Algorithmic Aspects of WQO (Well-Quasi-Ordering) Theory

Part I: Basics of WQO Theory

Philippe Schnoebelen

LSV, CNRS & ENS Cachan

Chennai Mathematical Institute, Jan. 2017

Based on joint work with Sylvain Schmitz, Prateek Karandikar, K. Narayan Kumar, Alain Finkel, ..

Lecture notes & exercises available via `www.lsv.ens-cachan.fr/~phs`

MOTIVATIONS FOR THE COURSE

- Well-quasi-orderings (wqo's) proved to be a powerful tool for decidability/termination in logic, AI, program verification, etc. *NB: they can be seen as a version of well-orderings with more flexibility*

- In program verification, wqo's are prominent in well-structured transition systems (WSTS's), a generic framework for infinite-state systems with good decidability properties.

- Analysing the complexity of wqo-based algorithms is still one of the dark arts …

- Purposes of these lectures = to disseminate the basic concepts and tools one uses for the wqo-based algorithms and their complexity analysis.

MOTIVATIONS FOR THE COURSE

- Well-quasi-orderings (wqo's) proved to be a powerful tool for decidability/termination in logic, AI, program verification, etc. *NB: they can be seen as a version of well-orderings with more flexibility*

- In program verification, wqo's are prominent in well-structured transition systems (WSTS's), a generic framework for infinite-state systems with good decidability properties.

- Analysing the complexity of wqo-based algorithms is still one of the dark arts ...

- Purposes of these lectures = to disseminate the basic concepts and tools one uses for the wqo-based algorithms and their complexity analysis.

Outline of the course

- (This) Lecture 1 = **Basics of WQO's.** Rather basic material: explaining and illustrating the definition of wqo's. Building new wqo's from simpler ones.
- Lecture 2 = **WQO-based reasoning.** Well-Structured Transition Systems (WSTS's), termination proofs, decidable logics, etc.
- Lecture 3 = **Fast-growing complexity I.** The Fast-growing hierarchy, Length function theorems for proving upper bounds.
- Lecture 4 = **Fast-growing complexity II.** Hardy computations for proving lower bounds.
- Lecture 5 = **Ideals of WQO's.** Basic concepts, Effective representations, Algorithms.

(Recalls) Ordered Sets

Def. A non-empty $(X, \leqslant)$ is a quasi-ordering (qo) $\overset{\text{def}}{\Leftrightarrow}$ $\leqslant$ is a reflexive and transitive relation.

($\approx$ a partial ordering without requiring antisymmetry, technically simpler but essentially equivalent)

Examples. $(\mathbb{N}, \leqslant)$, also $(\mathbb{R}, \leqslant)$, $(\mathbb{N} \cup \{\omega\}, \leqslant)$, ...

divisibility: $(\mathbb{Z}, \_ \mid \_)$ where $x \mid y \overset{\text{def}}{\Leftrightarrow} \exists a : a.x = y$

tuples: $(\mathbb{N}^3, \leqslant_{\text{prod}})$, or simply $(\mathbb{N}^3, \leqslant_\times)$, where $(0,1,2) <_\times (10,1,5)$ and $(1,2,3) \#_\times (3,1,2)$.

words: $(\Sigma^*, \leqslant_{\text{pref}})$ for some alphabet $\Sigma = \{a, b, \ldots\}$ and $ab <_{\text{pref}} abba$.

$(\Sigma^*, \leqslant_{\text{lex}})$ with e.g. $abba \leqslant_{\text{lex}} abc$ (NB: this assumes Σ is linearly ordered: $a < b < c$)

$(\Sigma^*, \leqslant_{\text{subword}})$, or simply $(\Sigma^*, \leqslant_*)$, with $aba \leqslant_* b\underline{a}a\underline{b}ba\underline{a}$.

(Recalls) Ordered Sets

Def. A non-empty $(X, \leqslant)$ is a quasi-ordering (qo) $\overset{\text{def}}{\Leftrightarrow}$ $\leqslant$ is a reflexive and transitive relation.

($\approx$ a partial ordering without requiring antisymmetry, technically simpler but essentially equivalent)

Examples. $(\mathbb{N}, \leqslant)$, also $(\mathbb{R}, \leqslant)$, $(\mathbb{N} \cup \{\omega\}, \leqslant)$, ...

divisibility: $(\mathbb{Z}, \_ \mid \_)$ where $x \mid y \overset{\text{def}}{\Leftrightarrow} \exists a : a.x = y$

tuples: $(\mathbb{N}^3, \leqslant_{\text{prod}})$, or simply $(\mathbb{N}^3, \leqslant_\times)$, where $(0,1,2) <_\times (10,1,5)$ and $(1,2,3) \#_\times (3,1,2)$.

words: $(\Sigma^*, \leqslant_{\text{pref}})$ for some alphabet $\Sigma = \{a, b, \ldots\}$ and $ab <_{\text{pref}} abba$.

$(\Sigma^*, \leqslant_{\text{lex}})$ with e.g. $abba \leqslant_{\text{lex}} abc$ (NB: this assumes Σ is linearly ordered: $a < b < c$)

$(\Sigma^*, \leqslant_{\text{subword}})$, or simply $(\Sigma^*, \leqslant_*)$, with $aba \leqslant_* b\underline{a}a\underline{b}ba\underline{a}$.

(RECALLS) ORDERED SETS

Def. A non-empty $(X,\leqslant)$ is a quasi-ordering (qo) $\stackrel{\text{def}}{\Leftrightarrow}$ $\leqslant$ is a reflexive and transitive relation.

($\approx$ a partial ordering without requiring antisymmetry, technically simpler but essentially equivalent)

Examples. $(\mathbb{N},\leqslant)$, also $(\mathbb{R},\leqslant)$, $(\mathbb{N}\cup\{\omega\},\leqslant)$, ...

divisibility: $(\mathbb{Z}, \_ \mid \_)$ where $x \mid y \stackrel{\text{def}}{\Leftrightarrow} \exists a : a.x = y$

tuples: $(\mathbb{N}^3, \leqslant_{\text{prod}})$, or simply $(\mathbb{N}^3, \leqslant_\times)$, where $(0,1,2) <_\times (10,1,5)$ and $(1,2,3) \#_\times (3,1,2)$.

words: $(\Sigma^*, \leqslant_{\text{pref}})$ for some alphabet $\Sigma = \{a,b,\ldots\}$ and $ab <_{\text{pref}} abba$.

$(\Sigma^*, \leqslant_{\text{lex}})$ with e.g. $abba \leqslant_{\text{lex}} abc$ (NB: this assumes Σ is linearly ordered: $a < b < c$)

$(\Sigma^*, \leqslant_{\text{subword}})$, or simply $(\Sigma^*, \leqslant_*)$, with $aba \leqslant_* b\underline{a}a\underline{b}ba\underline{a}$.

(Recalls) Ordered Sets

Def. A non-empty $(X, \leqslant)$ is a quasi-ordering (qo) $\overset{\text{def}}{\Leftrightarrow}$ $\leqslant$ is a reflexive and transitive relation.

($\approx$ a partial ordering without requiring antisymmetry, technically simpler but essentially equivalent)

Examples. $(\mathbb{N}, \leqslant)$, also $(\mathbb{R}, \leqslant)$, $(\mathbb{N} \cup \{\omega\}, \leqslant)$, ...

divisibility: $(\mathbb{Z}, \_ \mid \_)$ where $x \mid y \overset{\text{def}}{\Leftrightarrow} \exists a : a.x = y$

tuples: $(\mathbb{N}^3, \leqslant_{\text{prod}})$, or simply $(\mathbb{N}^3, \leqslant_\times)$, where $(0,1,2) <_\times (10,1,5)$ and $(1,2,3) \#_\times (3,1,2)$.

words: $(\Sigma^*, \leqslant_{\text{pref}})$ for some alphabet $\Sigma = \{a, b, \ldots\}$ and $ab <_{\text{pref}} abba$.

