Complexité avancée - Homework 2

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NL alternative definition

A Turing machine with *certificate tape*, called a verifier, is a deterministic Turing machine with an extra read-only input tape called *the certificate tape*, which moreover is *read once* (*i.e.* the head on that tape can either remain on the same cell or move right, but never move left). A verifier takes as input a word x, along with a word u written in the certificate tape.

Define NL_{certif} to be the class of languages L such that there exists a polynomial $p : \mathbb{N} \to \mathbb{N}$ and a verifier V running in logarithmic space such that:

$$x \in L \text{ iff } \exists u, |u| \leq p(|x|) \text{ and } V \text{ accepts on input } (x, u) .$$

1. Show that $\mathsf{NL}_{certif} = \mathsf{NL}$.

2. What complexity class do you obtain if you remove the read-once constraint in the definition of a machine with certification tape ? Justify your answer. You may use the fact that SAT is NP-complete for logspace reductions.

Solution:

1. $\mathsf{NL}_{certif} \subseteq \mathsf{NL}$: Let A be a language in NL_{certif}. If V is a verifier for A, we can simulate the run of V on an input x by guessing the next letter of u and accepting x if and only if V would have accepted on (x, u). This runs in logarithmic space. Indeed, V runs in logarithmic space and we only need to remember one letter of u at a time, because the certificate tape is read only once.

   $\mathsf{NL} \subseteq \mathsf{NL}_{certif}$: Let there be A in NL, and M a non deterministic Turing Machine running in logarithmic space and recognizing A. For a given input x, a run of M on x can be characterized by all the non deterministic choices done in its execution tree, which can be encoded in a word y over $\{0, 1\}^*$ with $|y| \leq p_M(|x|)$ for some polynomial function p_M bounding the running time of M (which exists since M runs in logarithmic space hence polynomial time). Note that this polynomial function p_M does not depend on the input, but only on the Turing Machine M (specifically, its number of states, the size of its alphabet, and its number of work tape). We can then construct a verifier V simulating M that consults its certificate tape each time M uses non determinism. The runs of V are deterministic and require logarithmic space.

2. We call the new class NL'_{certif}.
   $\mathsf{NL}'_{certif} \subseteq \mathsf{NP}$:. Let there be A in NL'_{certif} and consider a verifier V for A. We

build a non-deterministic Turing Machine M running in polynomial time accepting A. First, M guesses the certificate (which is polynomial in the size of the input), and stores it in a worktape since perhaps it will have to be read several times. And then, M runs V on (x, u), which takes polynomial time since a logspace verifier like V runs in polynomial time.

$\mathsf{NP} \subseteq \mathsf{NL}'_{certif}$: First, let us show that $\mathsf{SAT} \in \mathsf{NL}'_{certif}$. Consider a propositional formula φ. We build a verifier V that checks if the certificate given in the certification tape that gives a boolean valuation $(a_1, \ldots, a_n)$ of the variables satisfies the formula φ. Specifically, V loops over all the clauses and over the literals in these clauses. The truth value of a literal is given by the certificate tape, that V checks to see the value of the literal. If one literal is satisfied by the valuation, V goes to the next clause, otherwise another literal is checked. If no literal is satisfied, V rejects. If all clauses are satisfied, V accepts. This takes logarithmic space since we only need a pointer on the clause and literal we are considering. Note that the certificate tape is consulted each time a literal is evaluated (which would not be possible in the read-only-once setting).

Let us now show that NL'_{certif} is closed under logspace reduction, that is if $L \in \mathsf{NL}'_{certif}$ and L' reduces in logspace to L, then $L' \in \mathsf{NL}'_{certif}$. Let us denote by f the reduction from L' to L that can be computed in logarithmic space and by V a verifier accepting L. We build a verifier V' running in logspace accepting L'. On an input w, V' simulates the verifier V on $f(w)$. $f(w)$ can be cnstructed in logarithmic space, however, as in the proof of Lemma 2.11 in the course, it is not possible, a priori, to copy $f(w)$ on a work tape since the space taken may exceed the logarithmic space bound. Instead, each time a bit of $f(w)$ is required by the simulation of V, it is computed once again in logarithmic space. This ensures that the space used is in $O(\log(|w|))$. By using the same trick than for the speed-up theorem, we obtain a verifier V' running in logarithmic space accepting L'.

We conclude that $\mathsf{NP} \subseteq \mathsf{NL}'_{certif}$ by using the fact that SAT is NP-complete for logarithmic space reductions.