Consensus Acceptor Games

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Games with imperfect information on finite graphs:

One player (against Nature): regular strategies are sufficient

Two or more players: what kind of strategies can they require?
Consensus acceptor games

Simplest game with imperfect information:
- two players against Nature on a finite graph
- every play is finite
- players have only one choice: yes/no when the play ends
- winning condition: consensus + in $\nu_+$ play +, in $\nu_-$ play −!

A concurrent safety game.
Acceptor Games:

Game over observation alphabet $\Gamma$

$$G = (V, E, (\beta^1, \beta^2), v_0, F)$$

- $(V, E)$ finite graph (states, transitions)
- observation functions $\beta^1, \beta^2 : V \rightarrow \Gamma$
- initial state $v_0$, final states $F \subseteq \{v_+, v_-, v_=\}$

Play:

- Nature chooses a path $\pi$ from $v_0$ to a final state
  $\leadsto$ observations $\beta^1(\pi), \beta^2(\pi)$
- Players choose decision $d^1, d^2 \in \{+, -\}$ simultaneously and independently
Consensus strategies

Winning condition:

- $d^1 = d^2$
- if final state is $v_+$, then $d^1 = d^2 = +$; same for $v_$

Strategy $s^i$ maps plays to $\{+, -\}$, s.t.
respects observations: if $\beta^i(\pi) = \beta^i(\pi')$ then $s^i(\pi) = s^i(\pi')$
write $\pi \sim^i \pi'$

Set $\sim^* := (\sim^1 \cup \sim^2)^*$
necessary for consensus: if $\pi \sim^* \pi'$, then $s(\pi) = s(\pi')$

"play +" must be common knowledge
Example: $a^n b^n$
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Example: \( a^n b^n \)
Fix $\Sigma \subseteq \Gamma$.

A strategy $s^1$ induces $L(s^1) := \{w \in \Sigma^* \mid s(#w#) = 1\}$

A game $G$ covers a language $L \subseteq \Sigma^*$ if:

- there exists a winning strategy $s$ such that $L(s) = L$
- for every winning strategy $s$, we have $L \subseteq L(s)$

A game $G$ characterises $L$ if for every winning strategy $s$, we have $L = L(s)$. 
Theorem.

1. Every context-sensitive language is characterised by a game
2. Deciding which decision to take in a play is PSPACE-hard
3. It is undecidable whether a game admits a winning strategy
4. If yes, winning strategies can be implemented by linear bounded automata
Characterisation: expressing corridor tilings with dominoes
Safe decision problem: membership in a context-sensitive language
Solvability problem: emptiness of a context-sensitive language (undecidable)
Implementation: Linear bounded automata can decide $\sim^*$
Domino systems and corridor tilings

Set of dominoes:

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\begin{array}{ccc}
  a & b & b \\
a & a & b \\
\end{array}
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\[
\begin{array}{ccc}
a & a & b \\
  b & b \\
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Domino systems and corridor tilings

Set of dominoes:

Tiling:
Domino systems and corridor tilings

Set of dominoes:

Corridor tiling:
Domino systems and corridor tilings

Set of dominoes:

Corridor tiling:

Frontier language
Domino systems and corridor tilings

Set of dominoes:

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Corridor tiling:

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Frontier language: $a^*b^*$
Theorem. [Latteux, Simplot, 97]
Every context-sensitive language is the frontier language of a domino system.
and vice versa

Given a domino system, construct a game $G$ that covers the frontier language
For every observation history $\pi$, accept if there exists a corridor tiling from $\pi$
Example: $a^n b^n$

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Example: $a^n b^n$
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Characterising

Given $L$ context-sensitive, construct $G$ that covers it. Construct $G'$ to cover the complement. Combine $G$ and $G'$ to characterise $L$. 
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Consequences

(2) Take a game characterising a PSPACE-hard language $L$, 
$$s(\pi) = 1 \iff \pi \in L$$

(3) Combine games characterising $L$ and the empty language 
$s(\pi) \equiv 0$ only candidate, winning if $L = \emptyset$

(4) At $\pi$, if exists $\pi' \sim^* \pi$ that ends at $v_-$, play $-$, else play $+$ 
winning strategy, (nondeterministic) linear space
Cover regular languages

From a finite automaton:
Cover regular languages

From a finite automaton:
Cover Context-free languages

Chomsky-Schützenberger theorem.
Every context-free language $L$ can be written as $h(R \cap D)$ with

1. $h$ is a homomorphism
2. $D$ is a Dyck language
3. $R$ is a regular language

Theorem. [Okhotin 12] Also works with

1. $h$ letter-to-letter homomorphism
2. $D$ Dyck language with neutral symbols
Construction ingredients

Construction is possible with ordered observations.
Order on $\Gamma$: $\beta^1(v) \geq \beta^2(v)$ for all $v$

Construction:
1. Homomorphic image: add a copy of the clique thread with $\beta^1(v) = h(\beta^2(v))$
2. Dyck language: easy, threads to erase matching parentheses
3. Intersection with $R$: product with an NFA
Intersection closure

With ordered observations we can capture more than context-free languages: 

**Lemma.** The class of games with ordered observations is closed under intersection.
Theorem. [Book, Greibach, 69] Every language in $NTIME(n)$ can be written as $h(K_1 \cap K_2 \cap K_3)$ with

1. $h$ is a letter-to-letter homomorphism
2. $K_1$, $K_2$ and $K_3$ are context-free languages

Corollary. Every language in $NTIME(n)$ can be covered by a consensus acceptor game with ordered observations