Exercise 1 Space hierarchy theorem

Consider two space-constructible functions $f$ and $g$ such that $f(n) = o(g(n))$. Prove that $\text{DSPACE}(f) \subseteq \text{DSPACE}(g)$.

Hint: You may consider a language $L = \{(M, w') \mid \text{the simulation of } M \text{ on } (M, w') \text{ rejects}\}$ with an appropriate restriction on the simulation of $M$.

Solution:
First, we have $\text{DSPACE}(f) \subseteq \text{DSPACE}(g)$ since $f(n) \leq g(n)$ for a high enough $n$. Let us show that this inclusion is strict.

We define the following language:

$L = \{(M, w') \mid \text{the simulation of } M \text{ on } (M, w') \text{ rejects using space } \leq g(|M, w'|)\}$

- First, we show that $L \in \text{SPACE}(g)$. We describe the steps taken by a Turing machine $M'$ on an input $w = M, w'$. $M'$ first computes $g(|w|)$ (which can be done in space $O(g(|w|))$ since $g$ is space constructible) and marks down an end of tape marker at position $g(|w|)$ on the work tape: if more space is used, $M'$ rejects. Then, $M'$ simulates $M$ on $w$ by rejecting if the number of steps taken is bigger than $|Q_M| \cdot g(|w|)^{k_M} \cdot |\Gamma_M|^{|\Gamma_M| g(|w|)}$ (where $Q_M$ is the set of states, $\Gamma_M$ is the alphabet and $k_M$ is the number of working tapes of the Turing machine $M$). Then, if $w$ is accepted by $M$, $M'$ rejects, otherwise $M'$ accepts. Then, this Turing Machine $M$ accepts the language $L$ and runs in space $O(g(|w|))$. We conclude by using the speed-up theorem.

- Second, we show that $L \notin \text{SPACE}(f)$. Let us assume towards a contradiction that there is a machine $M'$ recognizing $L$ in space $f$. Simulating $M'$ on an input $w$ takes space in $O(f(|w|)) = c \times f(|w|)$ where the constant $c$ only depends on the Turing Machine $M$ (its number of states, size of alphabet, number of work tapes). For a sufficiently long $w'$, we have $c \times f(|M', w'|) \leq g(|M', w'|)$. Then, if $(M', w') \in L$, the simulation of $M'$, and therefore $M'$ rejects $(M', w')$. However, since $M'$ accepts $L$, $M'$ also accepts $(M', w')$. Hence the contradiction. Let us now assume that $(M', w') \notin L$. Since the space used by the simulation of $M'$ is $c \times f(|M', w'|) \leq g(|M', w'|)$, we can conclude that $M'$ accepts $(M', w')$ by definition of $L$. But then, since the language $L$ is accepted by $M'$, we should have $(M', w') \in L$. Hence the contradiction. In fact, there is no such Turing Machine $M'$.

Exercise 2 Polylogarithmic space
1. Let \( \text{polyL} = \bigcup_{k \in \mathbb{N}} \text{SPACE}(\log^k) \). Show that \( \text{polyL} \) does not have a complete problem for logarithmic space reduction.\(^1\)

2. We recall that \( \text{PSPACE} = \bigcup_{k \in \mathbb{N}} \text{SPACE}(n^k) \). Does \( \text{PSPACE} \) have a complete problem for logarithmic space reduction? Why doesn’t the proof of the previous question apply to \( \text{PSPACE} \)?