$(\Sigma^*, \leqslant_{\text{lex}})$ with e.g. $abba \leqslant_{\text{lex}} abc$ (NB: this assumes Σ is linearly ordered: $a < b < c$)

$(\Sigma^*, \leqslant_{\text{subword}})$, or simply $(\Sigma^*, \leqslant_*)$, with $aba \leqslant_* b\underline{a}a\underline{b}ba\underline{a}$.

(Recalls) Ordered Sets

Def. A non-empty $(X,\leqslant)$ is a quasi-ordering (qo) $\overset{\text{def}}{\Leftrightarrow}$ $\leqslant$ is a reflexive and transitive relation.

($\approx$ a partial ordering without requiring antisymmetry, technically simpler but essentially equivalent)

Examples. $(\mathbb{N},\leqslant)$, also $(\mathbb{R},\leqslant)$, $(\mathbb{N}\cup\{\omega\},\leqslant)$, ...

divisibility: $(\mathbb{Z},\_\mid\_)$ where $x \mid y \overset{\text{def}}{\Leftrightarrow} \exists a : a.x = y$

tuples: $(\mathbb{N}^3,\leqslant_{\text{prod}})$, or simply $(\mathbb{N}^3,\leqslant_\times)$, where $(0,1,2) <_\times (10,1,5)$ and $(1,2,3)\#_\times(3,1,2)$.

words: $(\Sigma^*,\leqslant_{\text{pref}})$ for some alphabet $\Sigma=\{a,b,\ldots\}$ and $ab <_{\text{pref}} abba$.

$(\Sigma^*,\leqslant_{\text{lex}})$ with e.g. $abba \leqslant_{\text{lex}} abc$ (NB: this assumes Σ is linearly ordered: $a < b < c$)

$(\Sigma^*,\leqslant_{\text{subword}})$, or simply $(\Sigma^*,\leqslant_*)$, with $aba \leqslant_* b\underline{a}a\underline{b}ba\underline{a}$.

(Recalls) Ordered sets

Def. $(X, \leqslant)$ is linear, a.k.a. total, if for any $x, y \in X$ either $x \leqslant y$ or $y \leqslant x$. (I.e., there is no $x \# y$.)

Def. $(X, \leqslant)$ is well-founded if there is no infinite strictly decreasing sequence $x_0 > x_1 > x_2 > \cdots$

	linear?	well-founded?
$\mathbb{N}, \leqslant$		
$\mathbb{Z}, \mid$		
$\mathbb{N} \cup \{\omega\}, \leqslant$		
$\mathbb{N}^3, \leqslant_\times$		
$\Sigma^*, \leqslant_{\text{pref}}$		
$\Sigma^*, \leqslant_{\text{lex}}$		
$\Sigma^*, \leqslant_*$		

(Recalls) Ordered sets

Def. $(X, \leqslant)$ is linear, a.k.a. total, if for any $x, y \in X$ either $x \leqslant y$ or $y \leqslant x$. (I.e., there is no $x \# y$.)

Def. $(X, \leqslant)$ is well-founded if there is no infinite strictly decreasing sequence $x_0 > x_1 > x_2 > \cdots$

	linear?	well-founded?
$\mathbb{N}, \leqslant$	✓	
$\mathbb{Z}, \mid$	×	
$\mathbb{N} \cup \{\omega\}, \leqslant$	✓	
$\mathbb{N}^3, \leqslant_\times$	×	
$\Sigma^*, \leqslant_{\text{pref}}$	×	
$\Sigma^*, \leqslant_{\text{lex}}$	✓	
$\Sigma^*, \leqslant_*$	×	

(Recalls) Ordered sets

Def. $(X, \leqslant)$ is linear, a.k.a. total, if for any $x, y \in X$ either $x \leqslant y$ or $y \leqslant x$. (I.e., there is no $x \# y$.)

Def. $(X, \leqslant)$ is well-founded if there is no infinite strictly decreasing sequence $x_0 > x_1 > x_2 > \cdots$

	linear?	well-founded?
$\mathbb{N}, \leqslant$	✓	✓
$\mathbb{Z}, \mid$	×	✓
$\mathbb{N} \cup \{\omega\}, \leqslant$	✓	✓
$\mathbb{N}^3, \leqslant_\times$	×	✓
$\Sigma^*, \leqslant_{\text{pref}}$	×	✓
$\Sigma^*, \leqslant_{\text{lex}}$	✓	×
$\Sigma^*, \leqslant_*$	×	✓

WELL-QUASI-ORDERING (WQO)

Def1. $(X,\leqslant)$ is a wqo $\overset{\text{def}}{\Leftrightarrow}$ any infinite sequence $x_0,x_1,x_2,\ldots$ contains an increasing pair: $x_i \leqslant x_j$ for some $i<j$.

Def2. $(X,\leqslant)$ is a wqo $\overset{\text{def}}{\Leftrightarrow}$ any infinite sequence $x_0,x_1,x_2,\ldots$ contains an infinite increasing subsequence: $x_{n_0} \leqslant x_{n_1} \leqslant x_{n_2} \leqslant \ldots$

Def3. $(X,\leqslant)$ is a wqo $\overset{\text{def}}{\Leftrightarrow}$ there is no infinite strictly decreasing sequence $x_0 > x_1 > x_2 > \ldots$ —i.e., $(X,\leqslant)$ is well-founded— and no infinite set $\{x_0,x_1,x_2,\ldots\}$ of mutually incomparable elements $x_i \# x_j$ when $i \neq j$ —we say "$(X,\leqslant)$ has no infinite antichain"—.

Fact. These three definitions are equivalent.
Clearly, Def2 $\Rightarrow$ Def1 and Def1 $\Rightarrow$ Def3.
But the reverse implications are non-trivial.

To show Def3 $\Rightarrow$ Def2, we first recall the Infinite Ramsey Theorem: "Let X be some countably infinite set and colour the elements of $X^{(n)}$ (the subsets of X of size n) in c different colours. Then there exists some infinite subset M of X s.t. the size n subsets of M all have the same colour."

Well-quasi-ordering (wqo)

Def1. $(X,\leqslant)$ is a wqo $\overset{\text{def}}{\Leftrightarrow}$ any infinite sequence $x_0, x_1, x_2, \ldots$ contains an increasing pair: $x_i \leqslant x_j$ for some $i < j$.

Def2. $(X,\leqslant)$ is a wqo $\overset{\text{def}}{\Leftrightarrow}$ any infinite sequence $x_0, x_1, x_2, \ldots$ contains an infinite increasing subsequence: $x_{n_0} \leqslant x_{n_1} \leqslant x_{n_2} \leqslant \ldots$

Def3. $(X,\leqslant)$ is a wqo $\overset{\text{def}}{\Leftrightarrow}$ there is no infinite strictly decreasing sequence $x_0 > x_1 > x_2 > \ldots$ —i.e., $(X,\leqslant)$ is well-founded— and no infinite set $\{x_0, x_1, x_2, \ldots\}$ of mutually incomparable elements $x_i \# x_j$ when $i \neq j$ —we say "$(X,\leqslant)$ has no infinite antichain"—.

Fact. These three definitions are equivalent.
Clearly, Def2 $\Rightarrow$ Def1 and Def1 $\Rightarrow$ Def3.
But the reverse implications are non-trivial.

