}$?
- $a^* + b^* = \{\varepsilon, a, aa, aaa, \ldots, b, bb, bbb, \ldots\}$?
- $(a + b)^*$?

Lem. $I \cdot J \in \mathit{Idl}(\Sigma^*)$ for all $I, J \in \mathit{Idl}(\Sigma^*)$

Thm. The ideals of Σ^* are exactly the concatenation products $P = A_1 \cdot A_2 \cdots A_n$ for atoms of the form $A = \downarrow a = \{a, \varepsilon\}$ with $a \in \Sigma$ or $A = \Gamma^*$ with $\Gamma \subseteq \Sigma$.

Exercise. Use this to compute $\Sigma^* \smallsetminus \uparrow bad$

IDEALS FOR $(\Sigma^*, \leqslant_*)$?

Recall: $\downarrow w$ is an ideal for any $w \in \Sigma^*$.

E.g. $\downarrow abc = \{abc, ab, ac, bc, a, b, c, \varepsilon\}$

What else?

- Σ^* ?
- $(ab)^* = \{\varepsilon, ab, abab, ababab, \ldots\}$?
- $a^* + b^* = \{\varepsilon, a, aa, aaa, \ldots, b, bb, bbb, \ldots\}$?
- $(a + b)^*$?

Lem. $I \cdot J \in \mathit{Idl}(\Sigma^*)$ for all $I, J \in \mathit{Idl}(\Sigma^*)$

Thm. The ideals of Σ^* are exactly the concatenation products $P = A_1 \cdot A_2 \cdots A_n$ for atoms of the form $A = \downarrow a = \{a, \varepsilon\}$ with $a \in \Sigma$ or $A = \Gamma^*$ with $\Gamma \subseteq \Sigma$.

Exercise. Use this to compute $\Sigma^* \setminus \uparrow bad$

What is required for handling $(X, \leqslant)$?

Def. X is ideally effective $\overset{\text{def}}{\Leftrightarrow}$

(XR): X is recursive
(OR): $\leqslant$ is decidable over X
(IR): $\mathit{Idl}(X)$ is recursive
(II): $\subseteq$ is decidable over $\mathit{Idl}(X)$

(CF): $F = \uparrow x \mapsto \neg F = X \setminus F = I_1 \cup \cdots \cup I_n$ is recursive
(CI): $I \mapsto \neg I = \uparrow x_1 \cup \cdots \cup \uparrow x_n$ is recursive
(IF) & (II): $F_1, F_2 \mapsto F_1 \cap F_2 = \uparrow x_1 \cup \cdots$ and $I_1, I_2 \mapsto I_1 \cap I_2 = J_1 \cup \cdots$ are recursive
(IM): membership $x \in I$ is decidable over X and $\mathit{Idl}(X)$
(XF) & (XI): $X = F_1 \cup \cdots F_n$ and $X = I_1 \cup \cdots I_m$ are effective
(PI): $x \mapsto \downarrow x$ is recursive

Examples: Is $(\mathbb{N}, \leqslant)$ ideally effective?
What about $(\Sigma^*, \leqslant_*)$?

What is required for handling $(X, \leqslant)$?

Def. X is ideally effective $\overset{\text{def}}{\Leftrightarrow}$

(XR): X is recursive
(OR): $\leqslant$ is decidable over X
(IR): $Idl(X)$ is recursive
(II): $\subseteq$ is decidable over $Idl(X)$

(CF): $F = \uparrow x \mapsto \neg F = X \smallsetminus F = I_1 \cup \cdots \cup I_n$ is recursive
(CI): $I \mapsto \neg I = \uparrow x_1 \cup \cdots \cup \uparrow x_n$ is recursive
(IF) & (II): $F_1, F_2 \mapsto F_1 \cap F_2 = \uparrow x_1 \cup \cdots$ and $I_1, I_2 \mapsto I_1 \cap I_2 = J_1 \cup \cdots$ are recursive
(IM): membership $x \in I$ is decidable over X and $Idl(X)$
(XF) & (XI): $X = F_1 \cup \cdots F_n$ and $X = I_1 \cup \cdots I_m$ are effective
(PI): $x \mapsto \downarrow x$ is recursive

Examples: Is $(\mathbb{N}, \leqslant)$ ideally effective?
What about $(\Sigma^*, \leqslant_*)$?

