Algorithmic Aspects of WQO (Well-Quasi-Ordering) Theory

Part I: Basics of WQO Theory

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Lecture notes & exercises available at

`http://www.lsv.fr/~schmitz/teach/2016_esslli`

MOTIVATIONS FOR THE COURSE

- Well-quasi-orderings (wqo's) proved to be a powerful tool for decidability/termination in logic, AI, program verification, etc. *NB: they can be seen as a version of well-founded orderings with more flexibility*

- In program verification, wqo's are prominent in well-structured transition systems (WSTS's), a generic framework for infinite-state systems with good decidability properties.

- Analysing the complexity of wqo-based algorithms is still one of the dark arts . . .

- Purposes of these lectures = to disseminate the basic concepts and tools one uses for the wqo-based algorithms and their complexity analysis.

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Outline of the course

- (This) Lecture 1 = **Basics of WQO's.** Rather basic material: explaining and illustrating the definition of wqo's. Building new wqo's from simpler ones.

- Lecture 2 = **Algorithmic Applications of WQO's.** Well-Structured Transition Systems, Program Termination, Relevance Logic, etc.

- Lecture 3 = **Complexity Analysis for WQO's.** Fast-growing complexity, Hardy computations, Length function theorems.

- Lecture 4 = **Ideals of WQO's.** Basic concepts, Representations, Algorithms.

- Lecture 5 = **Application of Ideals.** Complete WSTS, Computation of downward-closures

(RECALLS) ORDERED SETS

Def. A non-empty $(X, \leqslant)$ is a quasi-ordering (qo) $\stackrel{\text{def}}{\Leftrightarrow}$ $\leqslant$ is a reflexive and transitive relation.

($\approx$ a partial ordering without requiring antisymmetry, technically simpler but essentially equivalent)

Examples. $(\mathbb{N}, \leqslant)$, also $(\mathbb{R}, \leqslant)$, $(\mathbb{N} \cup \{\omega\}, \leqslant)$, ...

divisibility: $(\mathbb{Z}, \_ \mid \_)$ where $x \mid y \stackrel{\text{def}}{\Leftrightarrow} \exists a : a.x = y$

tuples: $(\mathbb{N}^3, \leqslant_{\text{prod}})$, or simply $(\mathbb{N}^3, \leqslant_\times)$, where $(0,1,2) <_\times (10,1,5)$ and $(1,2,3) \#_\times (3,1,2)$.

words: $(\Sigma^*, \leqslant_{\text{pref}})$ for some alphabet $\Sigma = \{a, b, \ldots\}$ and $ab <_{\text{pref}} abba$.

$(\Sigma^*, \leqslant_{\text{lex}})$ with e.g. $abba \leqslant_{\text{lex}} abc$ (NB: this assumes Σ is linearly ordered: $a < b < c$)

$(\Sigma^*, \leqslant_{\text{subword}})$, or simply $(\Sigma^*, \leqslant_*)$, with $aba \leqslant_* b\underline{a}a\underline{b}ba\underline{a}$.

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Def. $(X, \leqslant)$ is linear if for any $x, y \in X$ either $x \leqslant y$ or $y \leqslant x$. (I.e., there is no $x \# y$.)

Def. $(X, \leqslant)$ is well-founded if there is no infinite strictly decreasing sequence $x_0 > x_1 > x_2 > \cdots$

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WELL-QUASI-ORDERING (WQO)

Def1. $(X,\leqslant)$ is a wqo $\overset{\text{def}}{\Leftrightarrow}$ any infinite sequence $x_0,x_1,x_2,\ldots$ contains an increasing pair: $x_i \leqslant x_j$ for some $i<j$.