To show Def3 $\Rightarrow$ Def2, we first recall the Infinite Ramsey Theorem:
"Let X be some countably infinite set and colour the elements of $X^{(n)}$ (the subsets of X of size n) in c different colours. Then there exists some infinite subset M of X s.t. the size n subsets of M all have the same colour."

WELL-QUASI-ORDERING (WQO)

Def1. $(X, \leqslant)$ is a wqo $\overset{\text{def}}{\Leftrightarrow}$ any infinite sequence $x_0, x_1, x_2, \ldots$ contains an increasing pair: $x_i \leqslant x_j$ for some $i < j$.

Def2. $(X, \leqslant)$ is a wqo $\overset{\text{def}}{\Leftrightarrow}$ any infinite sequence $x_0, x_1, x_2, \ldots$ contains an infinite increasing subsequence: $x_{n_0} \leqslant x_{n_1} \leqslant x_{n_2} \leqslant \ldots$

Def3. $(X, \leqslant)$ is a wqo $\overset{\text{def}}{\Leftrightarrow}$ there is no infinite strictly decreasing sequence $x_0 > x_1 > x_2 > \ldots$ —i.e., $(X, \leqslant)$ is well-founded— and no infinite set $\{x_0, x_1, x_2, \ldots\}$ of mutually incomparable elements $x_i \# x_j$ when $i \neq j$ —we say "$(X, \leqslant)$ has no infinite antichain"—.

Fact. These three definitions are equivalent.
Clearly, Def2 $\Rightarrow$ Def1 and Def1 $\Rightarrow$ Def3.
But the reverse implications are non-trivial.

To show Def3 $\Rightarrow$ Def2, we first recall the Infinite Ramsey Theorem:
"Let X be some countably infinite set and colour the elements of $X^{(n)}$ (the subsets of X of size n) in c different colours. Then there exists some infinite subset M of X s.t. the size n subsets of M all have the same colour."

Well-quasi-ordering (wqo)

Def1. $(X,\leqslant)$ is a wqo $\overset{\text{def}}{\Leftrightarrow}$ any infinite sequence $x_0,x_1,x_2,\ldots$ contains an increasing pair: $x_i \leqslant x_j$ for some $i<j$.

Def2. $(X,\leqslant)$ is a wqo $\overset{\text{def}}{\Leftrightarrow}$ any infinite sequence $x_0,x_1,x_2,\ldots$ contains an infinite increasing subsequence: $x_{n_0} \leqslant x_{n_1} \leqslant x_{n_2} \leqslant \ldots$

Def3. $(X,\leqslant)$ is a wqo $\overset{\text{def}}{\Leftrightarrow}$ there is no infinite strictly decreasing sequence $x_0 > x_1 > x_2 > \ldots$ —i.e., $(X,\leqslant)$ is well-founded— and no infinite set $\{x_0,x_1,x_2,\ldots\}$ of mutually incomparable elements $x_i \# x_j$ when $i \neq j$ —we say "$(X,\leqslant)$ has no infinite antichain"—.

Fact. These three definitions are equivalent.
Clearly, Def2 $\Rightarrow$ Def1 and Def1 $\Rightarrow$ Def3.
But the reverse implications are non-trivial.

To show Def3 $\Rightarrow$ Def2, we first recall the Infinite Ramsey Theorem: "Let X be some countably infinite set and colour the elements of $X^{(n)}$ (the subsets of X of size n) in c different colours. Then there exists some infinite subset M of X s.t. the size n subsets of M all have the same colour."

Well-quasi-ordering (wqo)

Def1. $(X, \leqslant)$ is a wqo $\overset{\text{def}}{\Leftrightarrow}$ any infinite sequence $x_0, x_1, x_2, \ldots$ contains an increasing pair: $x_i \leqslant x_j$ for some $i < j$.

Def2. $(X, \leqslant)$ is a wqo $\overset{\text{def}}{\Leftrightarrow}$ any infinite sequence $x_0, x_1, x_2, \ldots$ contains an infinite increasing subsequence: $x_{n_0} \leqslant x_{n_1} \leqslant x_{n_2} \leqslant \ldots$

Def3. $(X, \leqslant)$ is a wqo $\overset{\text{def}}{\Leftrightarrow}$ there is no infinite strictly decreasing sequence $x_0 > x_1 > x_2 > \ldots$ —i.e., $(X, \leqslant)$ is well-founded— and no infinite set $\{x_0, x_1, x_2, \ldots\}$ of mutually incomparable elements $x_i \# x_j$ when $i \neq j$ —we say "$(X, \leqslant)$ has no infinite antichain"—.

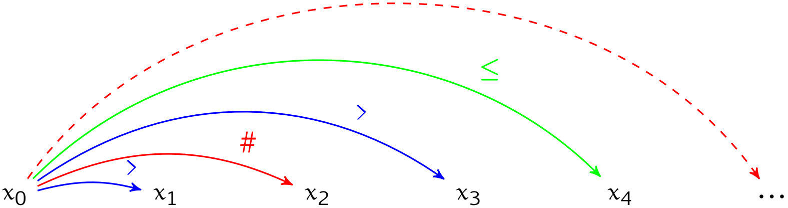
Fact. These three definitions are equivalent.
Clearly, Def2 $\Rightarrow$ Def1 and Def1 $\Rightarrow$ Def3.
But the reverse implications are non-trivial.

To show Def3 $\Rightarrow$ Def2, we first recall the Infinite Ramsey Theorem: "Let X be some countably infinite set and colour the elements of $X^{(n)}$ (the subsets of X of size n) in c different colours. Then there exists some infinite subset M of X s.t. the size n subsets of M all have the same colour."

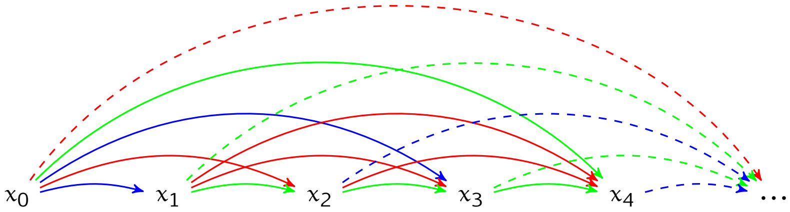
Proving Def3 $\Rightarrow$ Def2

$$x_0 \qquad x_1 \qquad x_2 \qquad x_3 \qquad x_4 \qquad \cdots$$

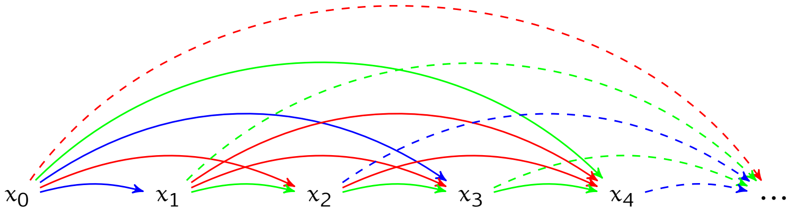
PROVING DEF3 $\Rightarrow$ DEF2



Proving Def3 ⇒ Def2



PROVING DEF3 $\Rightarrow$ DEF2



Infinite Ramsey Theorem:

there is an infinite subset $\{x_i\}_{i \in I}$ that is monochromatic

PROVING DEF3 $\Rightarrow$ DEF2

Infinite Ramsey Theorem:

there is an infinite subset $\{x_{n_i}\}_{i=0,1,2,\ldots}$ that is monochromatic

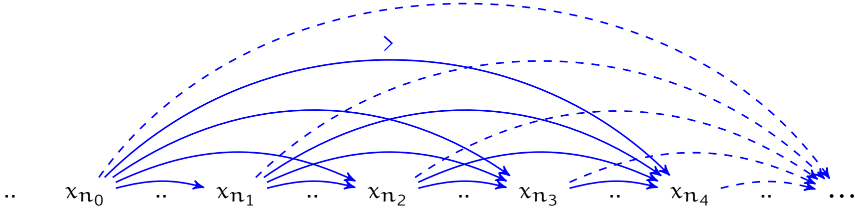
$$.. \quad x_{n_0} \quad .. \quad x_{n_1} \quad .. \quad x_{n_2} \quad .. \quad x_{n_3} \quad .. \quad x_{n_4} \quad .. \quad \cdots$$

What color?

