Examen final du cours (Calculabilité &) Complexité (L3)

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Date : 13 janv. 2019 à 10h30 / Durée : 2 heures

Note: when you give a reduction $r : L \to L'$ between two problems/languages, be sure to define the value of $r(x)$ in all cases, to argue the correctness of the reduction, and to argue its complexity.

Exercise 1: Closure properties of NP

1. Among the following four statements, say which are true and which are false (justify your answers):
   1. If L_1 and L_2 are in NP then also $L_1 \cap L_2$ is.
   2. If L_1 and L_2 are in NP then also $L_1 \cup L_2$ is.
   3. If L_1 and L_2 are NP-hard then also $L_1 \cap L_2$ is.
   4. If L_1 and L_2 are NP-hard then also $L_1 \cup L_2$ is.

Solution:

The first two statements are correct (easy proof omitted) while the last two are incorrect. For example, if L_1 is a NP-hard problem written with the alphabet $A = \{a, b, \ldots\}$, we can consider a disjoint copy $A' = \{a', b', \ldots\}$ of A and define L_2 as L_1 where $a, b, ..$ are replaced with $a', b', \ldots$ correspondingly. Then L_2 is NP-hard too but $L_1 \cap L_2 = \emptyset$ is not. If we now take $L_1' = L_1 \cup A^* A'(A \cup A')^*$ and $L_2' = L_2 \cup A'^* A(A \cup A')^*$, we again have two NP-hard languages. Now $L_1' \cup L_2' \cup \{\epsilon\}$ is $(A \cup A')^*$ hence is not NP-hard.

Exercise 2: A PSPACE-complete problem

2. Show that the following problem is PSPACE-complete:

   Input: A Turing Machine $\mathcal{M}$, a word x, a number k written in unary.

   Question: Does $\mathcal{M}$ accepts x using workspace at most k?

Solution:

Let us call the problem at hand PMk. The size of its input is $n = |\mathcal{M}| + |x| + k$.

PMk is in PSPACE: it is enough to simulate $\mathcal{M}$ on x while maintaining two counters: one checking that the simulation does not use more than k cells in the auxiliary memory, and one checking that $\mathcal{M}$ does not perform more than $|Q| \cdot \log |x| \cdot |\Sigma|^k$ steps, hence does not

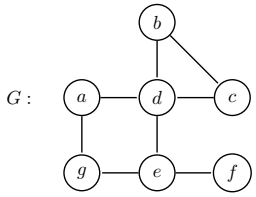
loop (this second check guarantees that the simulation terminates). The amount of memory used by this simulation is $k + \log k + \log |Q| + \log \log |x| + k \log |\Sigma|$, hence is in $O(n)$.

PMk is PSPACE-hard: If L is any language in PSPACE there is a TM $\mathcal{M}_L$ and a polynomial p_L such that $\mathcal{M}_L$ decides L using space $p_L(|x|)$. Thus $r_L : x \mapsto \mathcal{M}_L, x, 1^{p_L(|x|)}$ is a reduction proving $L \leq$ PMk since outputting $\mathcal{M}_L$ (fixed) and copying x is easy, while constructing $1^{p_L(|x|)}$ from x can be done in logspace.

Problem 3: Stable subsets of graphs

When $G = (V, E)$ is an undirected graph, a subset of vertices $X \subseteq V$ is called a *stable set* if no edge $u \in E$ has both extremities in X. In other words, if the subgraph G/X obtained by restricting G to vertices from X has no edge at all. (A stable set is sometimes called an *independent set*, or an *anticlique*.)

For example, $X = \{a, b, f\}$ is a stable of the graph G below.



The Stable problem is, given an undirected graph $G = (V, E)$ and an integer $k \in \mathbb{N}$ as inputs, to decide whether G has a stable subset of cardinal at least k. We assume that k is written in unary, so that the size n of the input is in $O(|V|^2 + k)$.