Valk-Jantzen-Goubault-Larrecq Algorithm

Thm. If $(X, \leqslant)$ satisfies the first 4 axioms above and (CF), (II), (PI),(XI) then it is ideally effective.

(XR): X is recursive
(OR): $\leqslant$ is decidable over X
(IR): $\mathit{Idl}(X)$ is recursive
(II): $\subseteq$ is decidable over $\mathit{Idl}(X)$

(CF): $F = \uparrow x \mapsto \neg F = X \setminus F = I_1 \cup \cdots \cup I_n$ is recursive
(CI): $I \mapsto \neg I = \uparrow x_1 \cup \cdots \cup \uparrow x_n$ is recursive
(IF) & (II): $F_1, F_2 \mapsto F_1 \cap F_2 = \uparrow x_1 \cup \cdots$ and $I_1, I_2 \mapsto I_1 \cap I_2 = J_1 \cup \cdots$ are recursive
(IM): membership $x \in I$ is decidable over X and $\mathit{Idl}(X)$
(XF) & (XI): $X = F_1 \cup \cdots F_n$ and $X = I_1 \cup \cdots I_m$ are effective
(PI): $x \mapsto \downarrow x$ is recursive

VALK-JANTZEN-GOUBAULT-LARRECQ ALGORITHM

(XR): X is recursive
(OR): $\leqslant$ is decidable over X
(IR): $\mathit{Idl}(X)$ is recursive
(II): $\subseteq$ is decidable over $\mathit{Idl}(X)$

(CF): $F = \uparrow x \mapsto \neg F = X \smallsetminus F = I_1 \cup \cdots \cup I_n$ is recursive
(CI): $I \mapsto \neg I = \uparrow x_1 \cup \cdots \cup \uparrow x_n$ is recursive
(IF) & (II): $F_1, F_2 \mapsto F_1 \cap F_2 = \uparrow x_1 \cup \cdots$ and $I_1, I_2 \mapsto I_1 \cap I_2 = J_1 \cup \cdots$ are recursive
(IM): membership $x \in I$ is decidable over X and $\mathit{Idl}(X)$
(XF) & (XI): $X = F_1 \cup \cdots F_n$ and $X = I_1 \cup \cdots I_m$ are effective
(PI): $x \mapsto \downarrow x$ is recursive

Proof. We first show (CD) $\overset{\text{def}}{\Leftrightarrow}$ one can design a recursive $D = I_1 \cup \cdots I_n \mapsto \neg D = U = \uparrow x_1 \cup \uparrow x_2 \cup \cdots$
For this, set $U_0 = \varnothing$ and, as long as $D \subsetneq \neg U_i$, we pick some x s.t. $D \not\ni x \notin U_i$ and set $U_{i+1} = U_i \cup \uparrow x$. Eventually $U_i = \neg D$ will happen

Valk-Jantzen-Goubault-Larrecq Algorithm

(XR): X is recursive
(OR): $\leqslant$ is decidable over X
(IR): $\mathit{Idl}(X)$ is recursive
(II): $\subseteq$ is decidable over $\mathit{Idl}(X)$

(CF): $F = \uparrow x \mapsto \neg F = X \setminus F = I_1 \cup \cdots \cup I_n$ is recursive
(CI): $I \mapsto \neg I = \uparrow x_1 \cup \cdots \cup \uparrow x_n$ is recursive
(IF) & (II): $F_1, F_2 \mapsto F_1 \cap F_2 = \uparrow x_1 \cup \cdots$ and $I_1, I_2 \mapsto I_1 \cap I_2 = J_1 \cup \cdots$ are recursive
(IM): membership $x \in I$ is decidable over X and $\mathit{Idl}(X)$
(XF) & (XI): $X = F_1 \cup \cdots F_n$ and $X = I_1 \cup \cdots I_m$ are effective
(PI): $x \mapsto \downarrow x$ is recursive

Proof. Then we get (IF) from (CD) and (CI), by expressing intersection as dual of union, (IM) from (PI) and (II), (XF) from (CD) by computing $\neg\varnothing$

Valk-Jantzen-Goubault-Larrecq Algorithm

(XR): X is recursive
(OR): $\leqslant$ is decidable over X
(IR): $\mathit{Idl}(X)$ is recursive
(II): $\subseteq$ is decidable over $\mathit{Idl}(X)$