Def2. $(X,\leqslant)$ is a wqo $\overset{\text{def}}{\Leftrightarrow}$ any infinite sequence $x_0,x_1,x_2,\ldots$ contains an infinite increasing subsequence: $x_{n_0} \leqslant x_{n_1} \leqslant x_{n_2} \leqslant \ldots$

Def3. $(X,\leqslant)$ is a wqo $\overset{\text{def}}{\Leftrightarrow}$ there is no infinite strictly decreasing sequence $x_0 > x_1 > x_2 > \ldots$ —i.e., $(X,\leqslant)$ is well-founded— and no infinite set $\{x_0,x_1,x_2,\ldots\}$ of mutually incomparable elements $x_i \# x_j$ when $i \neq j$ —we say "$(X,\leqslant)$ has no infinite antichain"—.

Fact. These three definitions are equivalent.
Clearly, Def2 $\Rightarrow$ Def1 and Def1 $\Rightarrow$ Def3 (think contrapositively). But the reverse implications are non-trivial.
Recall Infinite Ramsey Theorem: "Let X be some countably infinite set and colour the elements of $X^{(n)}$ (the subsets of X of size n) in c different colours. Then there exists some infinite subset M of X s.t. the size n subsets of M all have the same colour."

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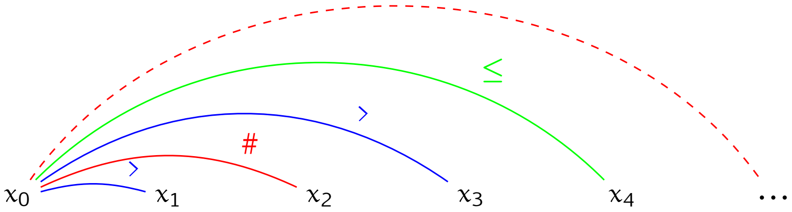
PROVING DEF3 $\Rightarrow$ DEF2

x_0 x_1 x_2 x_3 x_4 $\cdots$

Proving Def3 ⇒ Def2



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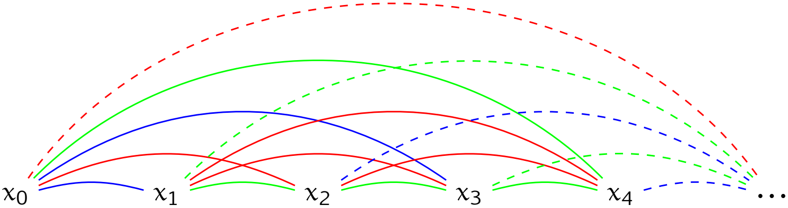

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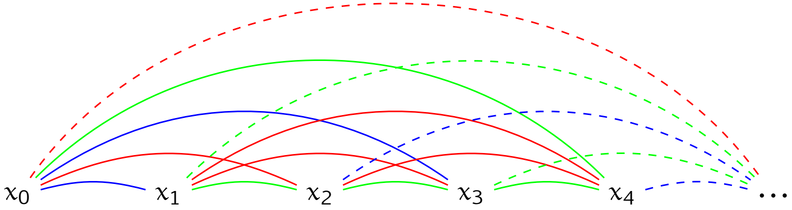

Infinite Ramsey Theorem:

there is an infinite subset $\{x_i\}_{i \in I}$ that is monochromatic

PROVING DEF3 $\Rightarrow$ DEF2

Infinite Ramsey Theorem:

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$$.. \quad x_{n_0} \quad .. \quad x_{n_1} \quad .. \quad x_{n_2} \quad .. \quad x_{n_3} \quad .. \quad x_{n_4} \quad .. \quad \ldots$$

What color?