PROVING DEF3 $\Rightarrow$ DEF2

Infinite Ramsey Theorem:

there is an infinite subset $\{x_{n_i}\}_{i=0,1,2,\ldots}$ that is monochromatic

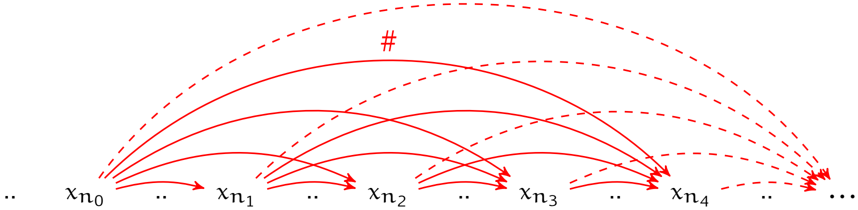


Blue $\Rightarrow$ infinite strictly decreasing sequence, contradicts WF

Proving Def3 $\Rightarrow$ Def2

Infinite Ramsey Theorem:

there is an infinite subset $\{x_{n_i}\}_{i=0,1,2,\ldots}$ that is monochromatic

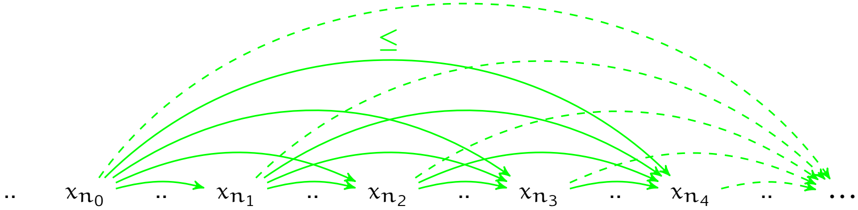


Red $\Rightarrow$ infinite antichain, contradicts FAC

PROVING DEF3 $\Rightarrow$ DEF2

Infinite Ramsey Theorem:

there is an infinite subset $\{x_{n_i}\}_{i=0,1,2,\ldots}$ that is monochromatic



Must be green $\Rightarrow$ infinite increasing sequence! QED

Spot the wqo's

	linear?	well-founded?	wqo?
$\mathbb{N}, \leqslant$	✓	✓	
$\mathbb{Z}, \mid$	×	✓	
$\mathbb{N} \cup \{\omega\}, \leqslant$	✓	✓	
$\mathbb{N}^3, \leqslant_\times$	×	✓	
$\Sigma^*, \leqslant_{\text{pref}}$	×	✓	
$\Sigma^*, \leqslant_{\text{lex}}$	✓	×	
$\Sigma^*, \leqslant_*$	×	✓	

SPOT THE WQO'S

	linear?	well-founded?	wqo?
$\mathbb{N}, \leqslant$	✓	✓	✓
$\mathbb{Z}, \vert$	×	✓	
$\mathbb{N} \cup \{\omega\}, \leqslant$	✓	✓	
$\mathbb{N}^3, \leqslant_\times$	×	✓	
$\Sigma^*, \leqslant_{\mathsf{pref}}$	×	✓	
$\Sigma^*, \leqslant_{\mathsf{lex}}$	✓	×	
$\Sigma^*, \leqslant_*$	×	✓	

Spot the wqo's

	linear?	well-founded?	wqo?
$\mathbb{N},\leqslant$	✓	✓	✓
$\mathbb{Z},\mid$	×	✓	
$\mathbb{N}\cup\{\omega\},\leqslant$	✓	✓	✓
$\mathbb{N}^3,\leqslant_\times$	×	✓	
$\Sigma^*,\leqslant_{\text{pref}}$	×	✓	
$\Sigma^*,\leqslant_{\text{lex}}$	✓	×	
$\Sigma^*,\leqslant_*$	×	✓	

More generally

Fact. For linear qo's: well-founded $\Leftrightarrow$ wqo.

Cor. Any ordinal is wqo.

Spot the wqo's

	linear?	well-founded?	wqo?
$\mathbb{N}, \leqslant$	✓	✓	✓
$\mathbb{Z}, \mid$	×	✓	×
$\mathbb{N} \cup \{\omega\}, \leqslant$	✓	✓	✓
$\mathbb{N}^3, \leqslant_\times$	×	✓	
$\Sigma^*, \leqslant_{\text{pref}}$	×	✓	
$\Sigma^*, \leqslant_{\text{lex}}$	✓	×	
$\Sigma^*, \leqslant_*$	×	✓	

$(\mathbb{Z}, \mid)$: The prime numbers $\{2, 3, 5, 7, 11, \ldots\}$ are an infinite antichain.

Spot the wqo's

	linear?	well-founded?	wqo?
$\mathbb{N}, \leqslant$	✓	✓	✓
$\mathbb{Z}, \mid$	×	✓	×
$\mathbb{N} \cup \{\omega\}, \leqslant$	✓	✓	✓
$\mathbb{N}^3, \leqslant_\times$	×	✓	✓
$\Sigma^*, \leqslant_{\text{pref}}$	×	✓	
$\Sigma^*, \leqslant_{\text{lex}}$	✓	×	
$\Sigma^*, \leqslant_*$	×	✓	

More generally

(Generalized) Dickson's lemma. If $(X_1, \leqslant_1), \ldots, (X_n, \leqslant_n)$'s are wqo's, then $\prod_{i=1}^n X_i, \leqslant_\times$ is wqo.

Proof. Easy with Def2. Otherwise, an application of the Infinite Ramsey Theorem.

(Usual) Dickson's Lemma. $(\mathbb{N}^k, \leqslant_\times)$ is wqo for any k.

SPOT THE WQO'S

	linear?	well-founded?	wqo?
$\mathbb{N}, \leqslant$	✓	✓	✓
$\mathbb{Z}, \mid$	×	✓	×
$\mathbb{N} \cup \{\omega\}, \leqslant$	✓	✓	✓
$\mathbb{N}^3, \leqslant_\times$	×	✓	✓
$\Sigma^*, \leqslant_{\text{pref}}$	×	✓	×
$\Sigma^*, \leqslant_{\text{lex}}$	✓	×	×
$\Sigma^*, \leqslant_*$	×	✓	

$(\Sigma^*, \leqslant_{\text{pref}})$ has an infinite antichain

$$\mathtt{b}, \mathtt{ab}, \mathtt{aab}, \mathtt{aaab}, \ldots$$

$(\Sigma^*, \leqslant_{\text{lex}})$ is not well-founded:

$$\mathtt{b} >_{\text{lex}} \mathtt{ab} >_{\text{lex}} \mathtt{aab} >_{\text{lex}} \mathtt{aaab} >_{\text{lex}} \cdots$$

SPOT THE WQO'S

	linear?	well-founded?	wqo?
$\mathbb{N}, \leqslant$	✓	✓	✓
$\mathbb{Z}, \mid$	×	✓	×
$\mathbb{N} \cup \{\omega\}, \leqslant$	✓	✓	✓
$\mathbb{N}^3, \leqslant_\times$	×	✓	✓
$\Sigma^*, \leqslant_{\text{pref}}$	×	✓	×
$\Sigma^*, \leqslant_{\text{lex}}$	✓	×	×
$\Sigma^*, \leqslant_*$	×	✓	✓

$(\Sigma^*, \leqslant_*)$ is wqo by Higman's Lemma (see next slide).

