Algorithmic Aspects of WQO (Well-Quasi-Ordering) Theory

Part II: Algorithmic Applications of WQOs

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LSV, CNRS & ENS Cachan

Chennai Mathematical Institute, Jan. 2017

Based on joint work with Sylvain Schmitz, Prateek Karandikar, K. Narayan Kumar, Alain Finkel, ..

Lecture notes & exercises available via `www.lsv.ens-cachan.fr/~phs`

If you missed Part I

$(X, \leqslant)$ is a well-quasi-ordering (a wqo) if any infinite sequence $x_0, x_1, x_2 \ldots$ over X contains an increasing pair $x_i \leqslant x_j$ (for some $i < j$)

Examples.

1. $(\mathbb{N}^k, \leqslant_\times)$ is a wqo (Dickson's Lemma)
   where, e.g., $(3,2,1) \leqslant_\times (5,2,2)$ but $(1,2,3) \not\leqslant_\times (5,2,2)$
2. $(\Sigma^*, \leqslant_*)$ is a wqo (Higman's Lemma)
   where, e.g., $abc \leqslant_* bacbc$ but $cba \not\leqslant_* bacbc$

Objectives for today's course:

- See algorithms that rely on wqos: verification of WSTS's
- Reduce complexity analysis to bounds on bad sequences

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- Well-structured transition systems (WSTS's)
- Deciding Termination
- Deciding Coverability
- (in lecture notes only:) other wqo-based algorithms: other termination proofs, relevance logic, Karp-Miller trees, ..

All of these are actual examples of algorithms that terminate thanks to wqo-theoretical arguments

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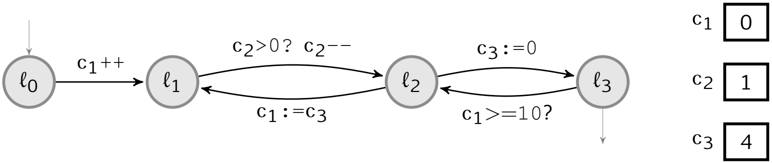
WSTS: Well-structured transition systems

In program verification, wqo's appear prominently under the guise of WSTS.

Def. A WSTS is a system $(S, \to, \leqslant)$ where

1. $(S, \to)$ with $\to \subseteq S \times S$ is a transition system
2. the set of states $(S, \leqslant)$ is wqo, and
3. the transition relation is compatible with the ordering (also called "monotonic"): $s \to t$ and $s \leqslant s'$ imply $s' \to t'$ for some $t' \geqslant t$

Some WSTS's: Monotonic Counter Machines



A run of M: $(\ell_0,0,1,4) \to (\ell_1,1,1,4) \to (\ell_2,1,0,4) \to (\ell_3,1,0,0)$

Ordering states: $(\ell_1,0,0,0) \leqslant (\ell_1,0,1,2)$ but $(\ell_1,0,0,0) \not\leqslant (\ell_2,0,1,2)$.
This is wqo as a product of wqo's: $(Loc,=) \times (\mathbb{N}^3,\leqslant_\times)$

Compatibility: easily checked when guards are upward-closed and assignments are monotonic functions of the variables.

NB. Other updates can be considered as long as they are monotonic.
Extending guards require using a finer ordering.

Question. How does this compare to Minsky (counter) machines?

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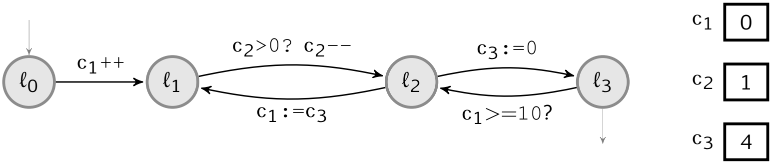

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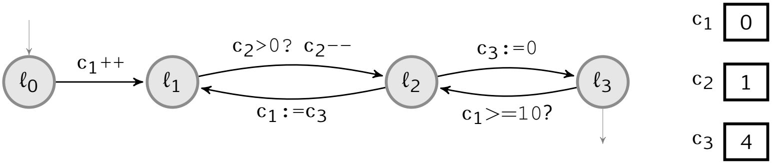

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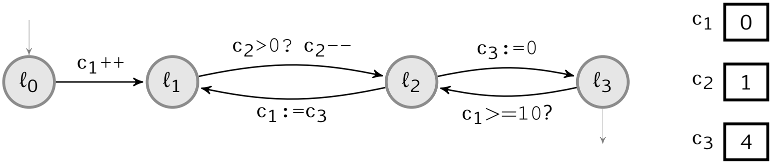

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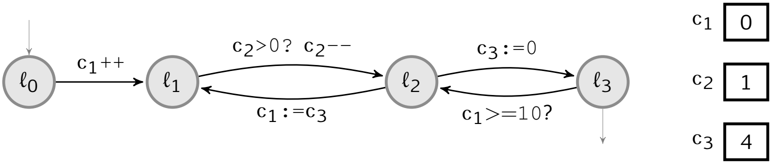

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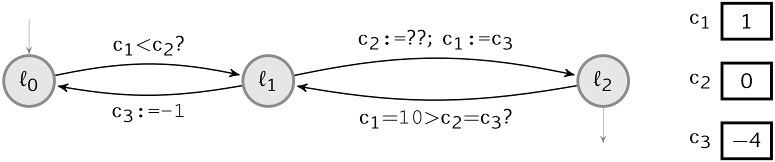

Guards: comparisons between counters and constants

Updates: assignments with counter values, constants, & "??"