(CF): $F = \uparrow x \mapsto \neg F = X \setminus F = I_1 \cup \cdots \cup I_n$ is recursive
(CI): $I \mapsto \neg I = \uparrow x_1 \cup \cdots \cup \uparrow x_n$ is recursive
(IF) & (II): $F_1, F_2 \mapsto F_1 \cap F_2 = \uparrow x_1 \cup \cdots$ and $I_1, I_2 \mapsto I_1 \cap I_2 = J_1 \cup \cdots$ are recursive
(IM): membership $x \in I$ is decidable over X and $\mathit{Idl}(X)$
(XF) & (XI): $X = F_1 \cup \cdots F_n$ and $X = I_1 \cup \cdots I_m$ are effective
(PI): $x \mapsto \downarrow x$ is recursive

Thm [Halfon]. There are no more redundancies in the blue axioms

Constructing ideally effective WQOs

- $(X \times Y, \leqslant_\times)$ is ideally effective when X and Y are.

- $(X^*, \leqslant_*)$ is ideally effective when X is. The ideals are the products of atoms $A = D^*$ for $D \in \mathit{Down}(X)$ and $A = {\downarrow} I$ for $I \in \mathit{Idl}(X)$

- $(X \sqcup Y, \leqslant_\sqcup)$ is ideally effective when X and Y are. $\mathit{Idl}(X \sqcup Y) \equiv \mathit{Idl}(X) \sqcup \mathit{Idl}(Y)$.

- $X \times_{\mathrm{lex}} Y$ and $X \sqcup_{\mathrm{lex}} Y$ are ideally effective when ..

- $\mathcal{P}_f(X)$ and $\mathcal{M}_f(X)$ and $(X^*, \leqslant_{\mathrm{st}})$ and $\cdots$ are ideally ..

- $\mathcal{T}(X)$ is ideally effective when X is but the ideals are more complex (see Goubault-Larrecq & Schmitz, ICALP 2016)

Constructing ideally effective WQOs

- $(X \times Y, \leqslant_\times)$ is ideally effective when X and Y are.

- $(X^*, \leqslant_*)$ is ideally effective when X is. The ideals are the products of atoms $A = D^*$ for $D \in \mathit{Down}(X)$ and $A = {\downarrow}I$ for $I \in \mathit{Idl}(X)$

- $(X \sqcup Y, \leqslant_\sqcup)$ is ideally effective when X and Y are. $\mathit{Idl}(X \sqcup Y) \equiv \mathit{Idl}(X) \sqcup \mathit{Idl}(Y)$.

- $X \times_{\text{lex}} Y$ and $X \sqcup_{\text{lex}} Y$ are ideally effective when ..

- $\mathcal{P}_f(X)$ and $\mathcal{M}_f(X)$ and $(X^*, \leqslant_{\text{st}})$ and $\cdots$ are ideally ..

- $\mathcal{T}(X)$ is ideally effective when X is but the ideals are more complex (see Goubault-Larrecq & Schmitz, ICALP 2016)

CONSTRUCTING IDEALLY EFFECTIVE WQOS

- $(X \times Y, \leqslant_\times)$ is ideally effective when X and Y are.

- $(X^*, \leqslant_*)$ is ideally effective when X is. The ideals are the products of atoms $A = D^*$ for $D \in \mathit{Down}(X)$ and $A = \downarrow I$ for $I \in \mathit{Idl}(X)$

- $(X \sqcup Y, \leqslant_\sqcup)$ is ideally effective when X and Y are. $\mathit{Idl}(X \sqcup Y) \equiv \mathit{Idl}(X) \sqcup \mathit{Idl}(Y)$.

- $X \times_{\text{lex}} Y$ and $X \sqcup_{\text{lex}} Y$ are ideally effective when ..

- $\mathcal{P}_f(X)$ and $\mathcal{M}_f(X)$ and $(X^*, \leqslant_{\text{st}})$ and $\cdots$ are ideally ..

- $\mathcal{T}(X)$ is ideally effective when X is but the ideals are more complex (see Goubault-Larrecq & Schmitz, ICALP 2016)

CONSTRUCTING IDEALLY EFFECTIVE WQOS

- $(X \times Y, \leqslant_\times)$ is ideally effective when X and Y are.