We can get some feeling by trying to build a bad sequence, i.e., some $w_0, w_1, w_2, \ldots$ without an increasing pair $w_i \leqslant_* w_j$.

Higman's Lemma

Def. The sequence extension of a qo $(X,\leqslant)$ is the qo $(X^*,\leqslant_*)$ of finite sequences over X ordered by embedding:

$$w = x_1 \dots x_n \leqslant_* y_1 \dots y_m = v \overset{\text{def}}{\Leftrightarrow} \begin{array}{l} x_1 \leqslant y_{l_1} \wedge \dots \wedge x_n \leqslant y_{l_n} \\ \text{for some } 1 \leqslant l_1 < l_2 < \dots < l_n \leqslant m \end{array}$$

$$\overset{\text{def}}{\Leftrightarrow} w \leqslant_\times v' \text{ for a length-}n\text{ subsequence } v' \text{ of } v$$

Higman's Lemma. $(X^*,\leqslant_*)$ is a wqo iff $(X,\leqslant)$ is.

With $(\Sigma^*,\leqslant_*)$, we are considering the sequence extension of $(\Sigma,=)$ which is finite, hence necessarily wqo.

Later we'll consider the sequence extension of more complex wqo's, e.g., $\mathbb{N}^2$:

$$\begin{vmatrix}0\\1\end{vmatrix}\begin{vmatrix}2\\0\end{vmatrix}\begin{vmatrix}0\\2\end{vmatrix} \leqslant_* ? \begin{vmatrix}2\\0\end{vmatrix}\begin{vmatrix}0\\2\end{vmatrix}\begin{vmatrix}0\\2\end{vmatrix}\begin{vmatrix}2\\2\end{vmatrix}\begin{vmatrix}2\\0\end{vmatrix}\begin{vmatrix}0\\1\end{vmatrix}$$

HIGMAN'S LEMMA

Def. The sequence extension of a qo $(X, \leqslant)$ is the qo $(X^*, \leqslant_*)$ of finite sequences over X ordered by embedding:

$$w = x_1 \dots x_n \leqslant_* y_1 \dots y_m = v \overset{\text{def}}{\Leftrightarrow} \begin{array}{l} x_1 \leqslant y_{l_1} \wedge \dots \wedge x_n \leqslant y_{l_n} \\ \text{for some } 1 \leqslant l_1 < l_2 < \dots < l_n \leqslant m \end{array}$$

$$\overset{\text{def}}{\Leftrightarrow} w \leqslant_\times v' \text{ for a length-}n\text{ subsequence } v' \text{ of } v$$

Higman's Lemma. $(X^*, \leqslant_*)$ is a wqo iff $(X, \leqslant)$ is.

With $(\Sigma^*, \leqslant_*)$, we are considering the sequence extension of $(\Sigma, =)$ which is finite, hence necessarily wqo.

Later we'll consider the sequence extension of more complex wqo's, e.g., $\mathbb{N}^2$:

$$\begin{vmatrix}\mathbf{0}\\\mathbf{1}\end{vmatrix}\begin{vmatrix}\mathbf{2}\\\mathbf{0}\end{vmatrix}\begin{vmatrix}\mathbf{0}\\\mathbf{2}\end{vmatrix} \leqslant_*? \begin{vmatrix}\mathbf{2}\\\mathbf{0}\end{vmatrix}\begin{vmatrix}\mathbf{0}\\\mathbf{2}\end{vmatrix}\begin{vmatrix}\mathbf{0}\\\mathbf{2}\end{vmatrix}\begin{vmatrix}\mathbf{2}\\\mathbf{2}\end{vmatrix}\begin{vmatrix}\mathbf{2}\\\mathbf{0}\end{vmatrix}\begin{vmatrix}\mathbf{0}\\\mathbf{1}\end{vmatrix}$$

PROOF OF HIGMAN'S LEMMA

Let $(X, \leqslant)$ be wqo and assume by way of contradiction that $(X^*, \leqslant_*)$ admits infinite bad sequences (sequences with no increasing pairs).

Let $w_0 \in X^*$ be a shortest word that can start an infinite bad sequence.

Let $w_1 \in X^*$ be a shortest word that can continue, i.e., such that there is an infinite bad sequence starting with w_0, w_1

Continue. This way we pick an infinite sequence $S = w_0, w_1, w_2, w_3, \ldots$

Claim. S too is bad

Write w_i under the form $w_i = x_i v_i$. Since X is wqo, there is an infinite increasing sequence $x_{n_0} \leqslant x_{n_1} \leqslant x_{n_2} \leqslant \cdots$ (here we use Def2)

Now consider $S' \stackrel{\text{def}}{=} w_0, w_1, \ldots, w_{n_0-1}, v_{n_0}, v_{n_1}, v_{n_2}, \ldots$

It cannot be bad (otherwise w_{n_0} would not have been shortest).

But an increasing pair like $v_n \leqslant_* v_m$ in S' leads to $x_n v_n \leqslant_* x_m v_m$, i.e., $w_n \leqslant_* w_m$, a contradiction.

Proof of Higman's Lemma

Let $(X, \leqslant)$ be wqo and assume by way of contradiction that $(X^*, \leqslant_*)$ admits infinite bad sequences (sequences with no increasing pairs).

Let $w_0 \in X^*$ be a shortest word that can start an infinite bad sequence.

Let $w_1 \in X^*$ be a shortest word that can continue, i.e., such that there is an infinite bad sequence starting with w_0, w_1

Continue. This way we pick an infinite sequence $S = w_0, w_1, w_2, w_3, \ldots$

Claim. S too is bad

Write w_i under the form $w_i = x_i v_i$. Since X is wqo, there is an infinite increasing sequence $x_{n_0} \leqslant x_{n_1} \leqslant x_{n_2} \leqslant \cdots$ (here we use Def2)

Now consider $S' \stackrel{\text{def}}{=} w_0, w_1, \ldots, w_{n_0-1}, v_{n_0}, v_{n_1}, v_{n_2}, \ldots$

It cannot be bad (otherwise w_{n_0} would not have been shortest).

But an increasing pair like $v_n \leqslant_* v_m$ in S' leads to $x_n v_n \leqslant_* x_m v_m$, i.e., $w_n \leqslant_* w_m$, a contradiction.

Proof of Higman's Lemma

Let $(X, \leqslant)$ be wqo and assume by way of contradiction that $(X^*, \leqslant_*)$ admits infinite bad sequences (sequences with no increasing pairs).

Let $w_0 \in X^*$ be a shortest word that can start an infinite bad sequence.

Let $w_1 \in X^*$ be a shortest word that can continue, i.e., such that there is an infinite bad sequence starting with w_0, w_1

Continue. This way we pick an infinite sequence $S = w_0, w_1, w_2, w_3, \ldots$

Claim. S too is bad

Write w_i under the form $w_i = x_i v_i$. Since X is wqo, there is an infinite increasing sequence $x_{n_0} \leqslant x_{n_1} \leqslant x_{n_2} \leqslant \cdots$ (here we use Def2)

Now consider $S' \stackrel{\text{def}}{=} w_0, w_1, \ldots, w_{n_0-1}, v_{n_0}, v_{n_1}, v_{n_2}, \ldots$

It cannot be bad (otherwise w_{n_0} would not have been shortest).