Solution:

1. Assume towards a contradiction that there exists a \( \text{polyL} \)-complete problem \( L \) for logspace reduction. Then, there exists \( k \in \mathbb{N} \) such that \( L \in \text{SPACE}(\log^k) \). Let us show that \( \text{SPACE}(\log^k) = \text{SPACE}(\log^{k+1}) \), which is a contradiction with the space hierarchy theorem. Let \( L' \in \text{SPACE}(\log^{k+1}) \subseteq \text{polyL} \). There exists a reduction \( f \) of \( L' \) to \( L \) that can be computed in logarithmic space since \( L \) is \( \text{polyL} \)-complete. Now, consider a Turing machine that, on an input \( w \), computes \( f(w) \) in logarithmic space and then simulates a Turing machine deciding \( L \) that runs in space \( \log^k \) on \( f(w) \). Note that here, it is important not store \( f(w) \) on a working tape as this could make the space used exceed the \( \log^k \) space bound. Instead, one must use a virtual tape where we only compute bits of \( f(w) \) when they are needed without remembering the whole computation. Then, note that \( |f(w)| = O(|w|^c) \) for some \( c \geq 0 \). Hence, the space used to check if \( f(w) \) is in \( L \) is lower than \( \log^k(|f(w)|) \) hence is in \( c^k \cdot \log^k(O(|w|)) = O(\log^k(|w|)) \). We conclude with the speed-up theorem to get that \( L' \in \text{SPACE}(\log^k) \). We get \( \text{SPACE}(\log^k) = \text{SPACE}(\log^{k+1}) \) which is in contradiction with the space hierarchy theorem. Hence \( L \) cannot exist.

2. \( \text{PSPACE} \) does have complete problems for logarithmic space reductions (such as \( \text{TQBF} \)). However, if we try to apply the previous proof to establish that \( \text{SPACE}(n^k) = \text{SPACE}(n^{k+1}) \), a problem arises: since \( |f(w)| \) is in \( O(|w|^c) \), we have \( |f(w)|^k \) in \( O(|w|^{c+k}) \neq O(|w|^k) \) if \( c > 1 \).

Exercise 3 Padding argument

1. Show that if \( \text{DSPACE}(n^c) \subseteq \text{NP} \) for some \( c > 0 \), then \( \text{PSPACE} \subseteq \text{NP} \).

   Hint: for \( L \in \text{DSPACE}(n^k) \) one may consider the language \( \tilde{L} = \{(x, 1|x|^{k/c}) \mid x \in L \} \).

2. Deduce that \( \text{DSPACE}(n^c) \neq \text{NP} \).

Solution:

1. Assume \( \text{DSPACE}(n^c) \subseteq \text{NP} \) and consider any \( L \in \text{PSPACE} \): we have to prove \( L \in \text{NP} \). For some \( k \), we have \( L \in \text{DSPACE}(n^k) \). Let \( M \) be a Turing Machine deciding \( L \) in space \( n^k \). Now, consider the language \( \tilde{L} = \{(x, 1|x|^k) \mid x \in L \} \) and consider the Turing machine \( \tilde{M} \) that, on an input \( w \), checks that it has the form \( w = (x, 1^\ell) \), verifies that \( \ell = |x|^k \), and if so launches a simulation of \( M \) on \( x \). Note that computing \( |x|^k \) only uses \( k/c \) nested loops going from 1 to \( |x| \), which can be done in logspace since \( k/c \) is a “constant” that depends on \( M \), not \( x \). Then, \( \tilde{M} \) accepts \( \tilde{L} \) and the space used by \( \tilde{M} \) is in \( |x|^k = |1|x|^k| \leq |w|^c \). Hence, \( \tilde{M} \) accepts \( \tilde{L} \) and the space used by \( \tilde{M} \) is in \( |x|^k = |1|x|^k| \leq |w|^c \). Hence,

\(^1\)From this, we can deduce that \( \text{polyL} \neq \text{P} \).
\( \tilde{L} \in \text{DSPACe}(n^c) \subseteq \text{NP}. \) Thus \( \tilde{L} \in \text{NP} \). As we can reduce \( L \) to \( \tilde{L} \) by transforming \( x \) into \((x, 1^{\lceil \varepsilon k / c \rceil})\) in logspace, we do have that \( L \in \text{NP} \).

2. Assume \( \text{DSPACe}(n^c) = \text{NP} \), then \( \text{DSPACe}(n^{c+1}) \subseteq \text{PSPACE} = \text{NP} = \text{DSPACe}(n^c) \) which is in contradiction with the space hierarchy theorem.