- $(X^*, \leqslant_*)$ is ideally effective when X is. The ideals are the products of atoms $A = D^*$ for $D \in \mathit{Down}(X)$ and $A = {\downarrow} I$ for $I \in \mathit{Idl}(X)$

- $(X \sqcup Y, \leqslant_\sqcup)$ is ideally effective when X and Y are. $\mathit{Idl}(X \sqcup Y) \equiv \mathit{Idl}(X) \sqcup \mathit{Idl}(Y)$.

- $X \times_{\text{lex}} Y$ and $X \sqcup_{\text{lex}} Y$ are ideally effective when ..

- $\mathcal{P}_f(X)$ and $\mathcal{M}_f(X)$ and $(X^*, \leqslant_{\text{st}})$ and $\cdots$ are ideally ..

- $\mathcal{T}(X)$ is ideally effective when X is but the ideals are more complex (see Goubault-Larrecq & Schmitz, ICALP 2016)

CONSTRUCTING IDEALLY EFFECTIVE WQOS

- $(X \times Y, \leqslant_\times)$ is ideally effective when X and Y are.

- $(X^*, \leqslant_*)$ is ideally effective when X is. The ideals are the products of atoms $A = D^*$ for $D \in \mathit{Down}(X)$ and $A = {\downarrow}I$ for $I \in \mathit{Idl}(X)$

- $(X \sqcup Y, \leqslant_\sqcup)$ is ideally effective when X and Y are. $\mathit{Idl}(X \sqcup Y) \equiv \mathit{Idl}(X) \sqcup \mathit{Idl}(Y)$.

- $X \times_{\mathsf{lex}} Y$ and $X \sqcup_{\mathsf{lex}} Y$ are ideally effective when ..

- $\mathcal{P}_f(X)$ and $\mathcal{M}_f(X)$ and $(X^*, \leqslant_{\mathsf{st}})$ and $\cdots$ are ideally ..

- $\mathcal{T}(X)$ is ideally effective when X is but the ideals are more complex (see Goubault-Larrecq & Schmitz, ICALP 2016)

Constructing ideally effective WQOs

- $(X \times Y, \leqslant_\times)$ is ideally effective when X and Y are.

- $(X^*, \leqslant_*)$ is ideally effective when X is. The ideals are the products of atoms $A = D^*$ for $D \in \mathit{Down}(X)$ and $A = {\downarrow} I$ for $I \in \mathit{Idl}(X)$

- $(X \sqcup Y, \leqslant_\sqcup)$ is ideally effective when X and Y are.
$\mathit{Idl}(X \sqcup Y) \equiv \mathit{Idl}(X) \sqcup \mathit{Idl}(Y)$.

- $X \times_{\text{lex}} Y$ and $X \sqcup_{\text{lex}} Y$ are ideally effective when ..

- $\mathcal{P}_f(X)$ and $\mathcal{M}_f(X)$ and $(X^*, \leqslant_{\text{st}})$ and $\cdots$ are ideally ..

- $\mathcal{T}(X)$ is ideally effective when X is but the ideals are more complex (see Goubault-Larrecq & Schmitz, ICALP 2016)

Constructing more ideally effective WQOs

1. Assume $(X, \leqslant')$ is an extension of $(X, \leqslant)$, i.e., $\leqslant \subseteq \leqslant'$.

Then $\mathit{Idl}(X, \leqslant') = \{\downarrow_{\leqslant'} I \mid I \in \mathit{Idl}(X, \leqslant)\}$.

Furthermore $(X, \leqslant')$ is ideally effective when $(X, \leqslant)$ is and the functions

$$I \mapsto \downarrow_{\leqslant'} I = I_1 \cup \cdots \cup I_\ell \quad \text{and} \quad \uparrow x = F \mapsto \uparrow_{\leqslant'} F = \uparrow x_1 \cup \cdots \cup \uparrow x_m$$

are recursive.