But an increasing pair like $v_n \leqslant_* v_m$ in S' leads to $x_n v_n \leqslant_* x_m v_m$, i.e., $w_n \leqslant_* w_m$, a contradiction.

Proof of Higman's Lemma

Let $(X,\leqslant)$ be wqo and assume by way of contradiction that $(X^*,\leqslant_*)$ admits infinite bad sequences (sequences with no increasing pairs).

Let $w_0 \in X^*$ be a shortest word that can start an infinite bad sequence.

Let $w_1 \in X^*$ be a shortest word that can continue, i.e., such that there is an infinite bad sequence starting with w_0, w_1

Continue. This way we pick an infinite sequence $S = w_0, w_1, w_2, w_3, \ldots$

Claim. S too is bad

Write w_i under the form $w_i = x_i v_i$. Since X is wqo, there is an infinite increasing sequence $x_{n_0} \leqslant x_{n_1} \leqslant x_{n_2} \leqslant \cdots$ (here we use Def2)

Now consider $S' \stackrel{\text{def}}{=} w_0, w_1, \ldots, w_{n_0-1}, v_{n_0}, v_{n_1}, v_{n_2}, \ldots$

It cannot be bad (otherwise w_{n_0} would not have been shortest).

But an increasing pair like $v_n \leqslant_* v_m$ in S' leads to $x_n v_n \leqslant_* x_m v_m$, i.e., $w_n \leqslant_* w_m$, a contradiction.

PROOF OF HIGMAN'S LEMMA

Let $(X, \leqslant)$ be wqo and assume by way of contradiction that $(X^*, \leqslant_*)$ admits infinite bad sequences (sequences with no increasing pairs).

Let $w_0 \in X^*$ be a shortest word that can start an infinite bad sequence.

Let $w_1 \in X^*$ be a shortest word that can continue, i.e., such that there is an infinite bad sequence starting with w_0, w_1

Continue. This way we pick an infinite sequence $S = w_0, w_1, w_2, w_3, \ldots$

Claim. S too is bad

Write w_i under the form $w_i = x_i v_i$. Since X is wqo, there is an infinite increasing sequence $x_{n_0} \leqslant x_{n_1} \leqslant x_{n_2} \leqslant \cdots$ (here we use Def2)

Now consider $S' \stackrel{\text{def}}{=} w_0, w_1, \ldots, w_{n_0-1}, v_{n_0}, v_{n_1}, v_{n_2}, \ldots$

It cannot be bad (otherwise w_{n_0} would not have been shortest).

But an increasing pair like $v_n \leqslant_* v_m$ in S' leads to $x_n v_n \leqslant_* x_m v_m$, i.e., $w_n \leqslant_* w_m$, a contradiction.

Proof of Higman's Lemma

Let $(X, \leqslant)$ be wqo and assume by way of contradiction that $(X^*, \leqslant_*)$ admits infinite bad sequences (sequences with no increasing pairs).

Let $w_0 \in X^*$ be a shortest word that can start an infinite bad sequence.

Let $w_1 \in X^*$ be a shortest word that can continue, i.e., such that there is an infinite bad sequence starting with w_0, w_1

Continue. This way we pick an infinite sequence $S = w_0, w_1, w_2, w_3, \ldots$

Claim. S too is bad

Write w_i under the form $w_i = x_i v_i$. Since X is wqo, there is an infinite increasing sequence $x_{n_0} \leqslant x_{n_1} \leqslant x_{n_2} \leqslant \cdots$ (here we use Def2)

Now consider $S' \stackrel{\text{def}}{=} w_0, w_1, \ldots, w_{n_0-1}, v_{n_0}, v_{n_1}, v_{n_2}, \ldots$

It cannot be bad (otherwise w_{n_0} would not have been shortest).

But an increasing pair like $v_n \leqslant_* v_m$ in S' leads to $x_n v_n \leqslant_* x_m v_m$, i.e., $w_n \leqslant_* w_m$, a contradiction.

Proof of Higman's Lemma

Let $(X,\leqslant)$ be wqo and assume by way of contradiction that $(X^*,\leqslant_*)$ admits infinite bad sequences (sequences with no increasing pairs).

Let $w_0 \in X^*$ be a shortest word that can start an infinite bad sequence.

Let $w_1 \in X^*$ be a shortest word that can continue, i.e., such that there is an infinite bad sequence starting with w_0, w_1

Continue. This way we pick an infinite sequence $S = w_0, w_1, w_2, w_3, \ldots$

Claim. S too is bad

Write w_i under the form $w_i = x_i v_i$. Since X is wqo, there is an infinite increasing sequence $x_{n_0} \leqslant x_{n_1} \leqslant x_{n_2} \leqslant \cdots$ (here we use Def2)

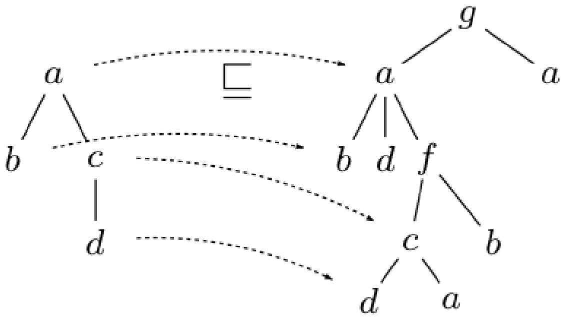
Now consider $S' \stackrel{\text{def}}{=} w_0, w_1, \ldots, w_{n_0-1}, v_{n_0}, v_{n_1}, v_{n_2}, \ldots$

It cannot be bad (otherwise w_{n_0} would not have been shortest).

But an increasing pair like $v_n \leqslant_* v_m$ in S' leads to $x_n v_n \leqslant_* x_m v_m$, i.e., $w_n \leqslant_* w_m$, a contradiction.

MORE WQO'S

- Finite Trees ordered by embeddings (Kruskal's Tree Theorem)



PROOF OF KRUSKAL'S TREE THEOREM

Let $(X, \leqslant)$ be wqo and assume, b.w.o.c., that $(\mathcal{T}(X), \sqsubseteq)$ is not wqo.

We pick a "minimal" bad sequence $S = t_0, t_1, t_2, \ldots$

Write every t_i under the form $t_i = f_i(u_{i,1}, \ldots, u_{i,k_i})$.

Claim. The set $U = \{u_{i,j}\}$ of the immediate subterms is wqo. (Indeed, an infinite bad sequence $u_{i_0,j_o}, u_{i_1,j_i}, ..$ could be used to show that t_{i_0} was not "shortest").

Since U is wqo, and using Higman's Lemma on U^*, there is some $(u_{n_1,1}, \ldots, u_{n_1,k_{n_1}}) \leqslant_* (u_{n_2,1}, \ldots, u_{n_2,k_{n_2}}) \leqslant_* (u_{n_3,1}, \ldots, u_{n_3,k_{n_3}}) \leqslant_* \cdots$

Further extracting some $f_{n_{i_1}} \leqslant f_{n_{i_2}} \leqslant \cdots$ exhibits an infinite increasing subsequence $t_{n_{i_1}} \sqsubseteq t_{n_{i_2}} \sqsubseteq \cdots$ in S, a contradiction

PROOF OF KRUSKAL'S TREE THEOREM

Let $(X,\leqslant)$ be wqo and assume, b.w.o.c., that $(\mathcal{T}(X),\sqsubseteq)$ is not wqo.

We pick a "minimal" bad sequence $S = t_0, t_1, t_2, \ldots$

Write every t_i under the form $t_i = f_i(u_{i,1}, \ldots, u_{i,k_i})$.