3. Show that Stable is in NP.

Solution:

A nondeterministic algorithm is to guess a subset X of V and check its size and stability. Obviously, X is polynomial-sized and checking that it is stable and has size $\geq k$ can be done in polynomial time.

4. Show that it is possible to solve Stable in time $n^{O(k)}$, i.e., in polynomial time *when k is fixed*.

Solution:

A possible algorithm is to enumerate all subsets of V having size k and check each of them for stability. There are $\binom{k}{|V|}$ such subsets, which is $\leq |V|^k$. Since checking some X takes polynomial time $|V|^{O(1)}$, the whole procedure is in $|V|^{k+O(1)}$, which is in $n^{O(k)}$.

5. Let $\text{Stable}_{\text{bin}}$ be the variant of Stable where k is given in binary, so that the size n of an input is in $O(|V|^2 + \log k)$. Show that Stable and $\text{Stable}_{\text{bin}}$ are logspace inter-reducible (i.e., there exist reductions in both directions).

Solution:

Reducing Stable to $\text{Stable}_{\text{bin}}$ just requires replacing a string 1^k with a string denoting k in binary. This can be done in logspace by maintaining an auxiliary counter (in binary) that

is incremented each time we read one "1" symbol in the unary input. When all the input has been read, the counter contains the binary representation of k and has size $\log k$.

Reducing $\mathsf{Stable}_{\text{bin}}$ to Stable is almost as simple. If $k \leq |V|$, we replace the binary representation of the input k with a string $\mathtt{1}^k$: the reduction is logspace in this case since $|V| \leq n$. If $k > |V|$, it may be impossible to produce $\mathtt{1}^k$ using only logspace but since no stable can have size $> |V|$, the reduction just has to produce some fixed negative Stable instance (or replace k with $\mathtt{1}^{|V|+1}$).

6. Show that 3SAT $\leq$ Stable. Deduce the complexity of Stable.

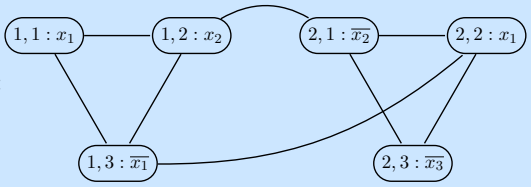
Solution:

With a 3SAT instance $\phi = C_1 \wedge C_2 \wedge \cdots \wedge C_m$ where each clause C_i is a disjunction $l_{i,1} \vee l_{i,2} \vee l_{i,3}$ of three literals over some set $V = \{x_1, \ldots, x_p\}$ of variables, we associate a Stable instance G_ϕ, k_ϕ with $k_\phi \stackrel{\text{def}}{=} m$ and $G_\phi \stackrel{\text{def}}{=} (V, E_1 \cup E_2)$ defined as follows:

$$
\begin{aligned}
V &\stackrel{\text{def}}{=} \left\{ (i,j) \;\middle|\; i = 1,2,\ldots,m \wedge j = 1,2,3 \right\}, \\
E_1 &\stackrel{\text{def}}{=} \left\{ \{(i,j),(i,j')\} \;\middle|\; 1 \leq i \leq m \wedge 1 \leq j < j' \leq 3 \right\}, \\
E_2 &\stackrel{\text{def}}{=} \left\{ \{(i,j),(i',j')\} \;\middle|\; 1 \leq i,i' \leq m \wedge l_{i,j} = \neg l_{i',j'} \right\}.
\end{aligned}
$$

The reduction is illustrated on the following example, where edges are drawn straight when from E_1 and bent when from E_2 (unless already in E_1):

$\phi: \; x_1 \vee x_2 \vee \overline{x_1} \bigwedge \overline{x_2} \vee x_1 \vee \overline{x_3} \quad \mapsto \quad G_\phi:$



Let us prove that the reduction is correct: If ϕ is satisfiable, say via some valuation v, we select in each clause C_i one literal l_{i,r_i} satisfied by v and let $X = \{(i, r_i) \mid i = 1, \ldots, m\}$. Two vertices in X cannot be connected by an E_1 edge since they come from different triangles. And they cannot be connected by an E_2 edge since E_2 connect vertices with opposite literals while X contains vertices with literals validated by a common v. Hence X is stable.