Proving Def3 $\Rightarrow$ Def2

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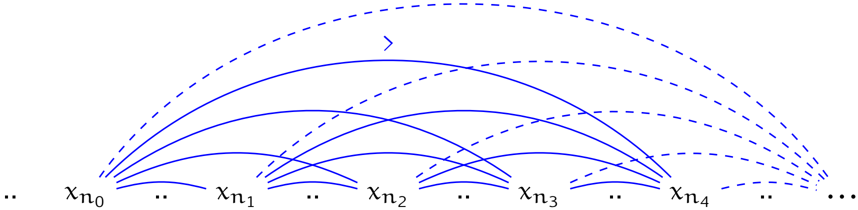

Blue $\Rightarrow$ infinite strictly decreasing sequence, contradicts WF

PROVING DEF3 ⇒ DEF2

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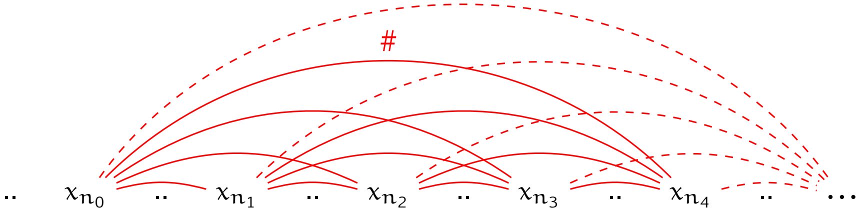

Red ⇒ infinite antichain, contradicts FAC

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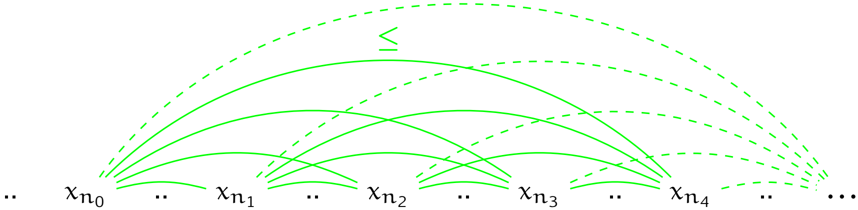

Must be green ⇒ infinite increasing sequence! QED

Spot the wqo's

	linear?	well-founded?	wqo?
$\mathbb{N}, \leqslant$	✓	✓	
$\mathbb{Z}, \vert$	×	✓	
$\mathbb{N} \cup \{\omega\}, \leqslant$	✓	✓	
$\mathbb{N}^3, \leqslant_\times$	×	✓	
$\Sigma^*, \leqslant_{\text{pref}}$	×	✓	
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More generally

Fact. For linear qo's: well-founded $\Leftrightarrow$ wqo.

Cor. Any ordinal is wqo.

SPOT THE WQO'S

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$(\mathbb{Z}, \mid)$: The prime numbers $\{2, 3, 5, 7, 11, \ldots\}$ are an infinite antichain.

SPOT THE WQO'S

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More generally

(Generalized) Dickson's lemma. If $(X_1, \leqslant_1), \ldots, (X_n, \leqslant_n)$'s are wqo's, then $\prod_{i=1}^n X_i, \leqslant_\times$ is wqo.

Proof. Easy with Def2. Otherwise, an application of the Infinite Ramsey Theorem.

(Usual) Dickson's Lemma. $(\mathbb{N}^k, \leqslant_\times)$ is wqo for any k.

SPOT THE WQO'S

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$\Sigma^*, \leqslant_{\text{lex}}$	✓	×	×
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$(\Sigma^*, \leqslant_{\text{pref}})$ has an infinite antichain

$$b, ab, aab, aaab, \ldots$$

$(\Sigma^*, \leqslant_{\text{lex}})$ is not well-founded:

$$b >_{\text{lex}} ab >_{\text{lex}} aab >_{\text{lex}} aaab >_{\text{lex}} \cdots$$

SPOT THE WQO'S

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$(\Sigma^*, \leqslant_*)$ is wqo by Higman's Lemma (see next slide).

We can get some feeling by trying to build a bad sequence, i.e., some $w_0, w_1, w_2, \ldots$ without an increasing pair $w_i \leqslant_* w_j$.