Claim. The set $U = \{u_{i,j}\}$ of the immediate subterms is wqo. (Indeed, an infinite bad sequence u_{i_0,j_o}, u_{i_1,j_i},.. could be used to show that t_{i_0} was not "shortest").

Since U is wqo, and using Higman's Lemma on U^*, there is some $(u_{n_1,1},\ldots,u_{n_1,k_{n_1}}) \leqslant_* (u_{n_2,1},\ldots,u_{n_2,k_{n_2}}) \leqslant_* (u_{n_3,1},\ldots,u_{n_3,k_{n_3}}) \leqslant_* \cdots$

Further extracting some $f_{n_{i_1}} \leqslant f_{n_{i_2}} \leqslant \cdots$ exhibits an infinite increasing subsequence $t_{n_{i_1}} \sqsubseteq t_{n_{i_2}} \sqsubseteq \cdots$ in S, a contradiction

Proof of Kruskal's Tree Theorem

Let $(X,\leqslant)$ be wqo and assume, b.w.o.c., that $(\mathcal{T}(X),\sqsubseteq)$ is not wqo.

We pick a "minimal" bad sequence $S = t_0, t_1, t_2, \ldots$

Write every t_i under the form $t_i = f_i(u_{i,1},\ldots,u_{i,k_i})$.

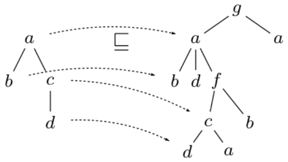
Claim. The set $U = \{u_{i,j}\}$ of the immediate subterms is wqo.
(Indeed, an infinite bad sequence u_{i_0,j_o}, u_{i_1,j_i},.. could be used to show that t_{i_0} was not "shortest").

Since U is wqo, and using Higman's Lemma on U^*, there is some $(u_{n_1,1},\ldots,u_{n_1,k_{n_1}}) \leqslant_* (u_{n_2,1},\ldots,u_{n_2,k_{n_2}}) \leqslant_* (u_{n_3,1},\ldots,u_{n_3,k_{n_3}}) \leqslant_* \cdots$

Further extracting some $f_{n_{i_1}} \leqslant f_{n_{i_2}} \leqslant \cdots$ exhibits an infinite increasing subsequence $t_{n_{i_1}} \sqsubseteq t_{n_{i_2}} \sqsubseteq \cdots$ in S, a contradiction

More wqo's

- Finite Trees ordered by embeddings (Kruskal's Tree Theorem)



- Finite Graphs ordered by minor (Robertson-Seymour Theorem)

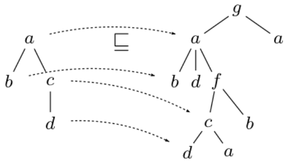
$$C_n \leqslant_{\text{minor}} K_n \text{ and } C_n \leqslant_{\text{minor}} C_{n+1}$$

- $(X^\omega, \leqslant_*)$ for X linear wqo.
- $(\mathcal{P}_f(X), \sqsubseteq_H)$ for X wqo, where

$$U \sqsubseteq_H V \overset{\text{def}}{\Leftrightarrow} \forall x \in U : \exists y \in V : x \leqslant y$$

More wqo's

- Finite Trees ordered by embeddings (Kruskal's Tree Theorem)



- Finite Graphs ordered by minor (Robertson-Seymour Theorem)

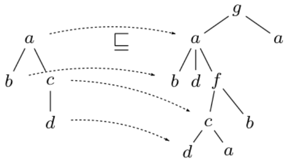
$$C_n \leqslant_{\text{minor}} K_n \text{ and } C_n \leqslant_{\text{minor}} C_{n+1}$$

- $(X^\omega, \leqslant_*)$ for X linear wqo.
- $(\mathcal{P}_f(X), \sqsubseteq_H)$ for X wqo, where

$$U \sqsubseteq_H V \overset{\text{def}}{\Leftrightarrow} \forall x \in U : \exists y \in V : x \leqslant y$$

More wqo's

- Finite Trees ordered by embeddings (Kruskal's Tree Theorem)



- Finite Graphs ordered by minor (Robertson-Seymour Theorem)

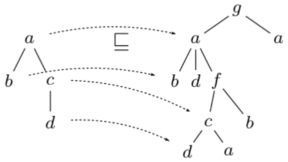
$$C_n \leqslant_{\text{minor}} K_n \text{ and } C_n \leqslant_{\text{minor}} C_{n+1}$$

- $(X^\omega, \leqslant_*)$ for X linear wqo.
- $(\mathcal{P}_f(X), \sqsubseteq_H)$ for X wqo, where

$$U \sqsubseteq_H V \overset{\text{def}}{\Leftrightarrow} \forall x \in U : \exists y \in V : x \leqslant y$$

MORE WQO'S

- Finite Trees ordered by embeddings (Kruskal's Tree Theorem)



- Finite Graphs ordered by minor (Robertson-Seymour Theorem)

$$C_n \leqslant_{\text{minor}} K_n \text{ and } C_n \leqslant_{\text{minor}} C_{n+1}$$

- $(X^\omega, \leqslant_*)$ for X linear wqo.
- $(\mathcal{P}_f(X), \sqsubseteq_H)$ for X wqo, where

$$U \sqsubseteq_H V \overset{\text{def}}{\Leftrightarrow} \forall x \in U : \exists y \in V : x \leqslant y$$

Finite-basis characterization

Defn. $(X, \leqslant)$ is a wqo $\overset{\text{def}}{\Leftrightarrow}$ every non-empty subset V of X has at least one and at most finitely many (non-equivalent) minimal elements.

Say $V \subseteq X$ is upward-closed if $x \geqslant y \in V$ implies $x \in V$. (There is a similar notion of downward-closed sets).

For $B \subseteq X$, the upward-closure $\uparrow B$ of B is $\{x \mid x \geqslant b \text{ for some } b \in B\}$. Note that $\uparrow(\bigcup_i B_i) = \bigcup_i \uparrow B_i$, and that V is upward-closed iff $V = \uparrow V$.

Cor1. Any upward-closed $U \subseteq X$ has a finite basis, i.e., U is some $\uparrow\{m_1, \ldots, m_k\}$.

Cor2. Any downward-closed $V \subseteq X$ can be defined by a finite set of excluded minors:

$$x \in V \Leftrightarrow m_1 \not\leqslant x \wedge \cdots \wedge m_k \not\leqslant x$$

FINITE-BASIS CHARACTERIZATION

Defn. $(X, \leqslant)$ is a wqo $\overset{\text{def}}{\Leftrightarrow}$ every non-empty subset V of X has at least one and at most finitely many (non-equivalent) minimal elements.

Say $V \subseteq X$ is upward-closed if $x \geqslant y \in V$ implies $x \in V$. (There is a similar notion of downward-closed sets).

For $B \subseteq X$, the upward-closure $\uparrow B$ of B is $\{x \mid x \geqslant b \text{ for some } b \in B\}$. Note that $\uparrow(\bigcup_i B_i) = \bigcup_i \uparrow B_i$, and that V is upward-closed iff $V = \uparrow V$.

Cor1. Any upward-closed $U \subseteq X$ has a finite basis, i.e., U is some $\uparrow\{m_1, \ldots, m_k\}$.

Cor2. Any downward-closed $V \subseteq X$ can be defined by a finite set of excluded minors:

$$x \in V \Leftrightarrow m_1 \not\leqslant x \wedge \cdots \wedge m_k \not\leqslant x$$

FINITE-BASIS CHARACTERIZATION

Defn. $(X, \leqslant)$ is a wqo $\overset{\text{def}}{\Leftrightarrow}$ every non-empty subset V of X has at least one and at most finitely many (non-equivalent) minimal elements.