Example. Subwords *cum* conjugacy:

$$abcd \leqslant_{\circlearrowleft} acbadbbdbdbdbadbc$$

Example. Quotienting $(X, \leqslant)$ by some equivalence $\approx$ such that $\approx \circ \leqslant = \leqslant \circ \approx$

Constructing more ideally effective WQOs

1. Assume $(X, \leqslant')$ is an extension of $(X, \leqslant)$, i.e., $\leqslant \subseteq \leqslant'$.

Then $Idl(X, \leqslant') = \{\downarrow_{\leqslant'} I \mid I \in Idl(X, \leqslant)\}$.

Furthermore $(X, \leqslant')$ is ideally effective when $(X, \leqslant)$ is and the functions

$$I \mapsto \downarrow_{\leqslant'} I = I_1 \cup \cdots \cup I_\ell \quad \text{and} \quad \uparrow x = F \mapsto \uparrow_{\leqslant'} F = \uparrow x_1 \cup \cdots \cup \uparrow x_m$$

are recursive.

Example. Subwords *cum* conjugacy:

$$abcd \leqslant_{\circlearrowleft} acbadbbdbdbdbadbc$$

Example. Quotienting $(X, \leqslant)$ by some equivalence $\approx$ such that $\approx \circ \leqslant = \leqslant \circ \approx$

CONSTRUCTING MORE IDEALLY EFFECTIVE WQOS

2. Assume $(Y, \leqslant_Y)$ is a subwqo of $(X, \leqslant_X)$, i.e., $Y \subseteq X$ and $\leqslant_Y = \leqslant_X \cap Y \times Y$.

Then $\mathit{Idl}(Y, \leqslant) = \{I \cap Y \mid I \in \mathit{Idl}(X) \text{ st. } I \subseteq {\downarrow_X} Y \wedge I \cap Y \neq \emptyset\}$.

Furthermore $(Y, \leqslant)$ is ideally effective when $(X, \leqslant)$ is and when Y and the functions

$$\begin{aligned} \mathit{Idl}(X) &\to \mathit{Down}(X) \\ I &\mapsto {\downarrow_X}(I \cap Y) = I_1 \cup \cdots I_\ell \end{aligned} \quad \text{and} \quad \begin{aligned} \mathit{Fil}(X) &\to \mathit{Up}(X) \\ {\uparrow}x = F &\mapsto {\uparrow_X}(F \cap Y) = {\uparrow}x_1 \cup \cdots {\uparrow}x_m \end{aligned}$$

are recursive.

Example. $(L, \leqslant_*)$ for a context-free $L \subseteq \Sigma^*$.

Example. Decreasing sequences in $\mathbb{N}^*$ with the subsequence ordering.

Constructing more ideally effective WQOs

2. Assume $(Y, \leqslant_Y)$ is a subwqo of $(X, \leqslant_X)$, i.e., $Y \subseteq X$ and $\leqslant_Y = \leqslant_X \cap Y \times Y$.

Then $\mathit{Idl}(Y, \leqslant) = \{I \cap Y \mid I \in \mathit{Idl}(X) \text{ st. } I \subseteq {\downarrow_X} Y \ \wedge\ I \cap Y \neq \emptyset\}$.

Furthermore $(Y, \leqslant)$ is ideally effective when $(X, \leqslant)$ is and when Y and the functions

$$\begin{array}{ll} \mathit{Idl}(X) & \to \mathit{Down}(X) \\ I & \mapsto {\downarrow_X}(I \cap Y) = I_1 \cup \cdots I_\ell \end{array} \quad \text{and} \quad \begin{array}{ll} \mathit{Fil}(X) & \to \mathit{Up}(X) \\ {\uparrow}x = F & \mapsto {\uparrow_X}(F \cap Y) = {\uparrow}x_1 \cup \cdots {\uparrow}x_m \end{array}$$

are recursive.

Example. $(L, \leqslant_*)$ for a context-free $L \subseteq \Sigma^*$.

Example. Decreasing sequences in $\mathbb{N}^*$ with the subsequence ordering.

Conclusion for Part V

Ideal-based algorithms already have several applications.

Handling WQO's raise many interesting algorithmic questions:

- Best algorithms for $(\Sigma^*, \leqslant_*)$? (Karandikar et al., TCS 2016)
- Best algorithms for $\left(\mathbb{N}^k\right)^*$?
- Fully generic library of data structures and algorithms?
- Separating the polynomial and the exponential cases?
- More constructions .. Beyond WQOs ..
- $\cdots$