Complexité avancée - TD 3

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Exercise 1 Space hierarchy theorem

Consider two space-constructible functions f and g such that $f(n) = o(g(n))$. Prove that $\mathsf{DSPACE}(f) \subsetneq \mathsf{DSPACE}(g)$.

Hint: You may consider a language $L = \{(M, w') \mid \text{ the simulation of } M \text{ on } (M, w') \text{ rejects } \}$ *with an appropriate restriction on the simulation of* M.

Solution:

First, we have $\mathsf{DSPACE}(f) \subseteq \mathsf{DSPACE}(g)$ since $f(n) \leq g(n)$ for a high enough n. Let us show that this inclusion is strict.

We define the following language:

$$L = \{(M, w') \mid \text{ the simulation of } M \text{ on } (M, w') \text{ rejects using space } \leq g(|M, w'|)\}$$

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- First, we show that $L \in \mathsf{SPACE}(g)$. We describe the steps taken by a Turing machine M' on an input $w = M, w'$. M' first computes $g(|w|)$ (which can be done in space $O(g(|w|))$ since g is space constructible) and marks down an end of tape marker at position $g(|w|)$ on the work tape: if more space is used, M' rejects. Then, M' simulates M on w by rejecting if the number of steps taken is bigger than $|Q_M| \cdot g(|w|)^{k_M} \cdot |\Gamma_M|^{k_M \cdot g(|w|)}$ (where Q_M is the set of states, Γ_M is the alphabet and k_M is the number of working tapes of the Turing machine M). Then, if w is accepted by M, M' rejects, otherwise M' accepts. Then, this Turing Machine M accepts the language L and runs in space $O(g(|w|))$. We conclude by using the speed-up theorem.

- Second, we show that $L \notin \mathsf{SPACE}(f)$. Let us assume towards a contradiction that there is a machine M' recognizing L in space f. Simulating M' on an input w takes space in $O(f(|w|)) = c \times f(|w|)$ where the constant c only depends on the Turing Machine M (its number of states, size of alphabet, number of work tapes). For a sufficiently long w', we have $c \times f(|M', w'|) \leq g(|M', w'|)$. Then, if $(M', w') \in L$, the simulation of M', and therefore M' rejects (M', w'). However, since M' accepts L, M' also accepts (M', w'). Hence the contradiction. Let us now assume that $(M', w') \notin L$. Since the space used by the simulation of M' is $c \times f(|M', w'|) \leq g(|M', w'|)$, we can conclude that M' accepts (M', w') by definition of L. But then, since the language L is accepted by M', we should have $(M', w') \in L$. Hence the contradiction. In fact, there is no such Turing Machine M'.

Exercise 2 Polylogarithmic space

1. Let $\mathsf{polyL} = \cup_{k\in\mathbb{N}}\mathsf{SPACE}(\log^k)$. Show that polyL does not have a complete problem for logarithmic space reduction.[1]
2. We recall that $\mathsf{PSPACE} = \cup_{k\in\mathbb{N}}\mathsf{SPACE}(n^k)$. Does PSPACE have a complete problem for logarithmic space reduction ? Why doesn't the proof of the previous question apply to PSPACE?

Solution:

1. Assume towards a contradiction that there exists a polyL-complete problem L for logspace reduction. Then, there exists $k \in \mathbb{N}$ such that $L \in \mathsf{SPACE}(\log^k)$. Let us show that $\mathsf{SPACE}(\log^k) = \mathsf{SPACE}(\log^{k+1})$, which is a contradiction with the space hierarchy theorem. Let $L' \in \mathsf{SPACE}(\log^{k+1}) \subseteq \mathsf{polyL}$. There exists a reduction f of L' to L that can be computed in logarithmic space since L is polyL-complete. Now, consider a Turing machine that, on an input w, computes $f(w)$ in logarithmic space and then simulates a Turing machine deciding L that runs in space $\log^k$ on $f(w)$. Note that here, it is important not store $f(w)$ on a working tape as this could make the space used exceed the $\log^k$ space bound. Instead, one must use a virtual tape where we only compute bits of $f(w)$ when they are needed without remembering the whole computation. Then, note that $|f(w)| = O(|w|^c)$ for some $c \geq 0$. Hence, the space used to check if $f(w)$ is in L is lower than $\log^k(|f(w)|)$ hence is in $c^k \cdot \log^k(O(|w|)) = O(\log^k(|w|))$. We conclude with the speed-up theorem to get that $L' \in \mathsf{SPACE}(\log^k)$. We get $\mathsf{SPACE}(\log^k) = \mathsf{SPACE}(\log^{k+1})$ which is in contradiction with the space hierarchy theorem. Hence L cannot exist.
2. PSPACE does have complete problems for logarithmic space reductions (such as TQBF). However, if we try to apply the previous proof to establish that $\mathsf{SPACE}(n^k) = \mathsf{SPACE}(n^{k+1})$, a problem arises: since $|f(w)|$ is in $O(|w|^c)$, we have $|f(w)|^k$ in $O(|w|^{c\cdot k}) \neq O(|w|^k)$ if $c > 1$.

Exercise 3 Padding argument

1. Show that if $\mathsf{DSPACE}(n^c) \subseteq \mathsf{NP}$ for some $c > 0$, then $\mathsf{PSPACE} \subseteq \mathsf{NP}$.

   Hint: for $\mathsf{L} \in \mathsf{DSPACE}(n^k)$ *one may consider the language* $\tilde{L} = \{(x, 1^{|x|^{k/c}}) \mid x \in L\}$.
2. Deduce that $\mathsf{DSPACE}(n^c) \neq \mathsf{NP}$.

Solution:

1. Assume $\mathsf{DSPACE}(n^c) \subseteq \mathsf{NP}$ and consider any $L \in \mathsf{PSPACE}$: we have to prove $L \in \mathsf{NP}$. For some k, we have $L \in \mathsf{DSPACE}(n^k)$. Let M be a Turing Machine deciding L in space n^k. Now, consider the language $\tilde{L} = \{(x, 1^{|x|^{k/c}}) \mid x \in L\}$ and consider the Turing machine $\tilde{M}$ that, on an input w, checks that it has the form $w = (x, 1^\ell)$, verifies that $\ell = |x|^{k/c}$, and if so launches a simulation of M on x. Note that computing $|x|^{k/c}$ only uses k/c nested loops going from 1 to $|x|$, which can be done in logspace since k/c is a "constant" that depends on M, not x. Then, $\tilde{M}$ accepts $\tilde{L}$ and the space used by $\tilde{M}$ is in $|x|^k = |1^{|x|^{k/c}}|^c \leq |w|^c$. Hence,

[1] From this, we can deduce that $\mathsf{polyL} \neq \mathsf{P}$.

$\tilde{L} \in \mathsf{DSPACE}(n^c) \subseteq \mathsf{NP}$. Thus $\tilde{L} \in \mathsf{NP}$. As we can reduce L to $\tilde{L}$ by transforming x into $(x, 1^{|x|^{k/c}})$ in logspace, we do have that $L \in \mathsf{NP}$.

2. Assume $\mathsf{DSPACE}(n^c) = \mathsf{NP}$, then $\mathsf{DSPACE}(n^{c+1}) \subseteq \mathsf{PSPACE} = \mathsf{NP} = \mathsf{DSPACE}(n^c)$ which is in contradiction with the space hierarchy theorem.