Higman's Lemma

Def. The sequence extension of a qo $(X, \leqslant)$ is the qo $(X^*, \leqslant_*)$ of finite sequences over X ordered by embedding:

$$w = x_1 \dots x_n \leqslant_* y_1 \dots y_m = v \overset{\text{def}}{\Leftrightarrow} x_1 \leqslant y_{l_1} \wedge \dots \wedge x_n \leqslant y_{l_n} \text{ for some } 1 \leqslant l_1 < l_2 < \dots < l_n \leqslant m$$

$$\overset{\text{def}}{\Leftrightarrow} w \leqslant_\times v' \text{ for a length-}n\text{ subsequence } v' \text{ of } v$$

Higman's Lemma. $(X^*, \leqslant_*)$ is a wqo iff $(X, \leqslant)$ is.

With $(\Sigma^*, \leqslant_*)$, we are considering the sequence extension of $(\Sigma, =)$ which is finite, hence necessarily wqo.

Later we'll consider the sequence extension of more complex wqo's, e.g., $\mathbb{N}^2$:

$$\begin{vmatrix}0\\1\end{vmatrix}\begin{vmatrix}2\\0\end{vmatrix}\begin{vmatrix}0\\2\end{vmatrix} \leqslant_* ? \begin{vmatrix}2\\0\end{vmatrix}\begin{vmatrix}0\\2\end{vmatrix}\begin{vmatrix}0\\2\end{vmatrix}\begin{vmatrix}2\\2\end{vmatrix}\begin{vmatrix}2\\0\end{vmatrix}\begin{vmatrix}0\\1\end{vmatrix}$$

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$$\Big|{}^{0}_{1}\ \Big|{}^{2}_{0}\ \Big|{}^{0}_{2} \ \leqslant_*?\ \Big|{}^{2}_{0}\ \Big|{}^{0}_{2}\ \Big|{}^{0}_{2}\ \Big|{}^{2}_{2}\ \Big|{}^{2}_{0}\ \Big|{}^{0}_{1}$$

Proof of Higman's Lemma

Let $(X, \leqslant)$ be wqo and assume by way of contradiction that $(X^*, \leqslant_*)$ admits infinite bad sequences (sequences with no increasing pairs).

Let $w_0 \in X^*$ be a shortest word that can start an infinite bad sequence.

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Continue. This way we pick an infinite sequence $S = w_0, w_1, w_2, w_3, \ldots$

Claim. S too is bad (easy with Def1)

Write w_i under the form $w_i = x_i v_i$. Since X is wqo, there is an infinite increasing sequence $x_{n_0} \leqslant x_{n_1} \leqslant x_{n_2} \leqslant \cdots$ (here we use Def2)

Now consider $S' \stackrel{\text{def}}{=} w_0, w_1, \ldots, w_{n_0 - 1}, v_{n_0}, v_{n_1}, v_{n_2}, \ldots$
It cannot be bad (otherwise w_{n_0} would not have been shortest).
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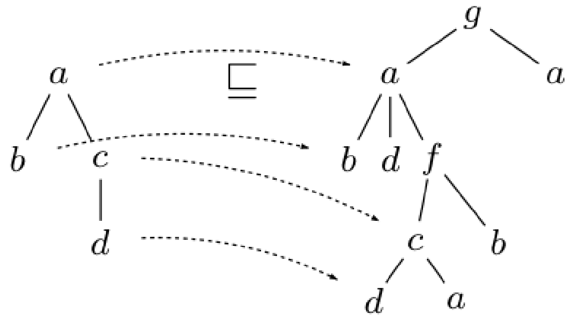
MORE WQO'S

- Finite Trees ordered by embeddings (Kruskal's Tree Theorem)



PROOF OF KRUSKAL'S TREE THEOREM

Let $(X,\leqslant)$ be wqo and assume, b.w.o.c., that $(\mathcal{T}(X),\sqsubseteq)$ is not wqo.

We pick a "minimal" bad sequence $S = t_0,t_1,t_2,\ldots$ —*Def1*

Write every t_i under the form $t_i = f_i(u_{i,1},\ldots,u_{i,k_i})$.