One does not use $\leqslant_\times$ to compare states!! Rather

$$(a_1,\ldots,a_k)\leqslant_{\text{sparse}}(b_1,\ldots,b_k)$$

$$\overset{\text{def}}{\Leftrightarrow} \forall i,j=1,\ldots,k:\big(a_i\leqslant a_j \text{ iff } b_i\leqslant b_j\big)\wedge\big(|a_i-a_j|\leqslant|b_i-b_j|\big)\,.$$

Fact. $(\mathbb{Z}^k,\leqslant_{\text{sparse}})$ is wqo

Compatibility: We use

$$(\ell,a_1,\ldots,a_k)\leqslant(\ell',b_1,\ldots,b_k) \overset{\text{def}}{\Leftrightarrow}$$

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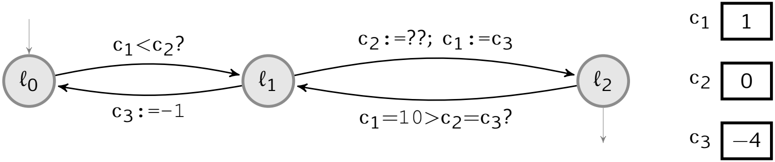
Some WSTS's: Relational automata



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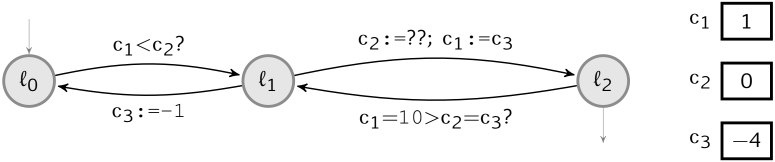

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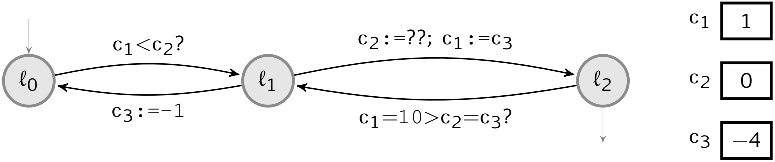

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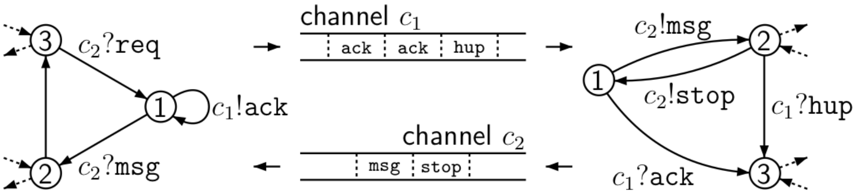
Some WSTS's: LCS / Lossy Channel Systems



A configuration $\sigma = (\ell_1, \ell_2, w_1, w_2)$ with $w_i \in \Sigma^*$.
E.g., $w_1 = \texttt{hup.ack.ack}$.

Reliable steps: $\sigma \rightarrow_{\text{rel}} \rho$ read in front of channels, write at end (FIFO)

Lossy steps: messages may be lost nondeterministically

$$\sigma \rightarrow \sigma' \overset{\text{def}}{\Leftrightarrow} \sigma \sqsupseteq \rho \rightarrow_{\text{rel}} \rho' \sqsupseteq \sigma' \text{ for some } \rho, \rho'$$

where $(S, \sqsubseteq)$ is the wqo $(Loc_1, =) \times (Loc_2, =) \times (\Sigma^*_{c_1}, \leqslant_*) \times (\Sigma^*_{c_2}, \leqslant_*)$

A model useful for concurrent protocols but also timed automata, metric temporal logic, products of modal logics, ...

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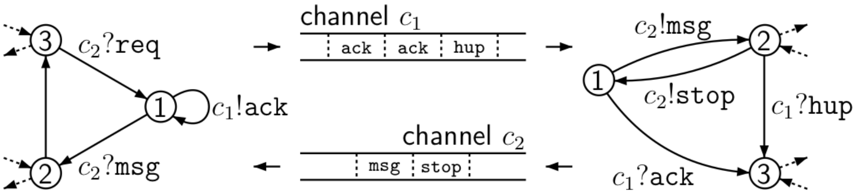

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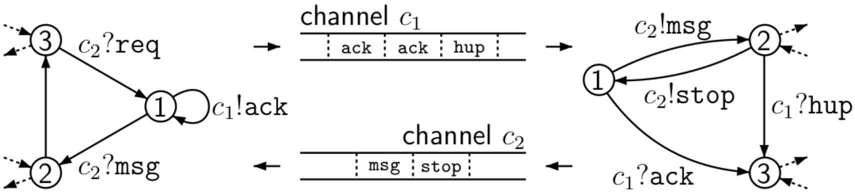

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Termination

Termination is the question, given a TS $(S, \to, \ldots)$ and a state $s_{\text{init}} \in S$, whether there are no infinite runs starting from s_{init}

Lem. [Finite Witnesses for Infinite Runs]
A WSTS $(S, \to, \leqslant)$ has an infinite run from s_{init} **iff** it has a finite run from s_{init} that is a good sequence.

