Complexité avancée - TD 3

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We recall the space-hierarchy theorem.

Theorem 1 (Space-hierarchy theorem)**.** *For two space-constructible functions f and g such that $f = o(g)$, we have $\mathsf{DSPACE}(f) \subsetneq \mathsf{DSPACE}(g)$.*

Exercise 1 (Poly-logarithmic space)**.**

1. *Let $\mathsf{polyL} = \cup_{k\in\mathbb{N}}\mathsf{SPACE}(\log^k)$. Show that polyL does not have a complete problem for logarithmic space reduction.*[1]

2. *Recall that $\mathsf{PSPACE} = \cup_{k\in\mathbb{N}}\mathsf{SPACE}(n^k)$. Does PSPACE have a complete problem for logarithmic space reduction ? Why doesn't the proof of the previous question apply to PSPACE?*

Exercise 2 (Padding argument)**.**

1. *Show that if $\mathsf{DSPACE}(n^c) \subseteq \mathsf{NP}$ for some $c > 0$, then $\mathsf{PSPACE} \subseteq \mathsf{NP}$.*

   Hint: for Ł $\in \mathsf{DSPACE}(n^k)$ one may consider the language $\tilde{L} = \{(x, w_x) \mid x \in L\}$. where w_x is a word written in unary.

2. *Deduce that $\mathsf{DSPACE}(n^c) \neq \mathsf{NP}$.*

Exercise 3 (On the existence of One-way function)**.** *A one-way function is a bijection f from k-bit integers to k-bit integers such that f is computable in polynomial time, but f^{-1} is not. Prove that for all one-way functions f, we have*

$$A := \{(x, y) \mid f^{-1}(x) < y\} \in (\mathsf{NP} \cap \mathsf{coNP})\backslash\mathsf{P}$$

Exercise 4 (Regular languages)**.** *Let REG denote the set regular/rational languages.*

1. *Show that for all $L \in \mathsf{REG}$, L is recognized by a TM running in space 0 and time $n + 1$.*[2]

2. *Exhibit a language recognized by a TM running in space $\log n$ and time $O(n)$ that is not in REG.*

Exercise 5 (Yet another NL-complete problem)**.** *For a finite set X, a subset $S \subseteq X$, and a binary operator $* : X \times X \to X$ defined on X, we inductively define $S_{0,*} := S$ and $S_{i+1,*} := S_{i,*} \cup \{x * y \mid x, y \in S_{i,*}\}$. The closure of S with regard to $*$ is the set $S_* = \cup_{i\in\mathbb{N}} S_{i,*}$.*

Show that the following problem is NL-complete.

- *Input: A finite set X, a binary operation $* : X \times X \to X$ that is associative (i.e. $(x * y) * z = x * (y * z)$ for all $x, y, z \in X$), a subset $S \subseteq X$ and a target $t \in X$.*

[1] Note that, from this, we can deduce that $\mathsf{polyL} \neq \mathsf{P}$.

[2] In fact, regular languages exactly correspond to languages that can be recognized in such a way.

- *Output: Yes if and only* $t \in S_*$.

Exercise 6 (Solving reachability games)**.** *A two player (turn-based) game is a directed graph* $G = (V, E)$ *where the set of vertices* $V = V_A \uplus V_B$ *is partitioned into vertices belonging to Player A (i.e.* V_A*) and vertices belonging to Player B (i.e.* V_B*) with a distinguished vertex* $v_0 \in V$ *that is the starting vertex. The graph is non-blocking in the sense that every vertex has a successor, i.e.* $\mathsf{Succ}(v) = \{v' \in V \mid (v, v') \in E\} \neq \emptyset$ *for all* $v \in V$*. A play then corresponds to a finite or infinite path* $\rho = v_0 \cdot v_1 \cdots \in V^* \cup V^\omega$ *with* v_0 *is the starting vertex. If the play is at a vertex* $v_i \in V_A$ *then it is Player A's turn to choose the next vertex* $v_{i+1} \in \mathsf{Succ}(v_i)$*, while it is Player B's turn if* $v_i \in V_B$*. A winning condition determines when a play is winning for Player A (we consider win/loose games, hence if Player A does not win, Player B does). A Player* $C \in \{A, B\}$ *has a winning strategy (or wins) from a vertex* $v \in V$ *if she can choose the next move in all vertices in* V_C *such that she wins in any play that starts in* v*.*

1. *Assume that the winning condition is a reachability objective: given a target set of states* $T \subseteq V$*, Player A wins if and only if a state in* T *is seen at some point. Show that deciding the winner of a reachability game from the vertex* $v_0 \in V$ *can be done in polynomial time.*

   Hint: construct inductively the set of vertices from which Player A can ensure to get closer to the target T (that is called the attractor of the set T).

2. *Consider some* $k \in \mathbb{N}$*. A* k*-generalized reachability condition is the following: given* k *target sets of states* $T_1, \ldots, T_k \subseteq V$*, Player A wins if and only if, for all* $1 \leq i \leq k$*, a state in* T_i *is seen at some point. Show that deciding the winner of a* k*-generalized reachability game from the vertex* $v_0 \in V$ *can be done in polynomial time.*