Claim. The set $U = \{u_{i,j}\}$ of the immediate subterms is wqo. (Indeed, an infinite bad sequence $u_{i_0,j_o},u_{i_1,j_i},..$ could be used to show that t_{i_0} was not "shortest").

Since U is wqo, and using Higman's Lemma on U^*, there is some $(u_{n_1,1},\ldots,u_{n_1,k_{n_1}}) \leqslant_* (u_{n_2,1},\ldots,u_{n_2,k_{n_2}}) \leqslant_* (u_{n_3,1},\ldots,u_{n_3,k_{n_3}}) \leqslant_* \cdots$ —*Def2*

Further extracting some $f_{n_{i_1}} \leqslant f_{n_{i_2}} \leqslant \cdots$ exhibits an infinite increasing subsequence $t_{n_{i_1}} \sqsubseteq t_{n_{i_2}} \sqsubseteq \cdots$ in S, a contradiction

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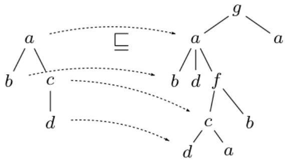

- Finite Graphs ordered by minor (Robertson-Seymour Theorem)

$$C_n \leqslant_{\text{minor}} K_n \text{ and } C_n \leqslant_{\text{minor}} C_{n+1}$$

- $(X^\omega, \leqslant_*)$ for X linear wqo.
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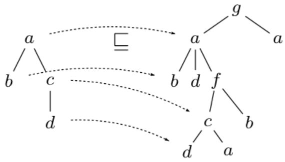

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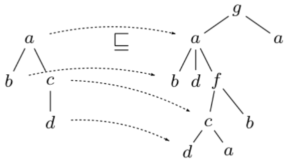

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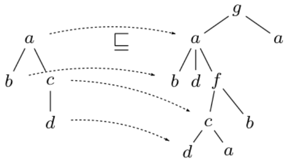

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FINITE-BASIS CHARACTERIZATION

Defn. $(X, \leqslant)$ is a wqo $\overset{\text{def}}{\Leftrightarrow}$ every non-empty subset V of X has at least one and at most finitely many (non-equivalent) minimal elements.

Say $V \subseteq X$ is upward-closed if $x \geqslant y \in V$ implies $x \in V$. (There is a similar notion of downward-closed sets).

For $B \subseteq X$, the upward-closure $\uparrow B$ of B is $\{x \mid x \geqslant b \text{ for some } b \in B\}$. Note that $\uparrow(\bigcup_i B_i) = \bigcup_i \uparrow B_i$, and that V is upward-closed iff $V = \uparrow V$.

Cor1. Any upward-closed $U \subseteq X$ has a finite basis, i.e., U is some $\uparrow\{m_1, \ldots, m_k\}$.

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E.g, Kuratowksi Theorem: a graph is planar iff it does not contain K_5 or $K_{3,3}$.

Gives polynomial-time characterization of closed sets.

Finite-basis characterization

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Fact. $\mathcal{P}(X)$ is well-founded iff X is wqo —Defn$'$

NB. X well-founded $\not\Rightarrow$ $\mathcal{P}(X)$ well-founded
Question. Does X wqo $\Rightarrow$ $\mathcal{P}(X)$ wqo? (Equivalently $\mathcal{P}_f(X)$ wqo?)

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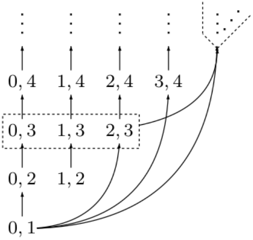

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Beyond wqo's

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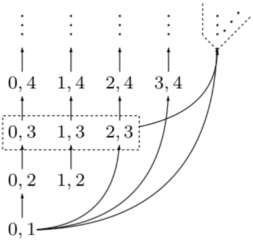

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Thm. 1. $(\mathcal{P}_f(X),\sqsubseteq_S)$ is not wqo: rows are incomparable
2. $(\mathcal{P}(Y),\sqsubseteq_S)$ is wqo iff Y does not contain X