Complexité avancée - Homework 4

Benjamin Bordais

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Closure under morphims Given a finite alphabet Σ, a function $f : \Sigma^* \to \Sigma^*$ is a morphism if $f(\Sigma) \subseteq \Sigma$ and for all $a = a_1 \cdots a_n \in \Sigma^*$, $f(a) = f(a_1) \cdots f(a_n)$ (f is uniquely determined by the value it takes on Σ).

Show that $\mathsf{P} = \mathsf{NP}$ if and only if P is closed under morphism.

Solution:

- Assume that $\mathsf{P} = \mathsf{NP}$. Consider f a morphism and $L \in \mathsf{P} = \mathsf{NP}$. Let us show that $f(L) \in \mathsf{NP} = \mathsf{P}$. We consider a non-deterministic Turing machine M that, on an input $w \in \Sigma^*$, guesses a word $a \in \Sigma^*$ such that $|a| = |w|$ and then checks that $f(a) = w$ and that $a \in L$ in polynomail time. It follows that $f(L) \in \mathsf{NP} = \mathsf{P}$ and P is closed under morphisms.

- Now, assume that P is closed under morphism. We show that $\mathsf{SAT} \in \mathsf{P}$, which proves that $\mathsf{NP} \subseteq \mathsf{P}$ since SAT is NP-complete for logspace reductions and P is closed under logarithmic space reductions. Consider the following language:

  $$L = \{(\phi, v) \mid v \text{ is a valuation satifying } \phi\}$$

  We have that $L \in \mathsf{P}$ as one can check in polynomial time that a valuation satisfies a boolean formula. Furthermore, we can assume that the alphabet Σ is equal to the disjoint union $\Sigma_\phi \uplus \Sigma_v$ and the symbols used to encode ϕ (resp. v) are in Σ_ϕ (resp. Σ_v). Then, if we consider the morphism f that ensures $f(a) = a$ for all $a \in \Sigma_\phi$ and $f(a) = 0$ for all $a \in \Sigma_v$. Then,

  $$f(L) = \{(\phi, 0^n) \mid \phi \text{ has n variables and is satistiable}\}$$

  By closure under morphism, it follows that $f(L) \in \mathsf{P}$. Since, an instance of SAT can be reduced in polynomial time (in fact, in logarithmic space) to an instance of $f(L)$, it follows that $\mathsf{SAT} \in \mathsf{P}$. Hence, $\mathsf{P} = \mathsf{NP}$.