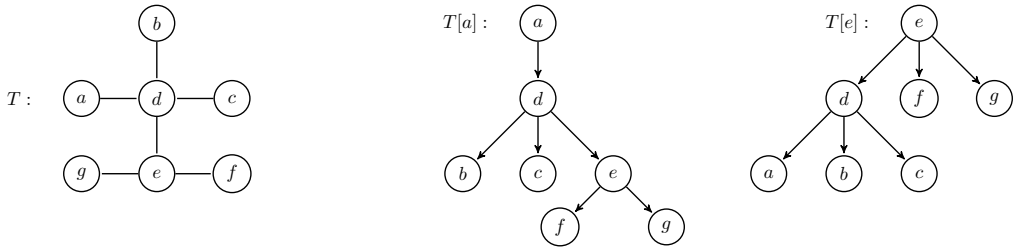
Reciprocally, if G_ϕ has a stable X of size m, then X must contain one vertex per triangle, hence one literal per clause. Now there is a valuation that validates all these literals (hence ϕ is satisfiable) since X cannot contain vertices with opposite literals (or it would not be stable w.r.t. E_2 edges).

We conclude by observing that the reduction is clearly logspace.

Now, with question 3 and since 3SAT is NP-complete, we deduce that Stable is NP-complete.

We are interested in $\mathsf{Stable}_{\text{tree}}$, the restriction of Stable to graphs that are trees. Recall that a *tree* is an undirected graph such that there is one and only one simple path between any two vertices $s, t \in V$. (A simple path between s and t is a list of pairwise distinct vertices $v_0, \ldots, v_\ell$ with $v_0 = s$, $v_\ell = t$, and such that for all $0 \leq i < \ell$, $\{v_i, v_{i+1}\}$ is an edge.)

A tree can be transformed into a *rooted directed tree* (a data structure commonly used in programs) by choosing any vertex as the root and by orienting every edge away from the root. Below we show a tree T and the two rooted trees obtained by choosing a or e as the root.



In a rooted directed tree with an edge $u \to v$, we say that v is *a child* of u, and u is *the parent* of v. A vertex is a *leaf* if it is has no children.

7. Show that if in a rooted directed tree T there is a stable set of size k, then there is a stable set that contains all the leaves and that has size at least k.

Solution:

Let $T = (V, E)$ be a rooted tree and X be a stable of size k in T. If u is a leaf in T that does not appear in X, we modify X by adding u and removing the father of u (if u has one). The new X is still stable (a leaf is only connected to its father) and its cardinality has not decreased. Repeating this process as long as some leaf is not in X, we finally obtain a stable of size $\geq k$ that contains all leaves.

8. Using the previous question, give an algorithm deciding $\mathsf{Stable}_{\text{tree}}$ in polynomial-time. Justify its correctness and your complexity analysis.

Solution:

Our algorithm first checks that its input G is a tree, and transforms it into a rooted tree T (easily done in polynomial-time). Then is uses the following recursive function for computing a stable set of T of maximum size and accepts if $|X_{\max}| \geq k$.

function `MaxStable` (T):
  `if` T is empty `then return` $\emptyset$
  `else let` $X_1 =$ all leaves of T
    `let` $T_1 = T$ without its leaves and their parents
    `in return` $X_1 \cup$ `MaxStable`(T_1)

The algorithm is polynomial-time since each recursive call is on a strictly smaller tree and since computing X_1, T_1 and building $X_1 \cup \cdots$ is polynomial-time. To prove correctness, we note that there is a stable set of T of maximum size that contains X_1 (by the previous question). Since X_1 joined with any stable of T_1 is a stable of T, and since if X is a stable of T that contains X_1, then $X \setminus X_1$ is a stable of T_1, one deduces (proof by induction on T) that X_1 joined with a maximum-sized stable of T_1 is a maximum-sized stable of T.