Say $V \subseteq X$ is upward-closed if $x \geqslant y \in V$ implies $x \in V$. (There is a similar notion of downward-closed sets).

For $B \subseteq X$, the upward-closure $\uparrow B$ of B is $\{x \mid x \geqslant b$ for some $b \in B\}$. Note that $\uparrow(\bigcup_i B_i) = \bigcup_i \uparrow B_i$, and that V is upward-closed iff $V = \uparrow V$.

Cor1. Any upward-closed $U \subseteq X$ has a finite basis, i.e., U is some $\uparrow\{m_1, \ldots, m_k\}$.

Cor2. Any downward-closed $V \subseteq X$ can be defined by a finite set of excluded minors:

$$x \in V \Leftrightarrow m_1 \not\leqslant x \wedge \cdots \wedge m_k \not\leqslant x$$

E.g, Kuratowksi Theorem: a graph is planar iff it does not contain K_5 or $K_{3,3}$.

Gives polynomial-time characterization of closed sets.

Finite-basis characterization

Defn. $(X, \leqslant)$ is a wqo $\overset{\text{def}}{\Leftrightarrow}$ every non-empty subset V of X has at least one and at most finitely many (non-equivalent) minimal elements.

Say $V \subseteq X$ is upward-closed if $x \geqslant y \in V$ implies $x \in V$. (There is a similar notion of downward-closed sets).

For $B \subseteq X$, the upward-closure $\uparrow B$ of B is $\{x \mid x \geqslant b \text{ for some } b \in B\}$. Note that $\uparrow(\bigcup_i B_i) = \bigcup_i \uparrow B_i$, and that V is upward-closed iff $V = \uparrow V$.

Cor1. Any upward-closed $U \subseteq X$ has a finite basis, i.e., U is some $\uparrow\{m_1, \ldots, m_k\}$.

Cor2. Any downward-closed $V \subseteq X$ can be defined by a finite set of excluded minors:

$$x \in V \Leftrightarrow m_1 \not\leqslant x \wedge \cdots \wedge m_k \not\leqslant x$$

Cor3. Any sequence $\uparrow V_0 \subseteq \uparrow V_1 \subseteq \uparrow V_2 \subseteq \cdots$ of upward-closed subsets converges in finite-time: $\exists m : (\bigcup_i \uparrow V_i) = \uparrow V_m = \uparrow V_{m+1} = \ldots$

Beyond wqo's

For $(X, \leqslant)$, we consider $(\mathcal{P}(X), \sqsubseteq_S)$ defined with

$$U \sqsubseteq_S V \stackrel{\text{def}}{\Leftrightarrow} \forall y \in V : \exists x \in U : x \leqslant y \qquad (\stackrel{\text{def}}{\Leftrightarrow} \uparrow U \supseteq \uparrow V)$$

Fact. $\mathcal{P}(X)$ is well-founded iff X is wqo —Defn'

NB. X well-founded $\not\Rightarrow$ $\mathcal{P}(X)$ well-founded
Question. Does X wqo $\Rightarrow$ $\mathcal{P}(X)$ wqo? (Equivalently $\mathcal{P}_f(X)$ wqo?)

Beyond wqo's

For $(X, \leqslant)$, we consider $(\mathcal{P}(X), \sqsubseteq_S)$ defined with

$$U \sqsubseteq_S V \overset{\text{def}}{\Leftrightarrow} \forall y \in V : \exists x \in U : x \leqslant y \qquad (\overset{\text{def}}{\Leftrightarrow} \uparrow U \supseteq \uparrow V)$$

Fact. $\mathcal{P}(X)$ is well-founded iff X is wqo —Defn$'$

NB. X well-founded $\not\Rightarrow$ $\mathcal{P}(X)$ well-founded
Question. Does X wqo $\Rightarrow$ $\mathcal{P}(X)$ wqo? (Equivalently $\mathcal{P}_f(X)$ wqo?)

Beyond wqo's

For $(X, \leqslant)$, we consider $(\mathcal{P}(X), \sqsubseteq_S)$ defined with

$$U \sqsubseteq_S V \overset{\text{def}}{\Leftrightarrow} \forall y \in V : \exists x \in U : x \leqslant y \qquad (\overset{\text{def}}{\Leftrightarrow} \uparrow U \supseteq \uparrow V)$$

Fact. $\mathcal{P}(X)$ is well-founded iff X is wqo —Defn$'$

NB. X well-founded $\not\Rightarrow \mathcal{P}(X)$ well-founded
Question. Does X wqo $\Rightarrow \mathcal{P}(X)$ wqo? (Equivalently $\mathcal{P}_f(X)$ wqo?)

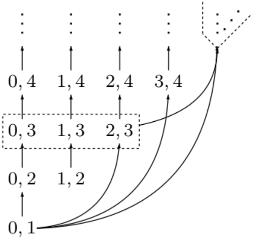
Beyond wqo's

For $(X,\leqslant)$, we consider $(\mathcal{P}(X),\sqsubseteq_S)$ defined with

$$U \sqsubseteq_S V \overset{\text{def}}{\Leftrightarrow} \forall y \in V : \exists x \in U : x \leqslant y \qquad (\overset{\text{def}}{\Leftrightarrow} \uparrow U \supseteq \uparrow V)$$

Fact. $\mathcal{P}(X)$ is well-founded iff X is wqo —Defn′

NB. X well-founded $\not\Rightarrow$ $\mathcal{P}(X)$ well-founded
Question. Does X wqo $\Rightarrow$ $\mathcal{P}(X)$ wqo? (Equivalently $\mathcal{P}_f(X)$ wqo?)



$$X \overset{\text{def}}{=} \{(a,b) \in \mathbb{N}^2 \mid a < b\}$$

$$(a,b) < (a',b') \overset{\text{def}}{\Leftrightarrow} \begin{cases} a = a' \text{ and } b < b' \\ \text{or } b < a' \end{cases}$$

Fact. $(X,\leqslant)$ is WQO

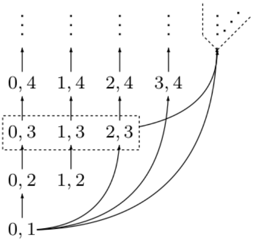
BEYOND WQO'S

For $(X, \leqslant)$, we consider $(\mathcal{P}(X), \sqsubseteq_S)$ defined with

$$U \sqsubseteq_S V \overset{\text{def}}{\Leftrightarrow} \forall y \in V : \exists x \in U : x \leqslant y \qquad (\overset{\text{def}}{\Leftrightarrow} \uparrow U \supseteq \uparrow V)$$

Fact. $\mathcal{P}(X)$ is well-founded iff X is wqo —Defn′

NB. X well-founded $\not\Rightarrow$ $\mathcal{P}(X)$ well-founded
Question. Does X wqo $\Rightarrow$ $\mathcal{P}(X)$ wqo? (Equivalently $\mathcal{P}_f(X)$ wqo?)



$$X \overset{\text{def}}{=} \{(a,b) \in \mathbb{N}^2 \mid a < b\}$$

$$(a,b) < (a',b') \overset{\text{def}}{\Leftrightarrow} \begin{cases} a = a' \text{ and } b < b' \\ \text{or } b < a' \end{cases}$$

Fact. $(X, \leqslant)$ is WQO

Thm. 1. $(\mathcal{P}_f(X), \sqsubseteq_S)$ is not wqo: rows are incomparable
2. $(\mathcal{P}(Y), \sqsubseteq_S)$ is wqo iff Y does not contain X