Recall: $s_0, s_1, s_2, \ldots, s_n$ is good $\overset{\text{def}}{\Leftrightarrow}$ there exist $i < j$ s.t. $s_i \leqslant s_j$

Coro. One can decide Termination for a WSTS by enumerating all finite runs from s_{init} and reject when/if a good sequence is found.

NB: This requires some minimal effectiveness assumptions on the WSTS, e.g., that the ordering is decidable

Algorithm extends and allows deciding inevitability, finiteness, and regular simulation

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$\Leftarrow$: good finite run $s_0 \xrightarrow{*} s_i \xrightarrow{+} s_j$ can be extended by simulating $s_i \xrightarrow{+} s_j$ from above: $s_j \xrightarrow{+} s_{2j-i}$, then $s_{2j-i} \xrightarrow{+} s_{3j-2i}$, etc.

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Coro. Termination is co-r.e.
Since it is also r.e. (for finitely branching systems), it is decidable.

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Coverability (in practice: Safety)

Coverability is the question, given $(S, \to, \dots)$, a state s_{init} and a target state t, whether there is a run $s_{\text{init}} \to s_1 \to s_2 \cdots \to s_n$ with $s_n \geqslant t$.

This is equivalent to having a pseudo-run $s_{\text{init}}, s_1, \dots, s_n$ with $s_n \geqslant t$, where a pseudo-run is a sequence of pseudo-steps $s_{i-1} \to s_i' \geqslant s_i$.

Picture $$\overbrace{s_0 \to s_1' \geqslant s_1}^{\text{1st pseudo-step}} \underbrace{\to s_2' \geqslant s_2}_{\text{2nd pseudo-step}} \to \cdots \geqslant \cdots \overbrace{s_{n-1} \to s_n' \geqslant s_n}^{\text{last pseudo-step}} \geqslant t$$

Lem. [Finite Witnesses for Covering] There is a pseudo-run $s_{\text{init}}, \dots, s_n$ covering t **iff** there is a minimal pseudo-run $s_0 \to\geqslant \cdots \to\geqslant s_{n'} = t$ from some $s_0 \leqslant s_{\text{init}}$ to t such that $s_{n'}, s_{n'-1}, \dots, s_0$ is a bad sequence.

NB. a pseudo-run $s_0, \dots, s_{n'}$ is minimal $\overset{\text{def}}{\Leftrightarrow}$ for all $0 \leqslant i < n'$, s_i is a minimal (pseudo) predecessor of s_{i+1}.

Coro. one can decide Coverability by enumerating all pseudo-runs ending in t (backward-chaining!) that are minimal & bad sequences.

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Picture
$$\overbrace{s_0 \to s'_1 \geqslant s_1}^{\text{1st pseudo-step}} \to s'_2 \geqslant s_2 \to \cdots \geqslant \cdots \overbrace{s_{n-1} \to s'_n \geqslant s_n}^{\text{last pseudo-step}} \geqslant t$$
$$s_1 \to s'_2 \geqslant s_2 \text{ : 2nd pseudo-step}$$

Lem. [Finite Witnesses for Covering] There is a pseudo-run $s_{\text{init}}, \ldots, s_n$ covering t **iff** there is a minimal pseudo-run $s_0 \to\geqslant \cdots \to\geqslant s_{n'} = t$ from some $s_0 \leqslant s_{\text{init}}$ to t such that $s_{n'}, s_{n'-1}, \ldots, s_0$ is a bad sequence.

NB. a pseudo-run $s_0, \ldots, s_{n'}$ is minimal $\overset{\text{def}}{\Leftrightarrow}$ for all $0 \leqslant i < n'$, s_i is a minimal (pseudo) predecessor of s_{i+1}.

Coro. one can decide Coverability by enumerating all pseudo-runs ending in t (backward-chaining!) that are minimal & bad sequences.

COMPLEXITY ANALYSIS

The two algorithms we have seen guess a finite sequence $s_0, s_1, \ldots, s_\ell$ that is bad (for Coverability) or almost bad (for non-Termination) and check that they are indeed correct witnesses.

We can give a complexity upper bound in $(\text{CO})\text{NTIME}(f(n))$ or $(\text{CO})\text{NSPACE}(f(n))$ if we can bound the size of the sequence —in practice: bound its length ℓ— as a function of the input $(S, \to, \leqslant), s_{\text{init}}, t, ..$

This is the topic for next course . . .

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